On Squeezing

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1. $\widetilde{K}_1$-squeezing (Quinn 70s, Ferry-Pedersen 90s ?)

2. L_n^h-squeezing (Pedersen-Y. 2006)

Squeezing: sometimes, we can deform a sufficiently “small” object as small as we like.

a trivial example:

X: a finite polyhedron, l: a loop in X

If l is sufficiently small, then we can shrink it as small as we like. Actually, we can shrink it to a point (size$= 0$!).

Vanishing: sometimes, a sufficiently “small” object represents a trivial element, as in the example above.

Review of $\widetilde{K}_0(R)$ and $\widetilde{K}_1(R)$

R: a ring with 1

$\widetilde{K}_0(R) = \{[P] - [Q] \mid P$, Q: f.g. projective R-modules$\}/\sim$

$$[P] + [Q] = [P \oplus Q], \quad [F] = 0 \ \ (F\text{: free})$$

If we allow infinitely generated modules, then $[P] = 0$.
(Eilenberg Swindle)

For P, choose Q s.t. $P \oplus Q \cong F$ (free). Then

$$\begin{aligned}[P] &= [P] + [F \oplus F \oplus \cdots] = [P \oplus Q \oplus P \oplus Q \oplus \cdots] \\ &= [F \oplus F \oplus F \oplus \cdots] = 0\end{aligned}$$

Reduced Projective Class of a Projective Chain Complex

$$P : 0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$$

(P_i: f.g. projective)

$$\Rightarrow \quad [P] = [P_0] - [P_1] + [P_2] - \cdots + (-1)^n [P_n] \in \widetilde{K}_0(R)$$

$[P] = 0 \quad \Leftrightarrow \quad P \simeq F$ (a finite f.g. free chain complex)

$GL(n,R) \subset GL(n+1,R); \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$

$GL(R) = \bigcup GL(n,R)$

$\widetilde{K}_1(R) = GL(R)/\sim$, where $\sim$ is gen. by the following elementary operations:

$$\begin{pmatrix} I & B \\ O & I \end{pmatrix} A \sim A \sim A \begin{pmatrix} I & B \\ O & I \end{pmatrix}, \quad (-1) \sim (1)$$

An element of $\widetilde{K}_1(R)$ can be thought of as a stable

automorphism α on a free R-module $R^n = \sum Re_i$:

$$\langle \alpha \rangle = (a_{ij}) \iff \alpha(e_i) = \sum a_{ij} e_j$$

a Geometric R-Module on a metric space X

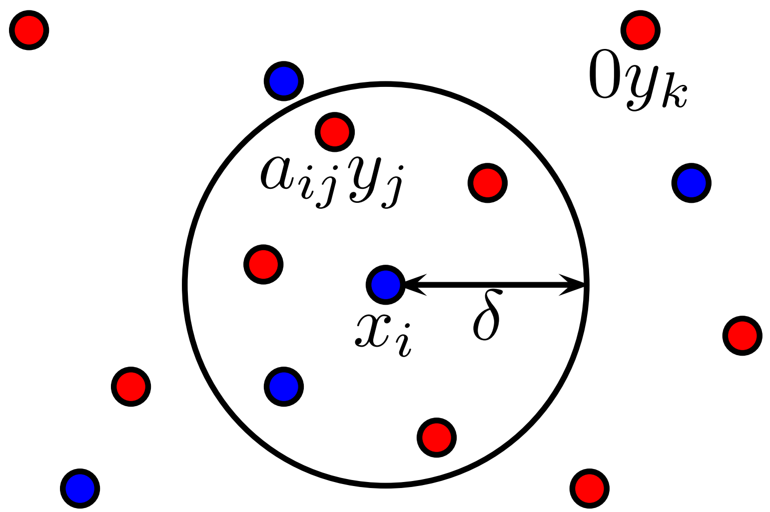
$=$ a free R-module with a basis $\{x_i\}$

together with a map $\{x_i\} \to X$

We pretend that x_i's are points in X.

Def. A homo. $\alpha : M = \sum Rx_i \to N = \sum Ry_j$ has radius δ

$$\iff \alpha(x_i) = \sum_{d(x_i, y_j) \le \delta} a_{ij} y_j$$



$\alpha : M = \sum Rx_i \to M$: an automorphism of radius δ

Def. α is a δ-automorphism $\iff$ α^{-1} also has radius δ.

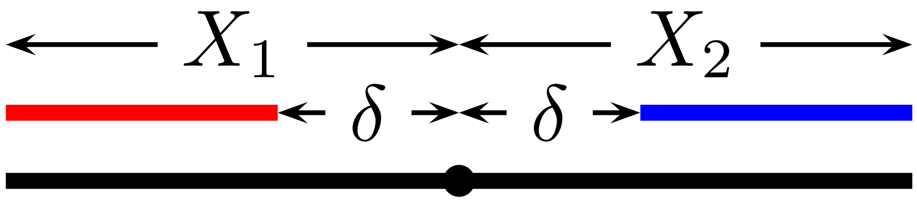
Def. α is δ-elementary $\iff$ $\langle \alpha \rangle = \begin{pmatrix} I_k & B \\ 0 & I_l \end{pmatrix}$ (w.r.t. some order)

Its inverse α^{-1} is automatically δ-elementary.

Assume: α is δ-elementary, as above, and $Y \subset X$.
A new automorphism $\bar{\alpha} : M \to M$ obtained from α by defining $\bar{\alpha}(x_i)$ to be $\alpha(x_i)$ if $x_i \in Y$, and x_i if $x_i \in X - Y$.
$\langle \bar{\alpha} \rangle$ is obtained by replacing those entries b_{ij} of B corresponding to the basis elements x_i contained in $X - Y$ by 0's, so it is still δ-elementary. $\bar{\alpha}$ will be called the localization at Y of α.

