“The 35th Symposium on Transformation Groups”
Abstracts of the Talks

November 11(Tue) – 13(Thu), 2008

November 11 (Tue)

Two Rigidity Conjectures from Transformation Groups

Yoshinobu Kamishima (Tokyo Metropolitan University)

Recall two types of Rigidity Conjecture/Result.

I. The Topological Borel conjecture (A. Borel 1960).
II. The Conformal rigidity of compact Riemannian manifold (Obata & Lelong-Ferrand 1970).

This is supported by the Vague conjecture by D'ambra and Gromov 1990. We shall develop this conjecture into the framework of Geometric topology from the viewpoint of Transformation groups. In this talk, we study the following problems related to I,II.

- The Smooth Borel conjecture: Which compact aspherical smooth manifolds with isomorphic fundamental groups must be *diffeomorphic*?
- The Obata & Lelong-Ferrand theorem to *Lorentz manifolds* - If a closed group $\mathbb{R}$ acts conformally on a compact special Lorentz manifold, then is it conformal to the conformally flat Lorentz model $S^{n-1,1} \approx S^{n-1} \times S^1/\mathbb{Z}$?

Degeneracy Loci Formulae from a Point of View of Equivariant Cohomology

Takeshi Ikeda (Okayama University of Science)

Let E, F be two vector bundles over a manifold X and $\varphi : E \to F$ a generic vector bundle homomorphism. A question posed by Thom is, which cohomology class is defined by the set $\Sigma_d(\varphi) \subset M$ consisting of points x where the linear map $\varphi_x : E_x \to F_x$ has corank d? The answer, due to Porteous, is a determinant in terms of Chern classes of the bundles E, F. There are many generalization of this results. In this talk, I would like to discuss these "degeneracy loci formulas" from a point of view of torus equivariant cohomology.

Finite Transformation Groups and the Equivariant Determinant of Elliptic Operators

Kenji Tsuboi (Tokyo University of Marine Science and Technology)

Assumption

(1) A finite group G acts smoothly and effectively on a closed smooth manifold M.
(2) M has an almost complex structure and the G-action preserves the almost complex structure.
(3) G contains an element g of order p whose fixed point set consists of points.

Definition   Let $\{q_1, \cdots, q_n\}$ be the fixed point set of g. Then g acts on the tangent space $T_{q_i}M$ via multiplication by a diagonal matrix with diagonal entries $(\alpha_p^{\tau_{i1}}, \cdots, \alpha_p^{\tau_{im}})$ where α_p is the primitive p-th root of unity.
We call the set $\tau = ((\tau_{11}, \cdots, \tau_{1m}), \cdots, (\tau_{n1}, \cdots, \tau_{nm}))$ the **rotation angle** of g.

Equivalence of rotation angles is defined by

$$\tau = ((\tau_{11}, \cdots, \tau_{1m}), \cdots, (\tau_{n1}, \cdots, \tau_{nm})) \sim \tau' = (\underbrace{(\overbrace{z\tau_{11}, \cdots, z\tau_{1m}}^{\text{permutation } \eta_1}), \cdots, (\overbrace{z\tau_{n1}, \cdots, z\tau_{nm}}^{\text{permutation } \eta_n})}_{\text{permutation } \rho})$$

where $1 \leq z \leq p-1$.

Equivariant determinant Let D be a G-equivariant elliptic operator. Then the equivariant determinant $\det(D, g)$ of D evaluated at $g \in G$ is defined by

$$\det(D, g) := \det(g|\ker D)/\det(g|\mathrm{coker}\ D) \in S^1 \subset \mathbb{C}^* .$$

Define a homomorphism $I_D(\bullet) : G \longrightarrow \mathbb{R}/\mathbb{Z}$ by

$$I_D(g) := \frac{1}{2\pi\sqrt{-1}} \log\det(D, g) .$$

Let L_ℓ be a complex G-line bundle defined by $L_\ell = (\wedge^m_{\mathbb{C}} TM)^\ell$,

$$D_\ell : \Gamma(S_+ \otimes L_\ell) \longrightarrow \Gamma(S_- \otimes L_\ell) \quad (S_\pm : \text{half spinor bundles})$$

the Dirac operator defined by the natural Spin^c-structure of M, $\mathrm{Td}(TM)$ the Todd class of TM and $[M]$ the fundamental cycle of M .

Theorem Assume that p is an odd prime number. Then for $1 \leq z \leq p-1$, we have

$$\begin{aligned} I_{D_\ell}(g^z) \equiv\ & \frac{p-1}{2p} e^{\ell c_1(TM)} \mathrm{Td}(TM)[M] \\ & + \frac{1}{p^{m+2}} \sum_{i=1}^{n} \left\{ \left(z \prod_{j=1}^{m} \kappa_{ij}^{j} \right) \sum_{s=0}^{p-1} f_{m,p}(s) - p \sum_{s=0}^{p-1} f_{m,p}(s) \Phi_s(z\tau) \right\} \pmod{\mathbb{Z}} \end{aligned}$$

where κ_{ij} are integers determined by

$$\begin{aligned} & \kappa_{i1} = \tau_{i1} , \quad 1 \leq \kappa_{ij} \leq p-1 , \ \kappa_{ij} \equiv \overline{\tau}_{i\,j-1}\tau_{ij} \pmod{p} \quad (i \geq 2) \\ & (1 \leq \overline{\tau}_{i\,j-1} \leq p-1 , \ \tau_{i\,j-1}\overline{\tau}_{i,j-1} \equiv 1 \pmod{p}) , \end{aligned}$$

