Two rigidity conjectures from Transformation groups

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Introduction

We recall two types of Rigidity Conjecture/Theorem posed in the days of topology.

I. The *Borel conjecture* (A. Borel 1960)
It expects that *any* two compact aspherical topological manifolds with isomorphic fundamental groups must be *homeomorphic*.

II. *The Conformal rigidity*
(Obata & Lelong-Ferrand 1970)
If a closed noncompact group $\mathbb{R}$ acts conformally on a compact Riemannian manifold of dimension $n > 3$, then it is conformal to S^n.

Supporting Evidence to the topological case I

- Any two closed aspherical topological manifolds with isomorphic fundamental groups of virtually abelian groups are homeomorphic in dimension $n \neq 4$ (Farrell-Hsiang 1978).
- Any two closed aspherical topological manifolds with isomorphic fundamental groups of virtually nilpotent groups are homeomorphic in dimension $n \neq 4$ (Farrell-Hsiang 1983).
- Any two closed aspherical topological manifolds with isomorphic fundamental groups π are homeomorphic in dimension $n \neq 4$ (Farrell-Jones 1998). Here π is isomorphic to a discrete subgroup of $\mathrm{GL}(m, \mathbb{R})$ (m large).

Supporting Evidence to the topological case

Note that this last statement covers the previous results. The method to the proofs is based on the topological surgery theory and the calculation of L-groups. The previous results to the topological case were inspired by the following smooth classical results.

Supporting Evidence to the smooth case I

- Bieberbach Theorem (1911) - If two compact Riemannian flat manifolds are homotopic, then they are affinely diffeomorphic.
- Mal'cev Theorem (1949) - If two compact nilmanifolds are homotopic, then they are *isomorphic*.
- Mostow Theorem (1954) - If two compact solvmanifolds are homotopic, then they are *diffeomorphic*.
- If two compact infranilmanifolds are homotopic, then they are affinely diffeomorphic (Auslander 1970's, Kamishima-Lee-Raymond 1983).
- If two compact infrasolvmanifolds are homotopic, then they are diffeomorphic (Baues 2004, Farrell-Jones 1997).

Supporting Evidence I to the smooth case I

Another class satisfying rigidity:

- Mostow $\mathbb{K}$-hyperbolic rigidity ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$) 1973.
- Gromov-Margulis rigidity (Ballman-Schroeder's book 1985): If a compact Riemannian manifold of nonpositive sectional curvature with flat dimension ≥ 2 is homotopic to a compact *locally symmetric* Riemannian manifold, then two such Riemannian manifolds are isometric.

Supporting Evidence to O& L-F II

- (CR-analogue of O&L-F) If a closed noncompact group $\mathbb{R}$ acts as CR transformations on a compact CR-manifold of $2n+1>3$, then it is CR-isomorphic to S^{2n+1} (Kamishima, J. Lee, R. Schoen 1996).
- (Quaternionic CR-analogue of O&L-F) If $\mathbb{R}$ acts as quaternionic CR transformations on a compact quaternionic CR-manifold of $4n+3>3$, it is pseudo-conformally isomorphic to S^{4n+3}. (1996, 2007.)
- (Quaternion Kähler analogue of O&L-F) If $\mathbb{R}$ acts as projective transformations on a compact quaternionic Kähler manifold of $4n>4$, it is projectively isomorphic to $\mathbb{HP}^n$ (Alekseevsky -Marchiafava 1990's).

Aim

We shall develop these conjecture/theorem into the framework of Geometric Topology from the viewpoint of Transformation groups. That is,

— The *Vague conjecture* —

(D'ambra and Gromov 1990)

If there exists a global geometric flow (relatively big Lie group) on a compact geometric manifold M, then M is rigid, i.e. *isomorphic to the standard model with flat G-structure*.

Aim

In order to make *vague conjecture* clear, we formulate the following two problems specifying the above conjecture/theorem I,II.

(SI) The Smooth Borel conjecture: Any two compact aspherical *smooth* manifolds with isomorphic fundamental groups must be *diffeomorphic*.

(LII) The Obata & Lelong-Ferrand theorem to Lorentz manifolds - If a closed group $\mathbb{R}$ acts conformally on a compact Lorentz manifold of dimension $n > 3$, then it is conformal to the conformally flat Lorentz model $S^{n-1,1} \approx S^{n-1} \times S^1/\mathbb{Z}_2$.

At once we see that these two problems SI, LII are false.

Aim

Nevertheless our purpose of this talk is to study the following problems to establish affirmative results.