Notation: $Y \subset X$, $\delta > 0 \Rightarrow Y^\delta =$ the closed δ-nbhd of Y

• X is split into two subsets X_1 and $X_2 = X - X_1$.



• δ-automorphisms β and γ on a geometric module M on X are related by a δ-elementary automorphism α.

The localization $\bar{\alpha}$ of α at X_1 satisfies $\bar{\alpha} = \begin{cases} \alpha & \text{on } X_1 \\ 1 & \text{on } X_2 \end{cases}$.

So, if we apply deformation corresponding to $\bar{\alpha}$ to β, we obtain a new automorphism which is equal to γ on $X_1 - X_2^\delta$ and is equal to β on $X_2 - X_1^\delta$.

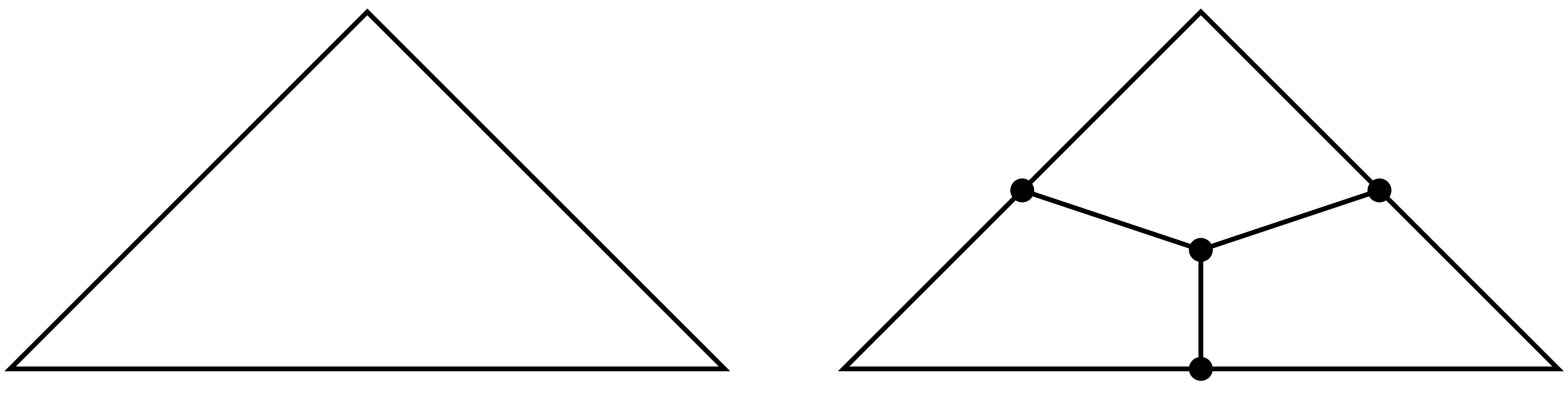
$\widetilde{K}_1$-squeezing

- F. Quinn, Ends of maps, I, Ann. of Math. (2) 110(1979)
- S. Ferry, a seminar talk at Univ. of Edinburgh, 1990
- E. Pedersen, Controlled algebraic K-theory, a survey, in 'Geometry and Toplogy: Aarhus (1998)' (AMS, 2000)

$\mathbb{R}^n$: the max metric, so the n-dim unit disk is $[-1,1]^n$

X: a cubical subcomplex of $S^n = \partial[-1,1]^n$

A triangulation induces an obvious cubical decomposition:



Thm. ($\widetilde{K}_1$-squeezing, Pedersen)

Suppose

$\delta \ll 6^{-\dim X}$, and

α is a δ-auto. on a f.g. geom. R-module M on X.

Then, for any $\varepsilon > 0$

$\exists N$: a f.g. geom. R-module on X

$\exists \beta$: an ε-automorphism on $M \oplus N$

s.t. $\alpha \oplus 1 \sim \beta$ (a $6^{\dim X+1}\delta$-deformation)

Let α be as above.

Let $C^*(X) = \{\ tx \in \mathbb{R}^n \mid x \in X,\ t \geq 1\ \}$.

($\widetilde{K}_1^{lf}$-vanishing on $C^*(X)$ ... Eilenberg Swindle!!)

$\exists$ a locally finetely generated geometric module G on $C^*(X)$

s.t. $\alpha \oplus 1_G \sim 1_M \oplus 1_G$

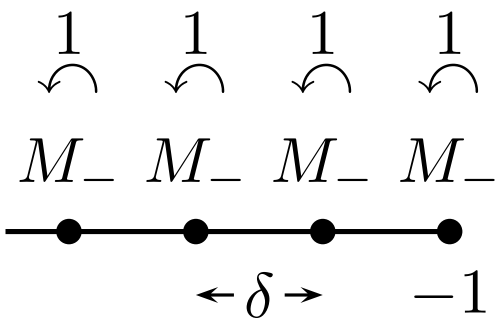
$n = 0$ case (assume $X = S^0$)
M is the direct sum of M_+ on $\{1\}$ and M_- on $\{-1\}$.
Since $\delta < 1 < 2 = d(-1,1)$, $\alpha : M \to M$ restricts to automorphisms of $M_\pm$:

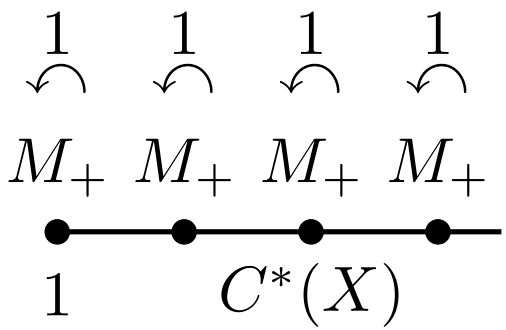
Put copies of $M_\pm$ along $C^*(X)$ and extend α by using the identitiy maps on the copies as in the picture below:

Apply copies of the following deformation to the above:

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \begin{pmatrix} 1 & -A^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix}$$

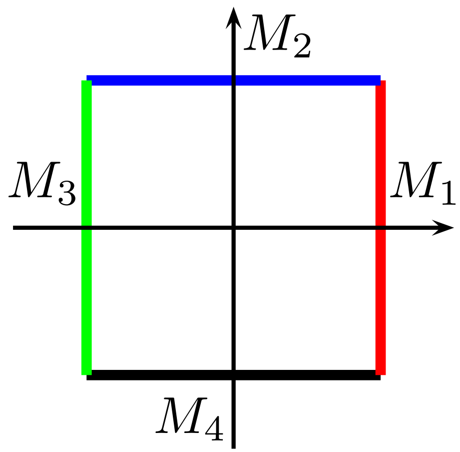
Apply copies of the same deformation again to get the identity map:





$n = 1$ case (assume $X = S^1$)

M splits into four submodules M_1, . . . , M_4 on the edges E_1, . . . ,E_4, but α does not have such a splitting.

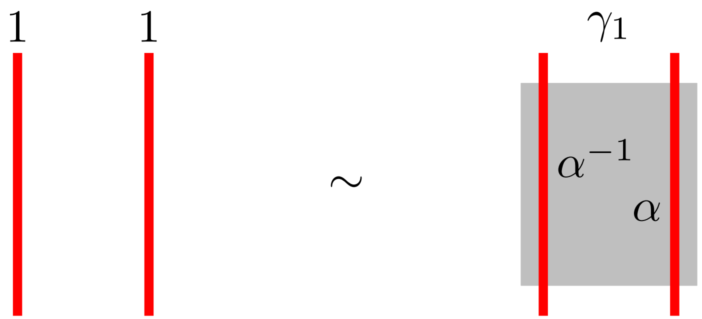


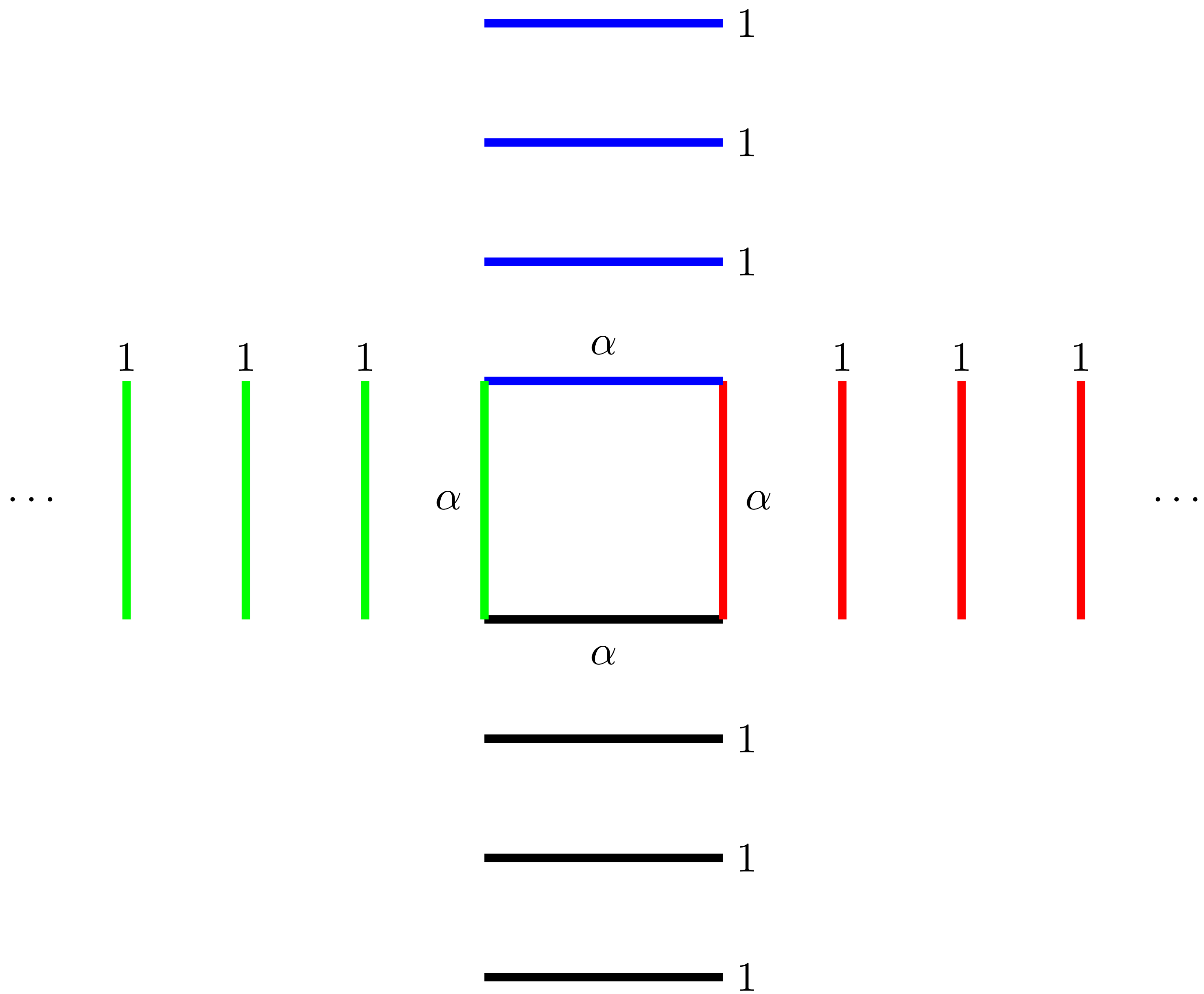
Let $A = \langle \alpha \rangle$, and consider the deformation on $M \oplus M$:

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \begin{pmatrix} 1 & -A^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix}$$

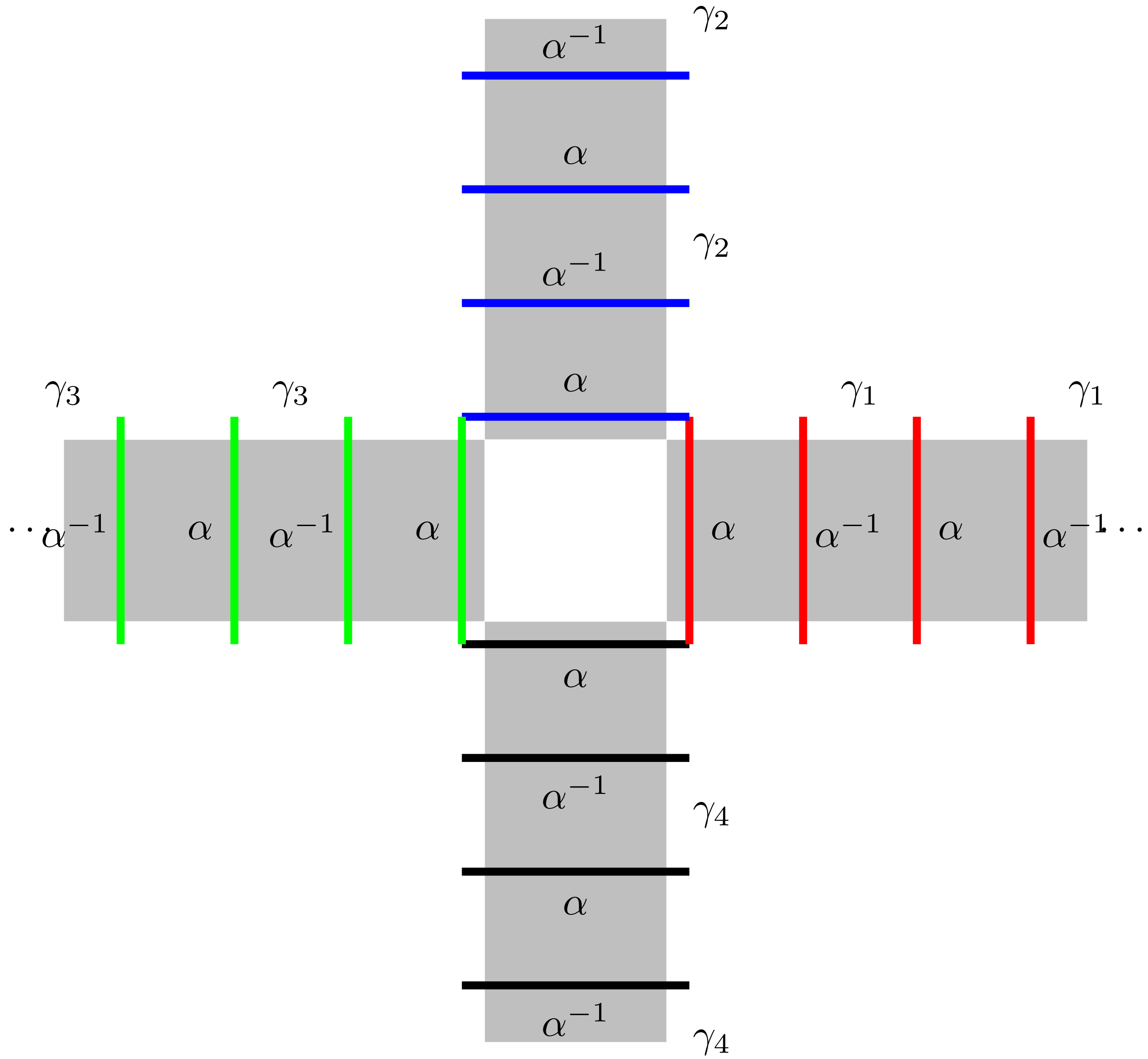
Take an edge E_i, and localize the six elementary deformations at $E_i - (\partial E_i)^\delta$.

On $(X - E_i) \cup (\partial E_i)^\delta$, it is the identity, and the localized deformation restricts to a deformation between the identity map on $M_i \oplus M_i$ and an automorphism γ_i on $M_i \oplus M_i$ which is equal to $\alpha \oplus \alpha^{-1}$ on $E_i - (\partial E_i)^{3\delta}$.