$f_{m,p}(s)$ is an integral polynomial defined by

$$\begin{aligned} f_{m,p}(s) = & \sum_{\kappa=0}^{m+1} \sum_{t=0}^{m-\kappa} (-1)^t \binom{m-\kappa}{t} \binom{-tp+s+m-p}{m} \\ & \sum_{u=\kappa}^{m+1} \binom{s}{m+1-u} \sum_{v=0}^{\kappa} (-1)^v \binom{\kappa}{v} \binom{pv}{u} \end{aligned}$$

and $\Phi_s(z\tau)$ is an integer defined by

$$\Phi_s(z\tau) = \# \left\{ \begin{pmatrix} j_{11}, j_{21}, j_{22}, \cdots \\ j_{m1}, \cdots, j_{mm} \end{pmatrix} \middle| \begin{array}{l} 0 \leq j_{11} < z\kappa_{i1} = z\tau_{i1} \\ 0 \leq j_{r1}, \cdots, j_{rr} < \kappa_{ir} \\ \qquad\qquad (2 \leq r \leq m) \\ 1 + (\ell+1)\sum_{j=1}^{m} z\tau_{ij} \\ +j_{11} + z\tau_{i1}(j_{21} + j_{22}) + \cdots \\ +z\tau_{i\,m-1}(j_{m1} + \cdots + j_{mm}) \\ +sz\tau_{im} \equiv 0 \pmod{p} \end{array} \right\} .$$

Using the theorem above, we can obtain information on the rotation angle.

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On the Isovariant Hopf Theorem

Ikumitsu Nagasaki (Kyoto Prefectural University of Medicine)

Let M be an n-dimensional connected orientable closed manifold. It is a well-known result, called the Hopf theorem or classification theorem of Hopf, that homotopy classes of continuous maps from M to the n-dimensional sphere S^n are classified by their degree. Namely, the degree of $f : M \to S^n$ induces the map $\deg : [M, S^n] \to \mathbb{Z}$, and then the Hopf theorem states that

Theorem. *The map* $\deg : [M, S^n] \to \mathbb{Z}$ *is a bijection.*

Equivariant versions of the Hopf theorem and related topics have been studied by Segal, Rubinsztein, Petrie, tom Dieck, Balanov, Ferrario, and others. We would like to discuss an isovariant version of the Hopf theorem under certain conditions.

Let G be a finite group and X, Y G-spaces.

Definition. A G-isovariant map $f : X \to Y$ is called G-isovariant if f preserves the isotropy groups, i.e., $G_{f(x)} = G_x$ for all $x \in X$.

A G-homotopy $F : X \times I \to Y$ is called an isovariant homotopy if F is G-isovariant.

Let M be a connected, orientable, closed G-manifold with free and orientation-preserving action, and SW a representation sphere (i.e., the unit sphere of unitary representation W of G). In this talk, using the multidegree introduced by [1], [2], we determine the isovariant homotopy set $[M, SW]_G^{\mathrm{isov}}$ under the Borsuk-Ulam inequality:

$$\dim M + 1 \leq \dim SW - \dim SW^{>1},$$

where $SW^{>1}$ denotes the singular set of SW.

This work is joint with F. Ushitaki.

[1] I. Nagasaki and F. Ushitaki, *Isovariant maps from free C_n-manifolds to representation spheres*, Topol. and its Appl. 155 (2008), 1066-1076.
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November 12 (Wed)

Lurie's Quasi Category Topos Theory

Norihiko Minami (Nagoya Institute of Technology)

In a series of papers:

- **[BOOK]:** Jacob Lurie, Higher Topos Theory, September 26, 2008, 731p.
- **[DAGI]:** Jacob Lurie, DAGI - Stable Infinity Categories, September 9, 2008, 72p.
- **[DAGII]:** Jacob Lurie, DAGII - Noncommutative Algebra, September 9, 2008, 170p.
- **[DAGIII]:** Jacob Lurie, DAGIII - Commutative Algebra, September 9, 2008, 80p.
- **[DAGIV]:** Jacob Lurie, DAGIV - Deformation Theory, September 9, 2008, 73p.
- **[DAGV]:** Jacob Lurie, DAGV - Structured Spaces, September 9, 2008, 137p;
- **[DAGVI]:** Lurie, J. Derived Algebraic Geometry VI: Spectral Schemes. In preparation.

Jacob Lurie embarked on the project of rewriting the Grothendieck theory of algebraic geometry, using ∞-category (which was first defined by Boardman-Vogt, and called "quasi category" by Joyal). In spite of their obvious importance, their long page-length made many reluctant to read these important series of works.

To remedy this situation, I wrote a survey paper on the ∞-category Yoneda's lemma presented in the thickest and most foundational [BOOK]:

- **[米田]:** 南範彦, "Lurie's quasi category Yoneda's lemma", 京都大学数理解析研究所講究録 **1612** , 変換群の幾何とその周辺, 21-40 (2008).