- Which class of aspherical smooth manifolds satisfies the smooth Borel conjecture?
- Which class of compact Lorentz manifolds satisfies the analogue of Obata & Lelong-Ferrand' theorem?
- Which kind of closed noncompact connected conformal transformations assures that a compact Lorentz manifold is conformal to $S^{n-1,1}$?

First we observe the counterexamples of (SI) smooth Borel rigidity and (LII) Lorentz Obata & Lelong-Ferrand theorem.

(SI)

The Smooth Borel conjecture.

Counterexamples SI

It is well known that

- The connected sum $T^7 \# \Sigma$ with an exotic sphere is not diffeomorphic to T^7 (Browder).

  In fact, we presume that Borel thought at first it is true for smooth aspherical manifolds but Browder, Wall have studied smooth structures of the connected sum of a manifold with homotopy spheres at late 60's. Then he knew a counterexample. Note that the connected sum of exotic spheres does not cover all the diffeomorphism classes of smooth 7-torus from the L-theory by the work of Wall. It is unknown how to construct the remaining explicitly.
  Virtual smooth Borel conjecture !
- A connected sum of closed hyperbolic manifold with an exotic sphere (Farrell and Jones 1980's).

Rigidity results to Problem SI (with Oliver Baues)

Affirmative results: Take the *aspherical homogeneous manifold* $M = G/H$. A systematic study of general aspherical homogeneous spaces was persisted by Gorbatsevich in a series of papers. We shall answer to his question.

Theorem A. *Let M, M' be compact aspherical homogeneous manifolds. If $\phi : \pi_1(M) \to \pi_1(M')$ is an isomorphism, then ϕ is induced by a diffeomorphism $\Phi : M \to M'$.*

Rigidity results to Problem SI (with Oliver Baues)

We start with a fiber space with *solv-geometry*: $(X, \mathcal{R}, H)$ where H acts properly on X and $\mathcal{R}$ is a simply connected solvable Lie group $\mathcal{R}$ $(H \rhd \mathcal{R})$ and $\exists$ principal bundle:

$$\mathcal{R} \longrightarrow X \stackrel{\mathsf{p}}{\longrightarrow} W = X/\mathcal{R},$$

Baues observed the T*-compatibility* condition for the reductive group T of the algebraic hull $A(\mathcal{R})$.
We prove that $\exists$ a simply connected nilpotent Lie group $\mathbf{U}$ such that the T-compatible triad $(X, \mathcal{R}, H)$ is equivalent to *standard* Γ*-fiber space* $(X, \mathbf{U}, \pi)$ whose fiber stabilising group Γ is a discrete standard subgroup of $\mathrm{Aff}(\mathbf{U})$.

We have the following rigidity for $1 \to \Gamma \to \pi \to \Theta \to 1$.

Rigidity results to Problem SI (with Oliver Baues)

Theorem B. *Let* $\rho : \pi \to \mathrm{Diff}_a(X, A(\Gamma))$, $\rho' : \pi' \to \mathrm{Diff}_a(X', A(\Gamma'))$ *be standard homomorphisms. If* ϕ *is an isomorphism of extensions, then every equivariant diffeomorphism* $(\bar{f}, \bar{\phi}) : (W, \Theta) \to (W', \Theta')$ *lifts to an equivariant diffeomorphism*

$$(f, \phi) : (X, \pi) \to (X', \pi').$$

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- Theorem A is obtained from Theorem B by showing Aspherical homogeneous manifold has the structure of T-compatible fiber space with *solv-geometry*.

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- Theorem A is obtained from Theorem B by showing Aspherical homogeneous manifold has the structure of T-compatible fiber space with *solv-geometry*.
- This kind of rigidity was originally proved by Conner and Raymond for the fiber space with abelian geometry $\mathbb{R}^n$ such that $\Gamma \subset \mathbb{R}^n$ (1970). In our case, $\Gamma \subset \mathrm{Aff}(\mathbb{R}^n)$ not necessarily lattice but standard.

Application-Double coset space

Let $G = \mathcal{R} \cdot \tilde{S}$ be a connected simply connected Lie group with Levi decomposition such that $\mathcal{R}$ is the radical and $\tilde{S}$ is a semisimple subgroup of noncompact type. Then there is the canonical exact sequence:

$$1 \to \mathcal{R} \to G \xrightarrow{p} S \to 1.$$

Here $S = p(\tilde{S})$ is a (centerless) connected semisimple Lie group without compact factor. Since G is simply connected, a maximal compact subgroup K maps isomorphically onto that of S and S/K is a simply connected noncompact Riemannian symmetric space.