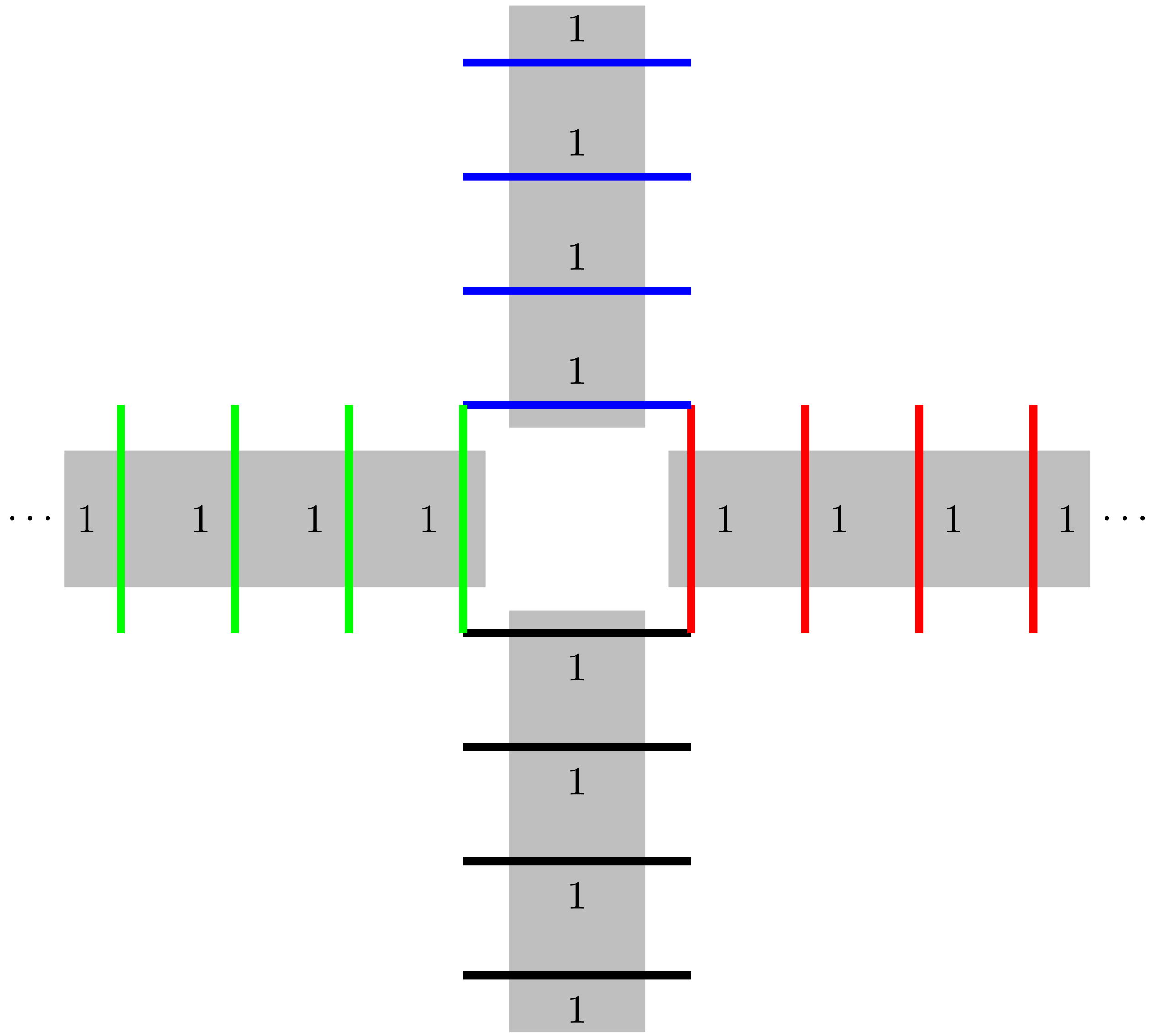




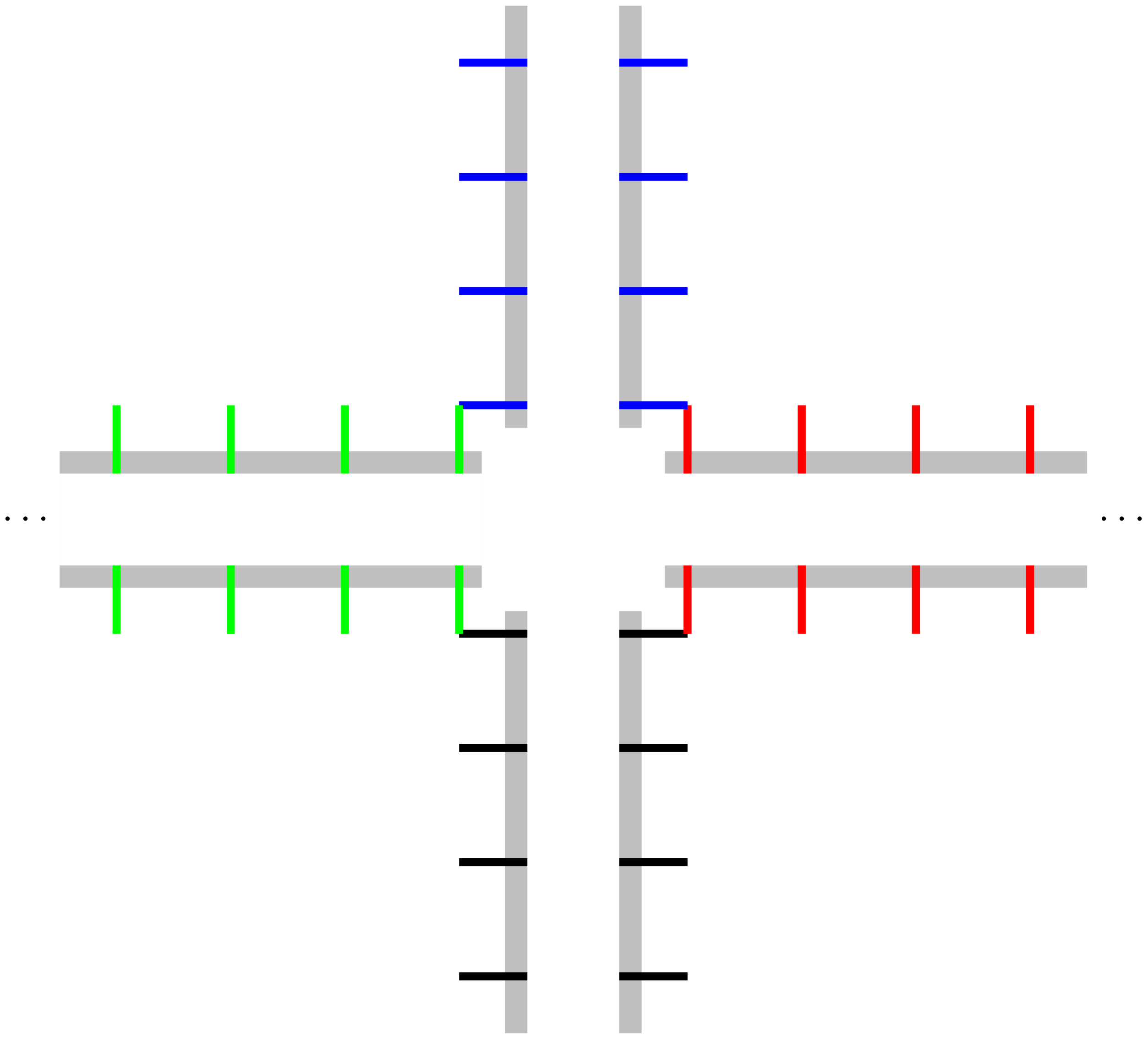
Put copies of M_i's along the axes and extend α by using the identitiy maps on the copies.