Now, the present talk is a continuation of this survey paper, and the purpose is to explain the following instructive analogies presented in the [BOOK]:

Vague analogies between Higher Category Theory and Algebra

Remark (BOOK, Remark 6.1.1.3.)**.** *Let $\mathcal{X}$ be an ∞-category. The assumption that colimits in $\mathcal{X}$ are universal can be viewed as a kind of distributive law. We have the following table of vague analogies:*

Higher Category Theory	*Algebra*
∞-*Category*	*Set*
Presentable ∞-*Category*	*Abelian Group*
Colimits	*Sums*
Limits	*Products*
$\lim_{\to}(X_\alpha) \times_S T \simeq \lim_{\to}(X_\alpha \times_S T)$	$(x+y)z = xz + yz$
∞-*Topos*	*Commutative Ring*

For this purpose, I shall give some ideas and background about the following definitions and theorems centering these analogies:

presentable ∞-category

Definition 1 (BOOK, Definition 5.5.0.18.)**.** An ∞-category $\mathcal{C}$ is ***presentable*** if $\mathcal{C}$ is accessible and admits small colimits.

Simpson's characterizqation of presentable ∞-category

Theorem 2 (BOOK, Theorem 5.5.1.1 (Simpson))**.** *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is presentable.*
(2) *The ∞-category $\mathcal{C}$ is accessible, and for every regular cardinal κ, the full subcategory $\mathcal{C}^{\kappa}$ admits κ-small colimits.*
(3) *There exists a regular cardinal κ such $\mathcal{C}$ is κ-accessible and $\mathcal{C}^{\kappa}$ admits κ-small colimits.*
(4) *There exists a regular cardinal κ, a small ∞-category $\mathcal{D}$ which admits κ-small colimits, and an equivalence $\mathrm{Ind}_{\kappa}\,\mathcal{D} \to \mathcal{C}$.*
(5) *There exists a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible localization of $\mathcal{P}(\mathcal{D})$.*
(6) *The ∞-category $\mathcal{C}$ is locally small, admits small colimits, and there exists a regular cardinal κ and a (small) set S of κ-compact objects of $\mathcal{C}$ such that every object of $\mathcal{C}$ is a colimit of a small diagram taking values in the full subcategory of $\mathcal{C}$ spanned by S.*

∞-topos

Definition 3 (BOOK, Definition 6.1.0.4.)**.** Let $\mathcal{X}$ be an ∞-category. We will say that $\mathcal{X}$ is an ***∞-topos*** if there exists a small ∞-category $\mathcal{C}$ and an accessible left exact localization functor $\mathcal{P}(\mathcal{C}) \to \mathcal{X}$.

Giraud's theorem of ∞-topos

Theorem 4 (BOOK, Theorem 6.1.0.6.)**.** *Let $\mathcal{X}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{X}$ is an ∞-topos.*
(2) *The ∞-category $\mathcal{X}$ is presentable, and for every small simplicial set K and every natural transformation $\overline{\alpha} : \overline{p} \to \overline{q}$ of diagrams $\overline{p}, \overline{q} : K^{\triangleright} \to \mathcal{X}$, the following condition is satisfied:*
   - If $\overline{q}$ is a colimit diagram and $\alpha = \overline{\alpha}|K$ is a Cartesian transformation, then $\overline{p}$ is a colimit diagram if and only if $\overline{\alpha}$ is a Cartesian transformation.
(3) *The ∞-category $\mathcal{X}$ satisfies the following ∞-categorical analogues of Giraud's axioms:*
   (i) *The ∞-category $\mathcal{X}$ is presentable.*
   (ii) *Colimits in $\mathcal{X}$ are universal.*
   (iii) *Coproducts in $\mathcal{X}$ are disjoint.*
   (iv) *Every groupoid object of $\mathcal{X}$ is effective.*

Finally, I do not claim any originality in my talk. However, since the concept of ∞-topos is, as the title of [BOOK] suggests, "the" most important concept in [BOOK], I hope this survey talk would make audicnce feel familiar with [BOOK].

On the Fourth Johnson Homomorphism of the Automorphism Group of a Free Group

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Keywords: Automrophism group of a free group, IA-automorphism group, Johnson homomorphism, Trace map

Abstract: In this talk we consider the Johnson homomorphism of the automorphism group of a free group with respect to the lower central series of the IA-automorphism group of a free group. In particular, we determine the ratoinal cokernel of the forth Johnson homomorphism, and show that there appears a new obsturustion for the surjectivity of the Johnson homomorphism. Furthermore we characterize this obstruction using trace maps.

Let F_n be a free group of rank $n \geq 2$, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. Let denote $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ the natural homomorphism induced from the abelianization H of F_n. The kernel of ρ is called the IA-automorphism group of F_n, denoted by IA_n. The IA-automorphism group IA_n reflects many richness and complexity of the structure of $\mathrm{Aut}\, F_n$, and plays important roles on various studies of $\mathrm{Aut}\, F_n$. The purpose of our research is to clarlify the group structure of IA_n.