Application-Double coset spaces

Let π be a discrete cocompact subgroup of G. If π is torsionfree, the double coset space $\pi\backslash G/K$ is a closed aspherical manifold.
As $Q = p(\pi) \subset S$ is a discrete cocompact subgroup, the Mostow rigidity theorem says that given an isomorphism $\bar{\phi} : Q \to Q'$, there exists an equivariant diffeomorphism $(\bar{f}, \bar{\phi}) : (S/K, Q) \to (S'/K', Q')$. Applying Theoem B,

Theorem C. *Suppose that* π *is isomorphic to* π'*. Then* $(G/K, \pi)$ *is equivariantly diffeomorphic to* $(G'/K', \pi')$*.*

LII

The Obata & Lelong-Ferrand analogue to Lorentz manifolds.

Counterexamples LII

Recall the definition of conformally flat Lorentz model.

- Let $P : \mathbb{R}^{n+2} - \{0\} \to \mathbb{RP}^{n+1}$ be the canonical projection.
- Take the quadric (Lorentz cone) in $\mathbb{R}^{n+2} - \{0\}$:

$$V_0 = \{(x_1, \ldots, x_n, y_1, y_2) |\ x_1^2 + \cdots + x_n^2 - y_1^2 - y_2^2 = 0\}.$$

- Define the Lorentz model to be

$$S^{n-1,1} = P(V_0).$$

- $S^{n-1,1} = S^{n-1} \times S^1/\mathbb{Z}_2$. The correspondence is given; $P(x, y) \to [x/|y|, y/|y|] \in S^{n-1} \times S^1/\mathbb{Z}_2$.

Counterexamples LII

- As *signature* of V_0 is $(n, 2)$, denote by $\mathrm{O}(n, 2)$ the subgroup of matrices preserving signature $(n, 2)$.
- The group $\mathrm{O}(n, 2)$ leaves V_0 invariant.
- $\mathrm{PO}(n, 2) = \mathrm{O}(n, 2)/\mathbb{Z}_2$ is the conformal group acting transitively on $S^{n-1,1} = P(V_0)$.
- Note that the real pseudo-hyperbolic space $\mathbb{H}_{\mathbb{R}}^{n,1}$ has the compactification in $\mathbb{RP}^{n+1}$:

$$\overline{\mathbb{H}_{\mathbb{R}}^{n,1}} = \mathbb{H}_{\mathbb{R}}^{n,1} \cup S^{n-1,1}.$$

(Pseudo-hyperbolic isometry of $\mathbb{H}_{\mathbb{R}}^{n,1}$ extends to conformal Lorentz transformation of $S^{n-1,1}$.)

3-dimensional counterexample LII

Proposition A. $\exists$ *compact* Lorentz standard space form $V^{2,1}_{-1}/\Gamma$ *on which a closed noncompact group* $\mathbb{R}$ *acts as isometries.*
In particular, the O& L-F conjecture to the Lorentz case is not true *only by the existence of conformal* $\mathbb{R}$.

Lorentz standard space form = complete Lorentz manifold of negative constant curvature. This is obtained as follows:

- $V^{2,1}_{-1} = \{(x_1, x_2, y_1, y_2) \mid x_1^2 + x_2^2 - y_1^2 - y_2^2 = -1\}$ which is identified with $\mathrm{SL}(2, \mathbb{R}) \approx S^1 \times \mathbb{R}^2$.
- The group $\mathrm{O}(2,2)^0 = \mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$ acts as Lorentz isometries on $V^{2,1}_{-1}$ (identified with $\mathrm{SL}(2,\mathbb{R})$) by

$$((g,h),x) = gxh^{-1} \quad (g,h,x \in \mathrm{SL}(2,\mathbb{R})).$$

3-dimensional counterexample LII

- Choose a torsionfree discrete cocompact subgroup Γ from $\{1\} \times \mathrm{SL}(2,\mathbb{R})$ so that $V^{2,1}_{-1}/\Gamma$ is a compact Lorentz manifold of curvature -1. (In particular, it is a conformally flat Lorentz manifold.)
- Take a closed subgroup $\mathbb{R} \subset \mathrm{SL}(2,\mathbb{R}) \times \{1\} \subset \mathrm{O}(2,2)^0$. As $\mathrm{O}(2,2)^0$ is the group of Lorentz isometries, $\mathbb{R}$ acts as conformally on $V^{2,1}_{-1}/\Gamma$.