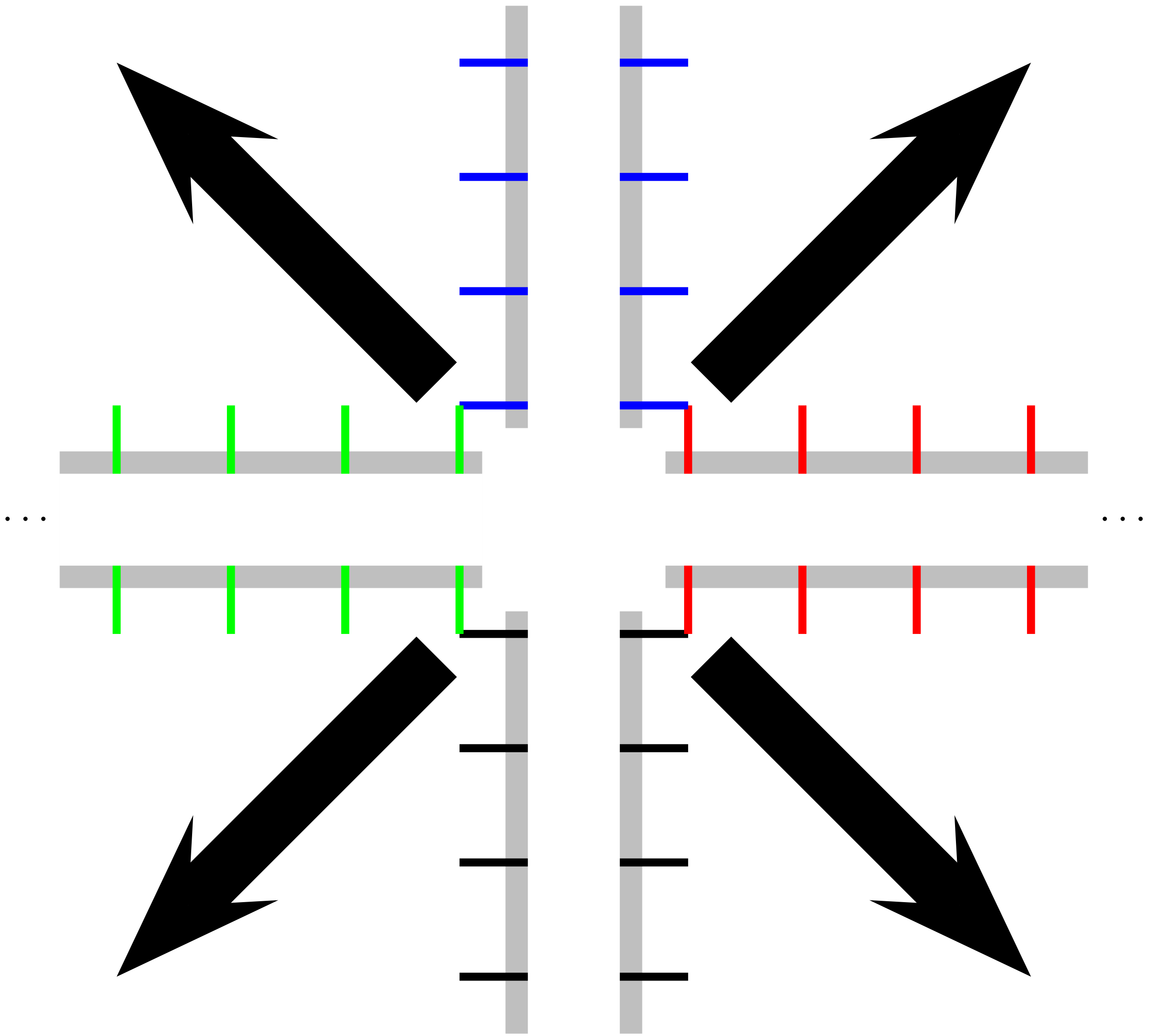
Apply copies of the localized deformation trick to get a new automorphism as above.



Now repeat a similar localization trick on a smaller nbhd of the axes to get identity maps in the gray region.