Although the study of the IA-automorphism group has a long history since its finitely many generators were obtained by Magnus [10] in 1935, the combinatorial group structure of IA_n is still quite complicated. For instance, any presentation for IA_n is not known in general. Nielsen [12] showed that IA_2 coincides with the inner automorphsim group $\mathrm{Inn}\, F_2$, hence, is a free group of rank 2. For $n \geq 3$, however, IA_n is much larger than the inner automorphism group $\mathrm{Inn}\, F_n$. Krstić and McCool [9] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not.

In this talk, in order to study the group structure of IA_n, we consider the Johnson filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

consisting of certain normal subgroups of $\mathrm{Aut}\, F_n$, and the Johnson homomorphisms

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

defined on each graded quotient of the Johnson filtration. The study of the Johnson homomorphisms was originally begun in 1980 by D. Johnson [6] who determined the abelianization of the Torelli subgroup of a mapping class group of a surface in [7]. Then the theory of the Johnson homomorphisms has been developed by many authors. Now, there is a broad range of results for it. (For example, see [5] and [11].)

By virtue of the injectivity of the Johnson homomorphisms, considering their images, we can divide IA_n into infinitely many free abelian groups of finite rank. These pieces are regarded as one by one approximations of IA_n. Hence, to determine the image of the Johnson homomorphisms, or equivalently to determine their cokernel is one of the most basic problems. In this talk, in particular, we are interested in the irreducible decomposition of the cokernel of $\tau_{k,\mathbf{Q}} = \tau_k \otimes \mathrm{id}_{\mathbf{Q}}$ as a $\mathrm{GL}(n, \mathbf{Z})$-module. Now, for $1 \leq k \leq 3$, the cokernel of $\tau_{k,\mathbf{Q}}$ is completely determined. (See [1], [13] and [14] for $k = 1$, 2 and 3 respectively.) In general, however it is quite hard problem to solve since we can not obtain an explicit generating system of each $\mathrm{gr}^k(\mathcal{A}_n)$ easily.

To avoid this difficulty, we consider the lower central series $\mathcal{A}'_n(1) = \mathrm{IA}_n$, $\mathcal{A}'_n(2)$, … of IA_n. Since the Johnson filtration is central, $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for $k \geq 1$. It is conjectured that $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis who showed $\mathcal{A}'_2(k) =$

$\mathcal{A}_2(k)$ for each $k \geq 1$ and $\mathcal{A}'_3(3) = \mathcal{A}_3(3)$ in [1]. Now, we have $\mathcal{A}'_n(2) = \mathcal{A}_n(2)$ due to Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [8]. Furthermore $\mathcal{A}'_n(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [13].

For each $k \geq 1$, set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. Since IA_n is finitely generated as above, each $\mathrm{gr}^k(\mathcal{A}'_n)$ is also fintely generated as an abelian group. Then we can define the Johnson homomorphisms

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

with respect to the lower central series of IA_n by a similar way of defining τ_k. Since $\mathrm{gr}^k(\mathcal{A}'_n)$ is fintely generated, it is easier to study the cokernel of τ'_k than that of τ_k. Furthermore, It is also important to determine $\mathrm{Coker}(\tau'_k)$ from the view point of the study of the difference between the Johnson filtration and the lower central series of IA_n. In this talk, as a consective result of our research [14], we determine the rational cokernel of the forth Johnson homomorphism $\tau'_{4,\mathbf{Q}} := \tau'_4 \otimes \mathrm{id}_{\mathbf{Q}}$.

Theorem 1. *For any $n \geq 6$,*

$$\mathrm{Coker}(\tau'_{4,\mathbf{Q}}) = S^4 H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[2,1^2]} \oplus H_{\mathbf{Q}}^{[2,2]}.$$

In the right hand side of the equation above, the first term $S^4 H_{\mathbf{Q}}$ is called the Morita obstruction for the surjectivity of the Johnson homomorphism, which can be detected by the Morita trace $\mathrm{Tr}_{[k]}$. The second term is an obstruction which can be detected by the trace map $\mathrm{Tr}_{[2,1^2]}$ constructed in our previous paper [14]. The finial term is an obstruction of new type. It seems that this obstruction appear due to the speciality of the degree 4. In this talk, we construct a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism $\mathrm{Tr}_{[2,2]}$ which detect $H_{\mathbf{Q}}^{[2,2]}$, and call it a trace map for $H_{\mathbf{Q}}^{[2,2]}$.

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Borsuk-Ulam Type Theorems and Equilibria in a Class of Games

Stanisław Spież (Polish Academy of Sciences)

In 1968, R. Aumann, M. Mashler and R. Sterns posed a problem, whether any undiscounted infinitely repeated two-person game of incomplete information on one side has a Nash equilibrium. A brief description of these games is as follows. A game between two players $\mathcal{A}$ and $\mathcal{B}$ proceeds in infinitely many successive stages. In the 0-stage a k is chosen from a finite set K of "states of nature" according to a probability distribution $p_0 \in \mathrm{int}\Delta(K)$. In any subsequent stage each of the players selects a "pure action" from a finite sets I (for $\mathcal{A}$) and J (for $\mathcal{B}$), gaining a stage-payoff $A_k(i,j)$ (for $\mathcal{A}$) or $B_k(i,j)$ (for $\mathcal{B}$), which depends only on the pure actions $i \in I$ and $j \in J$ selected in this stage and the "true state of nature" k, chosen at stage 0. At any stage the players also know the pure actions both of them took on proceedings stages and $\mathcal{A}$ (but not $\mathcal{B}$) knows the outcome $k \in K$ of the 0-stage. We settle in the positive the problem stated above. We also extend this result to more general games.

