On the isovariant Hopf theorem

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The Hopf theorem

We first review some well-known results on Hopf type theorems.

Let M be an n-dimensional connected orientable closed manifold, and S^n the n-dimensional sphere.

Let $[M, S^n]$ denote the set of (free) homotopy classes of continuous maps $f : M \to S^n$.

The degree of f induces a map $\deg : [M, S^n] \to \mathbb{Z}$. Then the Hopf theorem or classification theorem of Hopf states that

Theorem 1. The map $\deg : [M, S^n] \to \mathbb{Z}$ $(n \geq 1)$ is a bijection.

Proof of the Hopf theorem

The proof is divided into two steps.

(1) Application of obstruction theory.
The correspondence

$$\gamma : [M, S^n] \ni [f] \mapsto \gamma(c, f) \in H^n(M; \mathbb{Z}) \cong \mathbb{Z}$$

is a bijection, where c is a constant map.

(2) Calculation of the obstruction class.
In this case $\gamma(c, f) = f^*([S^n])$, and $f^*([S^n])$ is nothing but the degree of f.

Equivariant version

Equivariant versions of the Hopf theorem and related topics have been studied by Segal, Rubinsztein, Petrie, tom Dieck, Balanov, Ferrario, and others.

We recall a simple example of the equivariant Hopf theorem.

Let C_2 be a cyclic group of order 2, and S^n the n-sphere with antipodal C_2-action.

We want to know the C_2-homotopy set $[S^n, S^n]_{C_2}$.

Determination of $[S^n, S^n]_{C_2}$ ($n \geq 1$)

The argument is almost same as in the nonequivariant case.

(1) Equivariant obstruction theory shows that the correspondence

$$\gamma_{C_2} : [S^n, S^n]_{C_2} \ni [f] \to \gamma_{C_2}(id, f) \in \mathfrak{H}^n_{C_2}(S^n; \pi_n(S^n))$$

is a bijection.

(2) Identifying $\mathfrak{H}^n_{C_2}(S^n; \pi_n(S^n)) = \mathbb{Z}$, one can see that

$$\gamma_{C_2}(id, f) = (\deg f - 1)/2.$$

An equivariant Hopf type theorem

Consequently we have a Hopf type theorem for C_2-maps.

Theorem 2. By setting $D([f]) = (\deg f - 1)/2$, we have a bijection

$$D : [S^n, S^n]_{C_2} \to \mathbb{Z}.$$

Remark. $\deg : [S^n, S^n]_{C_2} \to \mathbb{Z}$ is injective, and its image is $1 + 2\mathbb{Z}$, i.e., the degree of a C_2-map is odd, and in particular the forgetful map $i : [S^n, S^n]_{C_2} \to [S^n, S^n]$ is injective.

A more general result

More generally, one can determine $[SV, SV]_G$ by equivariant obstruction theory. Here V is a unitary G-representation and SV denotes the unit sphere of V.

If V is large, e.g., $V \supset \mathbb{C}G$, then the correspondence

$$d : [SV, SV]_G \ni [f] \to (\deg f^H)_{(H)} \in \prod_{(H)} \mathbb{Z}$$

is injective, where $f^H : SV^H \to SV^H$, and the image of d is characterized by the Burnside ring relations. In particular,

$$[SV, SV]_G \cong A(G) \text{ (Burnside ring).}$$

Isovariant map

R. S. Palais introduced the notion of the isovariant map in order to study a classification problem of G-spaces.

Definition. A (continuous) G-map $f : X \to Y$ between G-spaces is called G-*isovariant* if f preserves the isotropy subgroups, i.e., $G_{f(x)} = G_x$ for all $x \in X$.

If a G-homotopy $F : X \times I \to Y$ is G-isovariant, then it is called a G-isovariant homotopy.

Let $[X, Y]_G^{\mathrm{isov}}$ denote the G-isovariant homotopy set, i.e., the set of isovariant homotopy classes of G-isovariant maps from X to Y.

Assumption

We would like to determine $[M, SW]_G^{\mathsf{isov}}$ under some assumptions.

We here assume the following.

- M is a connected, orientable, smooth closed G-manifold.
- The G-action on M is *free* and *orientation-preserving*.
- W is a faithful *unitary* G-representation.
- The Borsuk-Ulam inequality:

$$\dim M + 1 \leq \dim SW - \dim SW^{>1}.$$

Here $SW^{>1} = \bigcup_{1 \neq H \leq G} SW^H$, the singular set of SW. If $SW^{>1} = \emptyset$, then we set $\dim SW^{>1} = -1$.

Comments on the Borsuk-Ulam inequality

The Borsuk-Ulam inequality is related to a Borsuk-Ulam type theorem.