As a consequence, $(\mathbb{R}, V^{2,1}_{-1}/\Gamma)$ is a counterexample because $V^{2,1}_{-1}/\Gamma$ is never Lorentz model $S^{n-1,1}$ $(n=3)$.

Causal fields

Definition I. *Let ξ be a vector field on a Lorentz manifold (M, g).*

$$\begin{cases} \xi \text{ is spacelike} & g(\xi_x, \xi_x) > 0 \quad \text{whenever } \xi_x \neq 0. \\ \xi \text{ is lightlike} & g(\xi_x, \xi_x) = 0 \quad \text{whenever } \xi_x \neq 0. \\ \xi \text{ is timelike} & g(\xi_x, \xi_x) < 0 \quad \text{whenever } \xi_x \neq 0. \end{cases}$$

- Each ξ is called a *causal vector field* with respect to g.
- The group generated by causal vector field is said to be one-parameter group of causal transformations.

We can take the action $\mathbb{R}$ on $V^{2,1}_{-1}/\Gamma$ as lightlike group N or spacelike group A from the decomposition $\mathrm{SL}(2,\mathbb{R}) = \mathrm{SO}(2) \cdot A \cdot N$.

4-dimensional counterexample LII

Recall that $V^{2,1}_{-1}/\Gamma$ is a spherical CR manifold and $S^1 \times V^{2,1}_{-1}/\Gamma$ is an example of Fefferman Lorentz manifold. It is proved that

Proposition B. $S^1 \times V^{2,1}_{-1}/\Gamma$ *is a compact conformally flat Lorentz manifold which can admit a closed lightlike conformal Lorentz group* $\mathbb{C}^*$*. Here* $\mathbb{C}^* = S^1 \times N \subset S^1 \times (\mathrm{SL}(2,\mathbb{R}) \times \{1\})$*.*

4-dimensional counterexample LII

We observe that

- Conformal Riemann case $\overset{\text{key fact}}{\Longrightarrow}$ non-elliptic behaviour of noncompact $\mathbb{R}$ is determined.
- Conformal Lorentz case $\Longrightarrow$ not true. (The Lorentz structure does not control the non-ellipticity of noncompact groups.)
- We take into account Fefferman Lorentz manifold to define a class of *fine Lorentz structure* satisfying non-elliptic behaviour of noncompact groups.

We interpret the Fefferman Lorentz manifold in terms of G-structure.

Lorentz-CR structure

$$G = \{g = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & u^2 & a_1 & \cdots & a_{2n} \\ \hline & & & & \\ 0 & 0 & & u \cdot B & \\ & & & & \end{array}\right) \in \mathrm{GL}(2n+2, \mathbb{R})\},$$

$(\forall u \in \mathbb{R}^+, \forall B \in \mathrm{U}(n), \forall a_i \in \mathbb{R})$.

Definition II. *A $(2n+2)$-manifold M is said to be Lorentz-CR manifold if it admits a G-structure. That is, there is a reduction of the structure group of M to G.*

Note that if all $a_i = 0$, then the G-structure defines a conformal structure of Lorentz metrics: $\mathrm{O}(2n+1, 1) \times \mathbb{R}^+$.

Lorentz-CR structure

Recall the $(2n+2)$-dimensional conformally flat Lorentz geometry $(\mathrm{PO}(2n+2,2), S^{2n+1,1})$. Let $\mathrm{U}(n+1,1)$ be the unitary Lorentz group with center S^1. Put

$$\hat{\mathrm{U}}(n+1,1) = \mathrm{U}(n+1,1)/\mathbb{Z}_2$$

where $\mathbb{Z}_2 = \{\pm 1\} \subset S^1$.
The natural embedding $\mathrm{U}(n+1,1) \rightarrow \mathrm{O}(2n+2,2)$ induces the embedding of the Lie groups:

$$\hat{\mathrm{U}}(n+1,1) \rightarrow \mathrm{PO}(2n+2,2).$$

Uniformization

$(\mathrm{PO}(2n+2,2), S^{2n+1,1})$ restricts to a subgeometry

$$(\hat{\mathrm{U}}(n+1,1), S^{2n+1,1})$$

which is *conformally flat Lorentz-CR geometry*.

Remark that $\mathbb{C}^*$ of Proposition B is the group of conformal Lorentz transformations but not conformal Lorentz-CR transformations, $\mathbb{C}^* \not\subset \hat{\mathrm{U}}(n+1,1)$.