Several classical proofs in game theory depend on various fixed point and related theorems. Our proofs depend on new topological results. One of them, in its simplest form, states that if x_0 is a point of a compact subset C of $\mathbf{R}^n$ and $f : C \to Y$ is a mapping such that dimension of $f(\mathrm{int}C)$ is less then n, then in the boundary of C there exists a set C_0 mapped by f into a singleton and containing x_0 in its convex hull. The resemblance with Borsuk-Ulam theorem is that if C is is an n-ball and $Y = \mathbf{R}^{n-1}$, then the later says that C_0 may be taken so as to consist of two points only. We also prove a parametric version version of Borsuk-Ulam theorem.

This research is joint with T. Schick, R. S. Simon and H. Toruńczyk.

Shimizu's Lemma for Complex Hyperbolic Space and its Application

Shigeyasu Kamiya (Okayama University of Science)

It is important to find some conditions for a group to be discrete or non-discrete.

A well known result of Shimizu states the following:

Theorem. *Let G be a discrete subgroup of* $\mathrm{PSL}(2,\mathbf{R})$ *containing the parabolic element $A(z) = z + t$ for some $t > 0$. If $B(z) = (az+b)/(cz+d)$ is an element in G with $c \neq 0$, then $|c| \geq 1/t$.*

This is called Shimizu's Lemma.

Geometrically this result says that the radius r_B of the isometric circle $I(B)$ of B satisfies $r_B \leq t$. Equivalently, the horoball U_t of height t is precisely invariant under G_∞ in G.

In this talk we show Shimizu's lemma for complex hyperbolic space and we discuss its application to complex hyperbolic triangle groups of type (n, n, ∞).

Current Trends in the Study of the Smith Equivalent Representations
Krzysztof Pawałowski (Adam Mickiewicz University)

This is a joint work in progress of Krzysztof Pawałowski and Toshio Sumi.

Tangent spaces as group modules

Let G be a finite group. By a *real G-module* we mean a finite dimensional real vector space with a linear action of G given by a representation $G \to GL(n, \mathbb{R})$ of G. If G acts smoothly on a manifold M, then at any point $a \in M$ fixed under the action of G, the tangent space $T_a(M)$ can be considered as a real G-module. The action of G on $T_a(M)$ is defined by $gv = D_a\theta_g(v)$ for all $g \in G$, $v \in T_a(M)$, where $D_a\theta_g \colon T_a(M) \to T_a(M)$ is the derivative at the point a of the map $\theta_g : M \to M$ given by $\theta_g(x) = gx$ for all $x \in M$.

According to the Slice Theorem, the action of G on the tangent space $T_a(M)$ is equivalent to the action of G on some open neighborhood of a in M. Therefore, the G-module $T_a(M)$ describes the local behaviour of the action of G on M around the fixed point a.

The Smith isomorphism question

In 1960, P. A. Smith [16] posed the following question. If a finite group G acts smoothy on a sphere S with exactly two fixed points a and b, is it true that as real G-modules, the tangent spaces $T_a(S)$ and $T_b(S)$ are *always* isomorphic?

Let G be a finite group. Two real G-modules U and V are called *Smith equivalent* if as real G-modules,

$$U \cong T_a(S) \ \text{ and } \ V \cong T_b(S)$$

for a smooth action of G on a homotopy sphere S with exactly two fixed points a and b. If in addition the fixed point set S^g is connected for any element $g \in G$ of order 2^k for $k \geq 3$, then U and V are called *L-Smith equivalent*.

In 1996, Laitinen has posed the conjecture that for a finite Oliver group G, there exist pairs of noisomorphic, L-Smith equivalent real G-modules if and only if $r_G \geq 2$, where r_G denotes the number of real conjugacy classes in G of elements $g \in G$ not of prime power order.

The Smith question is answered in many cases of G. The articles [2], [3], [4], [12] and [13] contain long lists of related references, while [5], [6], [9], [10], [14] and [15] show the current trends in the study of the Smith equivalence problem and the Laitinen conjecture.

By [8], in the Laitinen conjecture, the condition that $r_G \geq 2$ is necessary, but Morimoto [9] proves that this condition is not sufficient for $G = \mathrm{Aut}(A_6)$, the automorphism group of the alternating group A_6. In [15], we show that this condition is not sufficient also for some finite solvable Oliver groups G, including the affine group $G = \mathrm{Aff}(2, 3)$.

Answers to Smith isomorphism question

Our goal is to impose a condition on G, stronger than $r_G \geq 2$, which implies that for any finite Oliver group G satisfying the condition, there exist pairs of nonisomorphic, L-Smith equivalent real G-modules. We describe this condition in the following way.

For any finite group G, we denote by G^{nil} the smallest normal subgroup of G such that the quotient group G/G^{nil} is nilpotent. We say that two elements x and y of G form a *nil-pair* if $xG^{\mathrm{nil}} = yG^{\mathrm{nil}}$ and the following two conclusions hold.