De-stabilize the automorphism by eliminating the identity part aruond the axes.



Now the automorphism is split into four pieces. Apply the Eilenberg swindle to each piece to get an identity map.

$n \geq 2$ cases can be handled in a similar way:

($\widetilde{K}_1^{lf}$-vanishing on $C^*(X)$)

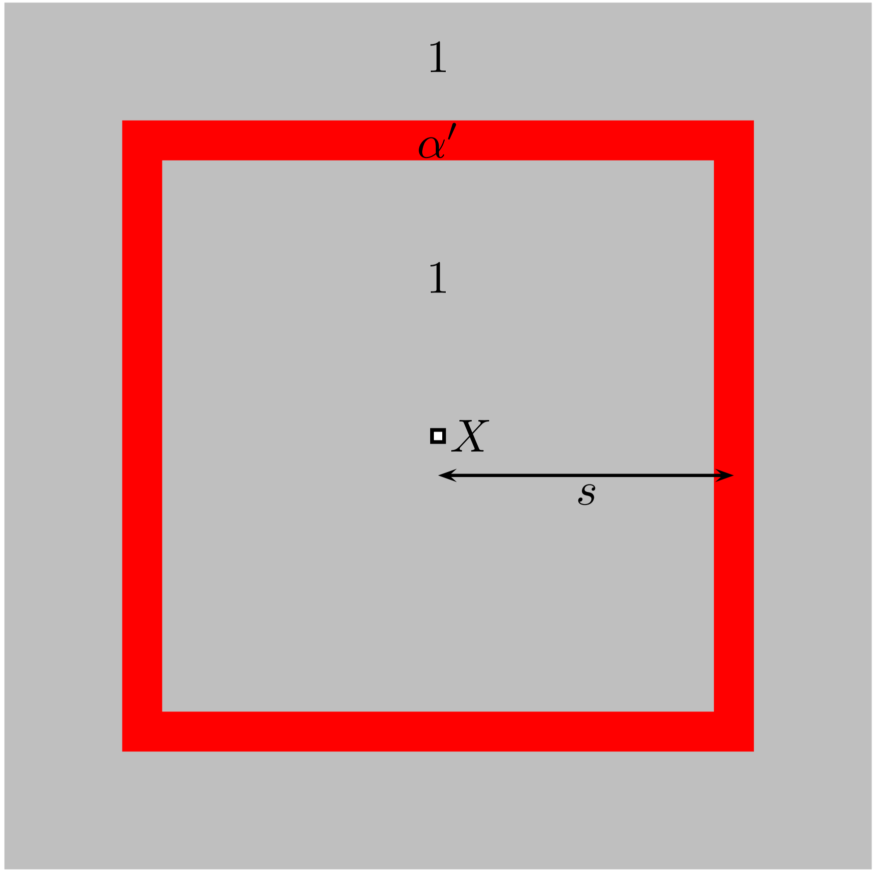
$\exists$ a locally finetely generated geom. module G on $C^*(X)$ s.t. $\alpha \oplus 1_G \ \sim \ 1_M \oplus 1_G$

Important Observation:

The deformation has a finite radius r , although the modules are infinitely generated.

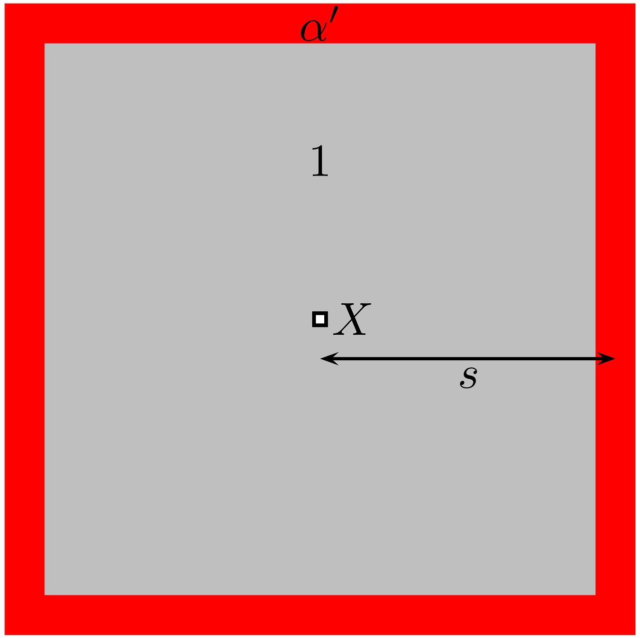
Choose a number $s \gg r$.

Now localize the deformation at the ball B of radius s, and use it to deform $\alpha \oplus 1_G$.



Then the result is the sum of an identity automorphism and an automorphism α' on a nbhd N of the boundary of B.

Throw away everything outside of $B \cup N$ to obtain an automorphism and a deformation on a f.g. module.



Now, radially shrink everything to X. The radius of the image β of α' can be made as small as we like by choosing a sufficiently large s. $\square$

Squeezing in L-theory

E. K. Pedersen-Y., Stability in Controlled L-theory, Geometry and Topology Monographs Vol.9: Exotic homology manifolds – Oberwolfach 2003 (2006)

L-theory = theory of Quadratic Complexes which are Poincaré (QPC)

a QC=an R-module chain complex + a quadratic structure

a QC C induces a symmetric structure and a duality chain map $\mathcal{D} : C^{n-*} \to C$, $n = \dim C$

a QC is Poincaré $\Longleftrightarrow$ $\mathcal{D}$ is a chain homotopy equivalence controlled L-theory = theory of geometric QPC's

Thm.
X: a finite polyhedron, $n > 0$
Then $\exists\ \delta_0 > 0$, $\exists\ K > 0$ which depend on X and n s.t.

if C is an n-dim. geom. QPC on X with radius $\delta \leq \delta_0$,

then, for any $\varepsilon > 0$,

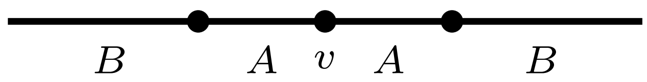
C is $K\delta$-cobordant to a geom. QPC of radius ε.