Theorem 3 (Borsuk-Ulam theorem)**.** For any continuous map $f : S^n \to \mathbb{R}^n$, there is $x \in S^n$ such that $f(x) = f(-x)$.

In an equivariant fashion, this is restated as follows.

Theorem 4. Assume that C_2 acts antipodally on spheres. If there is a C_2-map $f : S^m \to S^n$, then $m \leq n$.

A Borsuk-Ulam type theorem

The Borsuk-Ulam theorem has many generalizations. The following is one of Borsuk-Ulam type theorems.

Theorem 5 (Isovariant Borsuk-Ulam theorem)**.** Assume that M is a $\mathrm{mod}\,|G|$ homology sphere with free G-action,

$$H_*(M;\mathbb{Z}/|G|) \cong H_*(S^m;\mathbb{Z}/|G|),\ m = \dim M.$$

If there is a G-isovariant map $f : M \to SW$, then

$$\dim M + 1 \le \dim SW - \dim SW^{>1}.$$

Existence of an isovariant map

Set

$$d = \dim SW - \dim SW^{>1}.$$

Remark. Since W is faithful and unitary, d is even and ≥ 2.

By the isovariant Borsuk-Ulam theorem, if M is a $\mathrm{mod}\,|G|$ homology sphere and $\dim M > d-1$, then there is no isovariant map from M to SW, i.e., $[M, SW]_G^{\mathsf{isov}}$ is empty.

On the other hand,

Theorem 6. Let M be a closed free G-manifold.
If $\dim M \leq d-1$, then there is a G-isovariant map from M to SW, i.e., $[M, SW]_G^{\mathsf{isov}}$ is not empty.

Outline of proof

Set

$$SW_{\mathrm{free}} = SW \setminus SW^{>1}.$$

Since G acts freely on M, one can identify $[M, SW]_G^{\mathsf{isov}}$ with $[M, SW_{\mathrm{free}}]_G$. So one may consider G-maps $f : M \to SW_{\mathrm{free}}$.

Lemma 7. SW_{free} is $(d-2)$-connected.

This lemma shows that a G-map $\varphi : G \times S^{k-1} \to SW_{\mathrm{free}}$ can be extended to $\tilde{\varphi} : G \times D^k \to SW_{\mathrm{free}}$ for $k \leq d-1$. One can see the existence of a G-map $f : M \to SW_{\mathrm{free}}$ using a G-CW decomposition of M. □

Isovariant homotopy classes: $\dim M < d-1$

Similarly we have the following.

Theorem 8. If $\dim M < d-1$, then $[M, SW]_G^{\mathsf{isov}} = \{*\}$.

Namely all isovariant maps $f : M \to SW$ are isovariantly homotopic each other.

Outline of Proof. It suffices to show that any two G-maps f, $g : M \to SW_{\mathrm{free}}$ are G-homotopic.

Since $\dim M + 1 \leq d-1$ and SW_{free} is $(d-2)$-connected, the G-map $F_0 := f \coprod g : M \times \{0,1\} \to SW_{\mathrm{free}}$ can be extended to a G-homotopy $F : M \times I \to SW_{\mathrm{free}}$. $\square$

Isovariant homotopy classes: $\dim M = d - 1$

Hereafter we assume that

$$\dim M = d - 1 \quad (d = \dim SW - \dim SW^{>1}).$$

In order to determine $[M, SW]_G^{\mathsf{isov}}$, we introduce the notion of the *multidegree*. Set

$$\mathcal{A} = \{H \in \mathrm{Iso}\,(W) \mid \dim SW^H = \dim SW^{>1}\},$$

where $\mathrm{Iso}\,(W)$ is the set of isotropy subgroups of W.

Let $\mathcal{A}/G$ denote the set of conjugacy classes of subgroups in $\mathcal{A}$, i.e.,

$$\mathcal{A}/G = \{(H) \mid H \in \mathcal{A}\}.$$

Isomorphisms

Using the Mayer-Vietoris exact sequence, we have

$$H_{d-1}(SW_{\text{free}};\mathbb{Z}) \cong \bigoplus_{H\in\mathcal{A}} H_{d-1}(S(W^H)^\perp;\mathbb{Z}) \cong \bigoplus_{H\in\mathcal{A}} \mathbb{Z},$$

where $(W^H)^\perp$ is the orthogonal complement of W^H in W. Since $gS(W^H)^\perp = S(W^{gHg^{-1}})^\perp$ for $g \in G$, we have

Lemma 9. There is a $\mathbb{Z}G$-isomorphism

$$\Psi : H_{d-1}(SW_{\text{free}};\mathbb{Z}) \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}[G/NH],$$

where NH is the normalizer of H in G.