(1) The elements x and y are not real conjugate in G, and the orders $|x|$ and $|y|$ of x and y, respectively, are not prime powers (and thus $r_G \geq 2$).
(2) The elements x and y are in some gap subgroup of G, or the orders $|x|$ and $|y|$ are even and the involutions of $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G.

We say that a finite group G is of *nil-type*, or G is a *nil-type group*, if there exists a nil-pair of elements of G. Recall that a finite group G is called an *Oliver group* if there does not exist a series of normal subgroups $P \trianglelefteq H \trianglelefteq G$ such that P is a p-group, G/H is a q-group for some primes p and q, possibly $p = q$, and H/P is cyclic. According to Oliver [11], this algebraic condition is necessary and sufficient for G to have a smooth fixed point free action on a disk, or as it is shown in [7], to have a smooth one fixed point action on a sphere.

Theorem A. *For any finite Oliver nil-type group G, there exist pairs of nonisomorphic real G-modules U and V such that the following three conclusions hold.*

(1) $\mathrm{Res}^G_P(U) \cong \mathrm{Res}^G_P(V)$ *for $P \in \mathcal{P}(G)$, the family of prime power order subgroups of G.*
(2) $\dim U^L = \dim V^L = 0$ *for $L \in \mathcal{L}(G)$, the family of large subgroup of G.*
(3) *U and V both satisfy the weak gap condition and they are L-Smith equivalent.*

Recall that $r_G = 2$ in the case where $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2, 27)$, a semi-direct product of $PSL(2, 27)$ and the automorphism group of the field of 27 elements. As one can check directly, the two nonsolvable groups $\mathrm{Aut}(A_6)$ and $P\Sigma L(2, 27)$ are not of nil-type.

Theorem B. *Except for* $\mathrm{Aut}(A_6)$ *and $P\Sigma L(2, 27)$, a finite nonsolvable group G is of nil-type if and only if $r_G \geq 2$.*

For $G = \mathrm{Aut}(A_6)$, Morimoto [9] proves that any two Smith equivalent real G-modules are isomorphic. In contrast, for $G = P\Sigma L(2, 27)$, Morimoto [10] proves that there exist pairs of nonisomorphic, L-Smith equivalent real G-modules. As by the results of [8], the conditon that $r_G \geq 2$ is necessary in the Laitinen conjecture, Theorems A and B, and the papers [9] and [10] yield the following theorem.

Theorem C. *Except for the automorphism group of the alternating group on six letters, the Laitinen conjecture is true for any finite nonsolvable group.*

This extends the result of [8] showing that the Laitinen conjecture holds for any finite perfect group. Now, we are able to answer the Smith isomorphism question as follows.

Theorem D. *In either of the cases (1)–(6) below, there exist pairs of nonisomorphic, Smith equivalent real G-modules if and only if $r_G \geq 2$.*

(1) *G is a finite simple group.*
(2) *$G = PSL(n, q)$ or $SL(n, q)$ for any $n \geq 2$ and any prime power q.*
(3) *$G = PSp(n, q)$ or $Sp(n, q)$ for any even $n \geq 2$ and any prime power q.*
(4) *$G = A_n$ or S_n for any $n \geq 2$.*
(5) *$G = PGL(n, q)$ or $GL(n, q)$ for any $n \geq 2$ and any prime power q.*
(6) *$G = \mathrm{Aff}(n, q)$ for any $n \geq 2$ and any prime power q, and $G \neq \mathrm{Aff}(2, 3)$.*

By the work of Atiyah and Bott [1], for $G = \mathbb{Z}_p$ where p is prime, any two Smith equivalent real G-modules are isomorphic and by [13], the same conclusion is true for finite nonabelian simple groups G with $r_G = 0$ or 1. Therefore, in the case (1) of Theorem D, the condition that $r_G \geq 2$ is necessary. In the remaining five cases (2)–(6), the necessity of this condition follows from direct arguments for the values of n and q in the cases where $r_G = 0$ or 1.

In Theorem D, the sufficiency of the condition that $r_G \geq 2$ follows from equivariant surgery. The work [13] covers the cases (1)–(4) using surgery under the gap condition, and by applying surgery under the weak gap condition, we obtain the conclusion in the cases (5) and (6).

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November 13 (Thu)

Homeomorphism and Diffeomorphism Groups of Non-compact Manifolds with the Whitney Topology

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(Joint Work with T. Banakh, K. Mine, K. Sakai)

In this talk we discuss conditions which ensure that topological groups are locally homeomorphic to $\mathbb{R}^\infty \times \ell_2$, and apply this characterization to determine the local/gloal topological types of homeomorphism/diffeomorphism groups of noncompact manifolds endowed with the Whitney topology.

§1. Topological groups locally homeomorphic to $\mathbb{R}^\infty \times \ell_2$.

Suppose G is a topological group. Our main concern is the following fundamental problem.

