Assembly in Surgery

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1. Introduction

In [Y], I discussed gluing and splitting operations of geometric quadratic Poincaré complexes, and studied the $L^\infty$-theory assembly map

$$A : H_\ast(X; \mathbb{L}^\infty(p : E \to X)) \to L^\infty_\ast(\pi_1 E)$$

for certain polyhedral stratified systems of fibrations $p : E \to X$, following the general description of assembly maps by Quinn [Q, §8]. This assembly map was constructed in two steps: first we used the gluing operation to construct a map

$$\alpha : H_\ast(X; \mathbb{L}^\infty(p : E \to X)) \to L^\infty_\ast(p)$$

from the homology to the controlled $L$-group, and then composed it with the forget-control map

$$F : L^\infty_\ast(p : E \to X) \to L^\infty_\ast(E \to \{\ast\}) = L^\infty_\ast(\pi_1 E).$$

The following was claimed in (3.9) of [Y].

**Theorem.** If $p : E \to X$ is a polyhedral stratified system of fibrations on a finite polyhedron $X$, then the map $\alpha$ is an isomorphism.

The map $\alpha$ was constructed in the following way: an element of $H_b(K; \mathbb{L}^\infty(p : E \to X))$ can be thought of as a PL-triangulation $V$ of the product $S^N \times \Delta^k$ of a sphere $S^N$ ($N$ large) and the $k$-simplex $\Delta^k$ together with

1. a simplicial map $\phi : V \to X$, and
2. a compatible family $\{\rho(\Delta) \mid \Delta \in V\}$, where $\rho(\Delta)$ is a quadratic Poincaré (dim $\Delta + 2$)-ad on the pullback $q$ of $\bar{p} : \mathbb{R}^d \times E \to E \to X$ via the map $\Delta \to V \to X$, and $\rho(\Delta)$ is 0 if $\Delta$ is a simplex in the boundary.

I claimed that these ads $\rho(\Delta)$'s can be glued together to give a geometric quadratic Poincaré complex on $q$:
Theorem (Glueing over a manifold) [Y, 2.10] Let $L$ be the barycentric subdivision of a PL-triangulation $K$ of a compact $n$-dimensional manifold $M$ possibly with a non-empty boundary $\partial M$ and $p : E \to M$ be a map. And suppose each $n$-simplex $\Delta \in L$ is given an $m$-dimensional geometric quadratic Poincaré $(n+2)$-ad on $(p^{-1}(\Delta), p^{-1}(\partial \Delta))$ which are compatible on common faces. Then one can glue them together to get an $m$-dimensional geometric quadratic Poincaré pair on $(E, p^{-1}(\partial M))$.

If this is possible, then its functorial image on $\bar{p}$ gives a geometric quadratic complex on $\bar{p}$. By the ‘barycentric subdivision argument’ [Y, p.589], this assembled complex is equivalent to arbitrarily small complex and defines an element of $L^{-\infty}_\infty(p)$.

Unfortunately the argument given in [Y] is insufficient to prove this. The aim of this short note is to describe how to remedy this.

2. Glueing over a manifold

In [Y], I described the glueing operation of two quadratic Poincaré pairs along a common codimension 0 subcomplex of the boundaries. If there is an order of the $n$-simplices $\Delta_1, \ldots, \Delta_r$ of $L$ so that $(\Delta_1 \cup \ldots \cup \Delta_i) \cap \Delta_{i+1}$ is the union of $(n-1)$-simplices for each $i$, then we can successively glue the pieces in this linear order. But this seems very difficult to achieve. The strategy used in [Y] is the following:

For each vertex $v$ of $K$, consider its star $S(v)$ in $L$, i.e. the dual cone of $v$. Two such dual cones are either disjoint or meet along codimension 1 cells. The glueing problem over $S(v)$ can be solved by looking at the link $L(v)$ of $v$ in $L$. Note that $L(v)$ is an $(n-1)$-dimensional sphere or disk and the triangulation is the first barycentric subdivision of another. Thus we can keep on reducing the dimension until the link becomes a circle or an arc, and in this case there is an obvious order of 2-simplices and glueing can be done.

The fact is that the induction fails, since any two $n$-simplices of $S(v)$ have the vertex $v$ in common and are never disjoint.
There are two possible remedies for this. The first one is to use a different definition for the homology groups. This was actually done in [R].

Here I propose another remedy. Let us look at the dual cone at the vertex $v$. Let $c$ denote the quadratic Poincaré complex lying over $v$. Split each of the pieces of the dual cone so that the pieces near $v$ are of the form $c \otimes$ (a small simplex):

Here we do not need stabilization to split. We would like to glue the pieces away from $v$ first, and then fill in the hole with a piece of the form $c \otimes$ (a small copy of the dual cone):
To carry out the induction steps, we need to deal with holes of more complicated forms, and I have not worked out the details yet.

**Remarks.** (1) The control map should be a polyhedral stratified system of fibrations.
(2) The picture above may be misleading. The ‘hole’ itself lies over the vertex $v$, because $c \otimes (a small copy of the dual cone)$ can only live over $v$.
(3) Splitting needs a similar treatment.

**References**