Conclusion LII

We obtain the Obata & Lelong-Ferrand's theorem to the Lorentz-CR manifolds.

Theorem X (Lorentz analogue)**.** *Suppose that a compact Lorentz-*CR *manifold* M *admits a closed subgroup* $\mathbb{C}^*$ *consisting of* lightlike conformal Lorentz-CR *transformations. Then the universal covering* $\tilde{M}$ *is conformally isomorphic to the universal covering* $\tilde{S}^{2n+1,1}$ *of* $S^{2n+1,1}$*. Moreover,* M *is the quotient of* $\tilde{S}^{2n+1,1}$ *by an infinite cyclic subgroup* $\mathbb{Z}$*.*

More precisely, there exists a representation $\tilde{\rho} : \mathbb{Z} \rightarrow \mathbf{R} \times T^n$ such that $M \approx \tilde{S}^{2n+1,1}/\tilde{\rho}(\mathbb{Z})$.

Remark LII

Such representations $\tilde{\rho} : \mathbb{Z} \to \mathbf{R} \times T^n$ are determined by

$$\tilde{\rho}(m) = (a \cdot m, e^{\frac{2\pi \mathbf{i} p_1 \cdot m}{p}}, \ldots, e^{\frac{2\pi \mathbf{i} p_n \cdot m}{p}}) \ \ (\exists\, a \in \mathbf{R} - \{0\}).$$

The set of all such distinct homomorphisms is in one-to-one correspondence with

$$\begin{aligned} \mathcal{T} = \{(a, p, p_1, \ldots, p_n) \in \mathbb{R} - \{0\} \times \mathbb{N} \times (\mathbb{Z}_+)^n \\ \mid 0 \leq p_1 \leq \cdots \leq p_n < p, \ (p_i, p) = 1 \ (\exists\, i)\} \end{aligned}$$

The element $\tilde{\rho}_0 = (1, 1, 0, \ldots, 0)$ corresponds to $S^{2n+1,1} = \tilde{S}^{2n+1,1} / \tilde{\rho}_0(\mathbb{Z})$.

Fin

Part 2 - sketch of proofs

Proposition D. *Let M be a Lorentz-CR manifold of dimension $(2n+2)$ (admits a G-structure).*
If M is conformally flat, then M is conformally flat Lorentz-CR, i.e. it is uniformizable with respect to $(\hat{\mathrm{U}}(n+1,1), S^{2n+1,1})$.

Note that no particular Lorentz metric on M is specified.

Results LII

We prove our vague conjecture to the Lorentz-CR manifolds affirmatively.

First of all, the existence of *two dimensional lightlike vector fields* implies *conformally flatness*. *Two dimensional lightlike vector fields induces one timelike vector field.*

Proposition E. *Suppose that a compact Lorentz n-manifold (M, g) admits a closed subgroup $\mathbb{C}^*$ of conformal transformations.*
If $\mathbb{C}^*$ contains a one-parameter subgroup of timelike conformal transformations, then M is conformally flat.

Results LII (Sketch)

- Let ξ be the timelike vector field. A Riemannian metric h is defined on a domain $\mathcal{W}$ =the set of points at which Weyl curvature tensor nonzero in M:

$$h(X,Y) = \frac{2g(\xi,X)g(\xi,Y) - g(X,Y)g(\xi,\xi)}{g(\xi,\xi)^2}.$$

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- This defines $\mathbb{C}^* \subset \mathrm{Isom}(\mathcal{W})$ so there is a principal frame bundle $\mathrm{O}(n)\rightarrow P\rightarrow\mathcal{W}$
- Let $e \in P$. The orbit map: $h\rightarrow h_* e$ defines a proper embedding of $\mathbb{C}^*\rightarrow P$ whose image will be *compact* by the G-structure theory of finite type. It is a contradiction, $\mathcal{W} = \emptyset$.

Results LII

Suppose that a compact conformally flat Lorentz-CR manifold M admits a closed subgroup $\mathbb{C}^*$ consisting of *lightlike conformal Lorentz-*CR transformations.
By Proposition E, we have the developing pair:

$$(\tilde{\rho}, \mathrm{dev}) : (\tilde{\mathbb{C}}^*, \tilde{M}) \to (\mathrm{U}(n+1,1)^{\sim}, \tilde{S}^{2n+1,1}).$$

Here $\tilde{\mathbb{C}}^* = \tilde{S}^1 \times \mathbb{R}^+$. $\tilde{S}^{2n+1,1} = S^{2n+1} \times \mathbb{R}$.