Problem. *When is G locally homeomorphic to ($\approx_{loc}$) a topological linear space L ?*

The case $L = \mathbb{R}^n$ was solved by A. Gleason, D. Montgomery - L. Zippin and the case $L = \ell_2$ was studied by T. Dobrowolski - H. Toruńczyk. In this talk we consider the case $L = \mathbb{R}^\infty \times \ell_2$, where $\mathbb{R}^\infty$ is the direct limit of the sequence $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$

A tower of closed subgroups of G is a sequence $(G_n)_n$ of closed subgroups of G such that

$$G_1 \subset G_2 \subset \cdots, \qquad G = \bigcup_n G_n.$$

This tower yields the small box product $\boxdot_n G_n$ and the multiplication map

$$p : \boxdot_n G_n \to G : \quad p(x_1, x_2, \cdots) = x_1 \cdot x_2 \cdots .$$

We say that

(i) G carries the box topology with respect to (G_n) if the map p is an open map, and

(ii) (G_n) is (locally) topologically complemented (TC) in G if each quotient map $G_n \to G_n/G_{n-1}$ has a (local) section.

Theorem. *$G \approx_{(loc)} \mathbb{R}^\infty \times \ell_2$ if*

($*_1$) *G carries the box topology with respect to (G_n)*

($*_2$) *$(G_n)_n$ is (locally) TC in G*

($*_3$) *$G_n/G_{n-1} \approx_{(loc)} \ell_2$ $(n \geq 1)$*

§2. Homeomorphism and diffeomorphism groups of noncompact manifolds endowed with the Whitney topology.

§2.1. Homeomorphism groups.

Suppose M is a non-compact connected n-manifold without boundary. Let $\mathcal{H}(M)$ denote the group of homeomorphisms of M endowed with the Whitney topology and $\mathcal{H}_c(M)$ denote the subgroup of homeomorphisms of M with compact support. Since the identity component $\mathcal{H}_0(M)$ of $\mathcal{H}(M)$ is included in $\mathcal{H}_c(M)$, we have the mapping class group $\mathcal{M}_c(M) = \mathcal{H}_c(M)/\mathcal{H}_0(M)$ (with the discrete topology).

Proposition. *(1) $\mathcal{H}_c(M)$ is locally contractible. (2) $\mathcal{H}_c(M) \approx \mathcal{H}_0(M) \times \mathcal{M}_c(M)$.*

In the case $n = 1, 2$ we obtain the complete classification of topological types of $\mathcal{H}_c(M)$.

Theorem.

(1) $\mathcal{H}_c(\mathbb{R}) = \mathcal{H}_0(\mathbb{R}) \approx \mathbb{R}^\infty \times \ell_2$

(2) *the case* $n = 2$: $\mathcal{H}_0(M) \approx \mathbb{R}^\infty \times \ell_2$ $\mathcal{H}_c(M) \approx \begin{cases} \mathbb{R}^\infty \times \ell_2 & (\#) \\ \mathbb{R}^\infty \times \ell_2 \times \mathbb{Z} & \textit{all other cases} \end{cases}$

(#) $M \approx X \setminus K$, *where*
(i) $X =$ *an annulus, a disk or a Möbius band*
(ii) K *is a non-empty compact subset of a boundary circle of* X

§2.2. Diffeomorphism groups.

Suppose M is a non-compact connected smooth n-manifold without boundary. Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of M endowed with the Whitney C^∞-topology and $\mathcal{D}_c(M)$ denote the subgroup of diffeomorphisms of M with compact support. The identity component $\mathcal{D}_0(M)$ of $\mathcal{D}(M)$ is included in $\mathcal{D}_c(M)$ and induces the mapping class group $\mathcal{M}_c^\infty(M) = \mathcal{D}_c(M)/\mathcal{D}_0(M)$ (with the discrete topology).

Theorem. *(1)* $\mathcal{D}_c(M) \approx_{loc} \mathbb{R}^\infty \times \ell_2$. *(2)* $\mathcal{D}_c(M) \approx \mathcal{D}_0(M) \times \mathcal{M}_c^\infty(M)$.

Theorem.

(1) *the case* $n = 1, 2$: $\mathcal{D}_0(M) \approx \mathbb{R}^\infty \times \ell_2$.
(2) *the case* $n = 3$: *If* M *is orientable and irreducible, then* $\mathcal{D}_0(M) \approx \mathbb{R}^\infty \times \ell_2$.
(3) *If* X *is a compact connected smooth* n*-manifold with boundary and* $\mathcal{D}_0(X, \partial X)$ *has a homotopy type of a locally compact polyhedron* L*, then*

$$\mathcal{D}_0(\mathrm{Int}\, X) \approx \mathcal{D}_0(X, \partial X) \times \mathbb{R}^\infty \times \ell_2 \approx L \times \mathbb{R}^\infty \times \ell_2.$$

Recently, we have obtained some results on the spaces of maps endowed with the graph topology.

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A New Theorem to Find Smith Equivalent Representations

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Let G be a finite group. Two $\mathbb{R}G$-modules V and W are **Smith equivalent** if there exists a homotopy sphere Σ with a G-action such that $\Sigma^G = \{a, b\}$, $T_a(\Sigma) \cong_G V$ and $T_b(\Sigma) \cong_G W$. Smith equivalent pairs (V, W) obtained by various authors, e.g. T. Petrie, S. Cappell-J. Shaneson, K. Pawałowski-R. Solomon and etc., satisfy

$$V^N = 0 = W^N \text{ for all } N \triangleleft G \text{ with } |G/N| = \text{prime}.$$

Problem 1. Find Smith equivalent pairs (V, W) satisfying

$$V^N \neq 0 \text{ and } W^N = 0$$

for some $N \triangleleft G$ such that $|G/N|$ is a prime.