Hence we have

$$\Psi^G : H_{d-1}(SW_{\mathrm{free}};\mathbb{Z})^G \cong \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}[G/NH]^G.$$

Since

$$\mathbb{Z}[G/NH]^G = \mathbb{Z}\cdot\sigma_H \cong \mathbb{Z},$$

where $\sigma_H := \sum_{\bar{a}\in G/NH} \bar{a}$, we have an isomorphism

$$\Phi : H_{d-1}(SW_{\mathrm{free}};\mathbb{Z})^G \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}.$$

Multidegree

Let $f : M \to SW$ be a G-isovariant map (or equivalently $f : M \to SW_{\mathrm{free}}$ be a G-map). Then f induces a $\mathbb{Z}G$-homomorphism $f_* : H_{d-1}(M;\mathbb{Z}) \to H_{d-1}(SW_{\mathrm{free}};\mathbb{Z})$.

Since the G-action on M is orientation-preserving, the induced G-action on $H_{d-1}(M;\mathbb{Z}) \cong \mathbb{Z}$ is trivial, and so $f_*([M]) \in H_{d-1}(SW_{\mathrm{free}};\mathbb{Z})^G$, where $[M]$ is the fundamental class of M.

Definition. The multidegree of f is defined by

$$\mathrm{mDeg}\, f = \Phi(f_*([M])) \in \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}.$$

The multidegree is an isovariant invariant.

Isovariant Hopf theorem

Theorem 10. Under the assumption.

(1) $\mathrm{mDeg} : [M, SW]_G^{\mathrm{isov}} \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$ is injective.

(2) For any two G-isovariant maps f, $g : M \to SW$,

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, g \in \bigoplus_{(H)\in\mathcal{A}/G} |NH|\mathbb{Z}.$$

(3) Fix a G-isovariant map $f_0 : M \to SW$. For any $a \in \bigoplus_{(H)\in\mathcal{A}/G} |NH|\mathbb{Z}$, there exists a G-isovariant map $f : M \to SW$ such that

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, f_0 = a.$$

Isovariant Hopf theorem

Let $d_H(f)$ be the (H)-component of $d(f)$, i.e., $\mathrm{mDeg}\, f = (d_H(f))_{(H)} \in \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$. We define

$$\mathrm{D}_{f_0}(f) = \left(\frac{1}{|NH|}(d_H(f) - d_H(f_0))\right)_{(H)} \in \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z},$$

where f_0 is a fixed isovariant map. Then we have

Corollary 11. The map $\mathrm{D}_{f_0} : [M, SW]_G^{\mathrm{isov}} \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$ is a bijection.

Remark. When the action on SW is not free, then $[M, SW]_G = \{*\}$, and so the forgetful map $[M, SW]_G^{\mathsf{isov}} \to [M, SW]_G$ is surjective.

Example: Cyclic case

Let C_{pq} be a cyclic group of order pq, where p, q are distinct primes. Let g be a generator of C_{pq}.

Let U_k $(= \mathbb{C})$ be the C_{pq}-representation with the action $gz = z^k$.

Set $M = SU_1$ and $SW = S(U_p \oplus U_q)$.

In this case, $d = 2$ and $\mathcal{A} = \mathcal{A}/G = \{C_p, C_q\}$. So we have

$$[M, SW]^{\mathsf{isov}}_{C_{p,q}} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

More concretely, a C_{pq}-isovariant map

$$f_{\alpha,\beta} : SU_1 \to S(U_p \oplus U_q),\ \ \alpha, \beta \in \mathbb{Z},$$

is defined by

$$f_{\alpha,\beta}(z) = \frac{1}{\sqrt{2}}(z^{(1+\alpha q)p}, z^{(1+\beta p)q}).$$

These $f_{\alpha,\beta}$ are representatives of isovariant homotopy classes. In fact one can see that

$$\mathrm{D}_{f_{0,0}}([f_{\alpha,\beta}]) = (\beta, \alpha).$$

Example: Metacyclic case

Let $Z_{p,q}$ be the metacyclic group of order pq, where p, q are odd primes such that $q|p-1$, i.e., $Z_{p,q}$ has

$$1 \to C_p \to Z_{p,q} \to C_q \to 1 \text{ (split exact).}$$

Petrie first proved that $Z_{p,q}$ can act smoothly (but not linearly) and freely on some high-dimensional sphere, and finally Madsen,Thomas and Wall showed that $Z_{p,q}$ can act smoothly and freely on S^{2q-1}. Let Σ be such a free $Z_{p,q}$-sphere of dimension $2q-1$.

$Z_{p,q}$ has a complex q-dimensional representation R and a nontrivial 1-dimensional representation T.

We set $W_k = R \oplus kT$, $k \geq 1$.