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- The causality of lightlike fields implies that the lift of S^1 to the universal covering $\tilde{M}$ is $\tilde{S}^1 = \mathbb{R}$.
- $\tilde{\rho}(\tilde{S}^1) = \mathbf{R}$ which is the center of $\mathrm{U}(n+1,1)^{\sim}$.

Results LII

We obtain the following commutative diagram of fiber spaces:

$$
\begin{array}{ccc}
\mathbb{R} & \longrightarrow & \mathbf{R} \\
\downarrow & & \downarrow \\
(\pi, \mathbb{R} \times \mathbb{R}^+, \tilde{M}) & \xrightarrow{(\tilde{\rho},\mathrm{dev})} & (\mathrm{U}(n+1,1)^{\sim}, \mathbf{R} \times \mathbb{R}, S^{2n+1} \times \mathbf{R}) \\
\downarrow & & \downarrow \\
(Q, \mathbb{R}^+, W) & \xrightarrow{(\hat{\rho},\hat{\mathrm{dev}})} & (\mathrm{PU}(n+1,1), \mathbb{R}, S^{2n+1}).
\end{array}
$$

Here $\tilde{\rho}(\mathbb{R} \times \mathbb{R}^+) = \mathbb{R} \times \mathbf{R}$. $\tilde{S}^{2n+1,1} = S^{2n+1} \times \mathbf{R}$.

CR-analogue

As a consequence, the group Q acts properly discontinuously on W such that the quotient W/Q is an orbifold. In particular, a closed CR orbifold W/Q admits a noncompact CR-transformations $\mathbb{R}^+$.

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Now we use the analogue of Obata & Lelong-Ferrand's theorem to compact strictly pseudo-convex CR-manifolds.

- The developing map $\hat{\mathrm{dev}}$ is CR-isomorphic.

$$\hat{\mathrm{dev}} : W \cong S^{2n+1}.$$

In particular, M is conformally equivalent to $\tilde{S}^{2n+1,1}/\tilde{\rho}(\pi)$ by the diagram.

Conclusion LII

By the diagram, there is the exact sequence:

$$1 \to \mathbb{Z} \to \pi \to Q \to 1.$$

- As W/Q is a compact orbifold, Q is a finite subgroup. Moreover, we can prove that Q is a cyclic group.
- $\pi_1(M)$ is isomorphic to an infinite cyclic group $\mathbb{Z}$. As a consequence, M is the quotient of $\tilde{S}^{2n+1,1}$ by an infinite cyclic subgroup $\mathbb{Z}$; $M \approx \tilde{S}^{2n+1,1}/\tilde{\rho}(\mathbb{Z})$.

This proves Theorem X - Obata & Lelong-Ferrand's theorem

to the Lorentz-CR manifolds.

Rigidity results to Problem SI (with Oliver Baues)

We start with a fiber space with *solv-geometry*: $(X, \mathcal{R}, H)$ where H acts properly on X and normalises a simply connected solvable Lie group $\mathcal{R}$. $\exists$ A principal bundle:

$$\mathcal{R} \longrightarrow X \xrightarrow{\mathsf{p}} W = X/\mathcal{R}.$$

The fiber stabilising subgroup Δ of H contains a solvable subgroup of finite index and $\mathcal{R}/\Delta$ is compact. Associated to group extension

$$1 \rightarrow \Delta \rightarrow H \rightarrow \Theta \rightarrow 1\ ,$$

there is a singular fibration:

$$\Delta \backslash \mathcal{R} \longrightarrow X/H \xrightarrow{\mathsf{q}} W/\Theta.$$

Rigidity results to Problem SI (with Oliver Baues)

If Δ^0 is the connected component of Δ, then it induces a group extension:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta/\Delta^0 & \longrightarrow & H/\Delta^0 & \longrightarrow & \Theta & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \pi & \longrightarrow & \Theta & \longrightarrow & 1 \end{array}$$

where Γ is a virtually solvable group. Then Baues observed the T-*compatibility* condition for the reductive group T of the algebraic hull $A(\mathcal{R})$. We prove that $\exists$ a simply connected nilpotent Lie group $\mathbf{U}$ such that the T-compatible $(X, \mathcal{R}, H)$ is equivalent to *standard* Γ*-fiber space* $(X, \mathbf{U}, \pi)$ whose fiber stabilising group Γ is a discrete standard subgroup of $\mathrm{Aff}(\mathbf{U})$.