We will invoke the next hypothesis on the group G.

Hypothesis 1. G is an Oliver group such that $G^{\text{nil}} = G^{\{p\}} \neq G$ for some **odd prime** p.

We set $N = G^{\text{nil}}$ and $\mathcal{L}(G) = \{H \leq G \mid H \supseteq G^{\{p\}}\}$. We will invoke the next hypothesis on the $\mathbb{R}G$-modules V and W.

Hypothesis 2. V is an $\mathbb{R}G$-module such that $V^G = \mathbb{R}$. W is an $\mathcal{L}(G)$-free $\mathbb{R}G$-module $\mathcal{P}(G)$-matched with V.

Let
$$\mathbb{R}[G]_{\mathcal{L}} = \mathbb{R}[G] - \mathbb{R}[G/G^{\{p\}}].$$
Then we obtain the following results.

Lemma 1. For a sufficiently large integer a, there exists a smooth G-action on a disk D such that

(1) $D^G = \{x\}$,
(2) $D^L{}_x = P(V^N)^L$ (conn. component) for all $L \in \mathcal{L}(G)$,
(3) $(|\pi_1(D^Q)|, q) = 1$ for any $Q \in \mathcal{P}(G)$ and $|Q| = q^a > 1$,
(4) $T_x(D) = (V - V^G) \oplus \mathbb{R}[G]_{\mathcal{L}}{}^{\oplus a}$.

A $\mathcal{P}(G)$-matched pair (V, W) is called of **type** $\mathbb{R}[G/N]$ if
$$V^N \cong \mathbb{R}[G/N] \text{ and } W^N = 0.$$
A $\mathcal{P}(G)$-matched pair (V_1, V_2) is called of **type** $\mathbb{R}[G/N]_G{}^n$ if $V_2^N = 0$ and there exists a $\mathcal{P}(G)$-matched pair (V, W) of type $\mathbb{R}[G/N]$ such that
$$V_1^N = (V^N - V^G)^{\oplus n}.$$

Lemma 2. Suppose N has a subquotient $\cong D_{2qr}$ $(q \neq r)$. For each $i = 1, \ldots m$, let $(V_1^{(i)}, V_2^{(i)})$ be a $\mathcal{P}(G)$-matched pair of type $\mathbb{R}[G/N]_G{}^{n_i}$ with respect to $(U_1^{(i)}, U_2^{(i)})$ of type $\mathbb{R}[G/N]$. Suppose $V_1^{(1)}, \ldots, V_1^{(m)}$ are $\mathcal{P}(G)$-matched with one another. Then for a sufficiently large integer a there exits a smooth G-action on a disk D such that

(1) $D^G = \{x_1, \ldots, x_m\}$,
(2) $D^N{}_{x_i} = P(U_1^{(i)N})^{\times n_i}$ (conn. comp.), $1 \le i \le m$,
(3) $T_{x_i}(D) \cong V_1^{(i)} \oplus \mathbb{R}[G]_{\mathcal{L}}{}^{\oplus a}$.

Theorem 1. Let U be a gap G-module. Suppose N has a subquotient $\cong D_{2qr}$ $(q \neq r)$. For each $1 \le i \le m$, let $(V_1^{(i)}, V_2^{(i)})$ be a $\mathcal{P}(G)$-matched pair of type $\mathbb{R}[G/N]_G{}^{n_i}$ with respect to $(U_1^{(i)}, U_2^{(i)})$ of type $\mathbb{R}[G/N]$. Suppose $V_1^{(1)}, \ldots, V_1^{(m)}$ are $\mathcal{P}(G)$-matched with one another. Then for sufficiently large integers a and b, there exists a smooth G-action on a sphere S such that

(1) $S^G = \{x_1, \ldots, x_m\}$,
(2) $S^N{}_{x_i} = P(U_1^{(i)N})^{\times n_i}$ (conn.comp.), $1 \le i \le m$,
(3) $T_{x_i}(S) \cong V_1^{(i)} \oplus \mathbb{R}[G]_{\mathcal{L}}{}^{\oplus a} \oplus U^{\oplus b}$.

Corollary 1. Let G be an Oliver group such that

(1) $|G/G^{\mathrm{nil}}| = p$ (p odd prime),
(2) $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \neq 0$,
(3) G^{nil} has a subquotient $\cong D_{2qr}$ $(q \neq r)$.

Then $\mathbb{R}[G]_{\mathcal{L}(G)}$ is a gap module and $Sm(G) \supset \mathbb{Z}$. Particularly, if $p = 3$ and $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \cong \mathbb{Z}$ then $Sm(G) = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$.

We answer to problems posed by K. Pawałowski-R. Solomon and T. Sumi.

Corollary 2. If $G = P\Sigma L(2, 27)$, $SG(864, 2666)$, or $SG(864, 4666)$ then $Sm(G) = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \cong \mathbb{Z}$.