In this case $d = 2q$ and so $\dim \Sigma = d - 1$.

Moreover

$$\mathcal{A}/G = \begin{cases} \{(C_p), (C_q)\} & \text{if } k = 1 \\ \{(C_p)\} & \text{if } k > 1. \end{cases}$$

Hence we have

$$[\Sigma, SW_k]^{\mathsf{isov}}_{Z_{p,q}} \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 1 \\ \mathbb{Z} & \text{if } k > 1. \end{cases}$$

Proof — Equivariant cohomology

We give the outline of proof of the isovariant Hopf theorem (Theorem 10).

Let M be a free G-manifold and $C_*(M)$ its cellular chain complex. Note that $C_n(M)$ is a free $\mathbb{Z}G$-module.

Let π be a $\mathbb{Z}G$-module, and define the equivariant cochain complex $C^*_G(M;\pi) = \mathrm{Hom}_{\mathbb{Z}G}(C_*(M);\pi)$.

Definition. $\mathfrak{H}^n_G(M;\pi) = H^n(C^*_G(M;\pi))$.

Remark. $\mathfrak{H}^n_G(M;\pi) \cong H^n(M/G;\{\pi\})$, where $\{\pi\}$ denotes the local coefficients induced from the $\mathbb{Z}G$-module π.

Proof — From equivariant obstruction theory

Let f, $g : M \to SW_{\mathrm{free}}$ be G-maps and let $\gamma_G(f, g)$ denote the equivariant obstruction class to the existence of a G-homotopy between f and g.

Let $\pi_{d-1} = \pi_{d-1}(SW_{\mathrm{free}})$. Since SW_{free} is $(d-2)$-connected and $\dim M = d-1$, we have

Proposition 12. The correspondence $[f] \mapsto \gamma_G(f_0, f)$ gives a bijection $\gamma_G : [M, SW_{\mathrm{free}}]_G \to \mathfrak{H}_G^{d-1}(M; \pi_{d-1})$, where f_0 is a fixed isovariant map.

Remark. When $d = 2$, using the Borsuk-Ulam inequality, one can see that G is cyclic and π_1 is abelian.

Proof — Computation

Let

$$\varepsilon : \mathfrak{H}_G^{d-1}(M;\pi_{d-1}) \to H^{d-1}(M;\pi_{d-1})$$

be the forgetful map.

Proposition 13.

(1) $\mathfrak{H}_G^{d-1}(M;\pi_{d-1}) \cong \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$.

(2) $H_{d-1}(M;\pi_{d-1}) \cong_G \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}[G/NH]$.

(3) ε is injective.

(4) $\operatorname{Im}\varepsilon = \bigoplus_{(H)\in\mathcal{A}/G} |NH|\mathbb{Z}[G/NH]^G \cong \bigoplus_{(H)\in\mathcal{A}/G} |NH|\mathbb{Z}$.

Proof — Cohomological description of the multidegree

Proposition 14.

(1) $\pi_{d-1}(SW_{\mathrm{free}}) \cong_G \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}[G/NH]$, and

$\pi_{d-1}(SW_{\mathrm{free}})^G \cong \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$.

(2) Under the above identification, we have

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, g = \langle \varepsilon(\gamma_G(f, g)), [M]\rangle,$$

where $\langle\, - \,, [M]\rangle : H^{d-1}(M; \pi_{d-1}) \to \pi_{d-1}(SW_{\mathrm{free}})$ is the evaluation map, which is a $\mathbb{Z}G$-isomorphism.

Proof of the isovariant Hopf theorem

(1) $\mathrm{mDeg} : [M, SW]_G^{\mathrm{isov}} \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$ is injective.

Since

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, g = \langle \varepsilon(\gamma_G(f,g)), [M] \rangle,$$

if $\mathrm{mDeg}\, f = \mathrm{mDeg}\, g$, then $\varepsilon(\gamma_G(f,g)) = 0$. Since ε is injective, we have $\gamma_G(f,g) = 0$.

This implies that f and g are isovariantly homotopic. Hence mDeg is injective.

(2) For any two G-isovariant maps f, $g : M \to SW$,

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, g \in \bigoplus_{(H) \in \mathcal{A}/G} |NH|\mathbb{Z}.$$

(3) Fix a G-isovariant map $f_0 : M \to SW$. For any $a \in \bigoplus_{(H) \in \mathcal{A}/G} |NH|\mathbb{Z}$, there exists a G-isovariant map $f : M \to SW$ such that

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, f_0 = a.$$

Using the fact $\mathrm{Im}\, \varepsilon \cong \bigoplus_{(H) \in \mathcal{A}/G} |NH|\mathbb{Z}$, one can see (2) and (3). $\square$