The proof is quite similar to the K-theory case, but we avoid using infinitely generated objects.

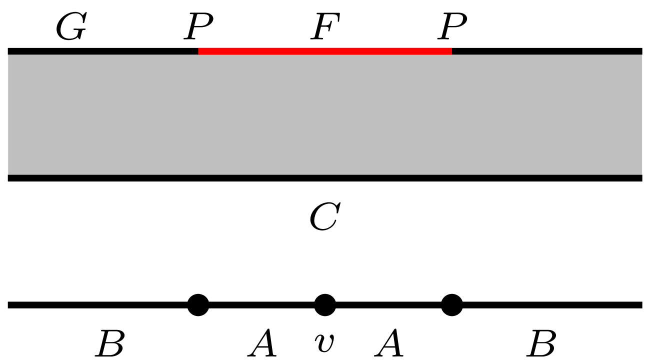
We do use a tower for Eilenberg Swindle but do not need an infinite tower.

Let us consider the $X = S^1$ case.

Pick a vertex v and set A to be its star nbhd, and B be the closure of the complement of A.

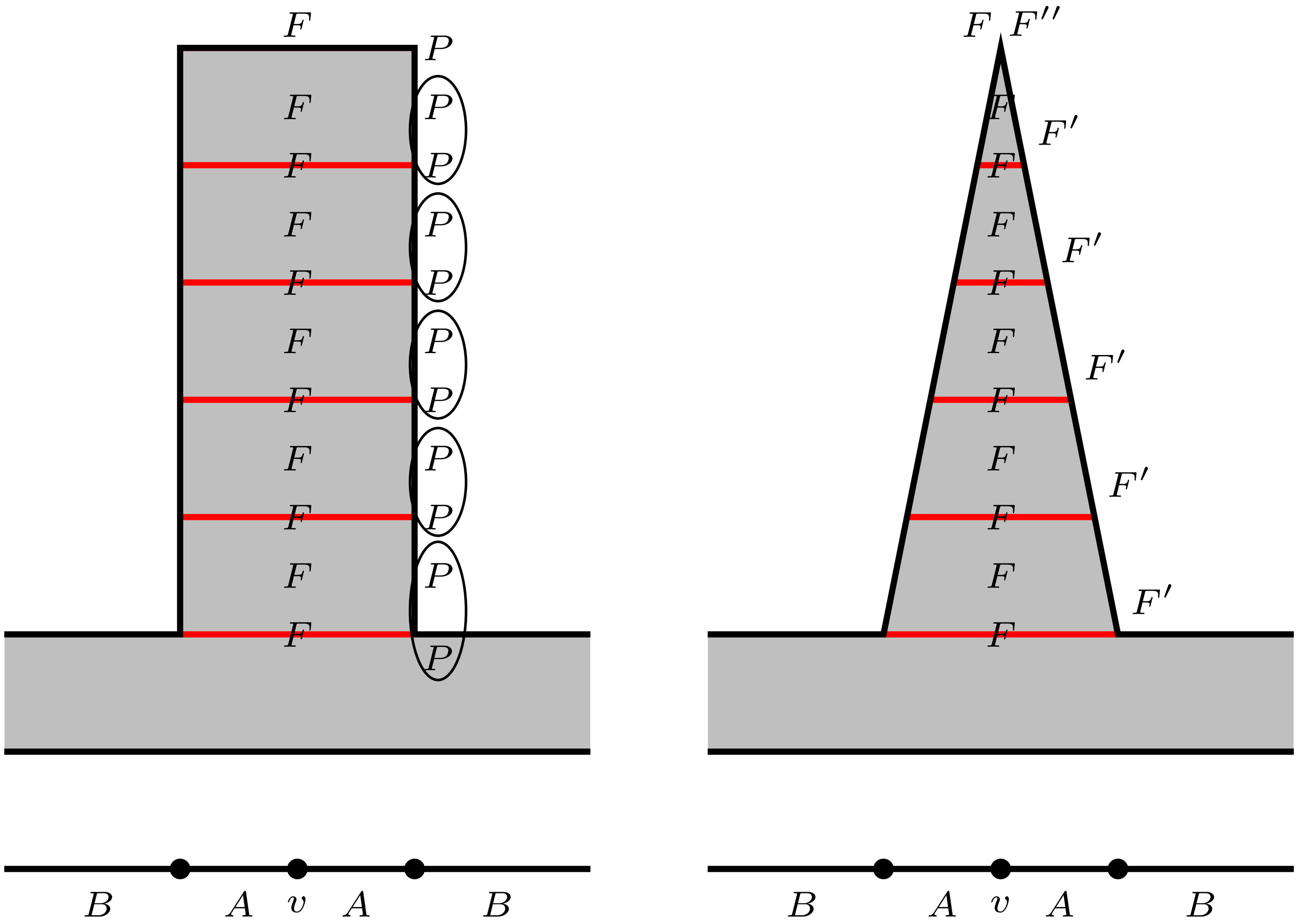


Let C be a geom. QPC on X with very small radius δ. Then, C is cobordant to the union of F on A and G on B with the common boundary P on ∂A:



P is a projective QPC on ∂A, but is chain equivalent to a free cx on A

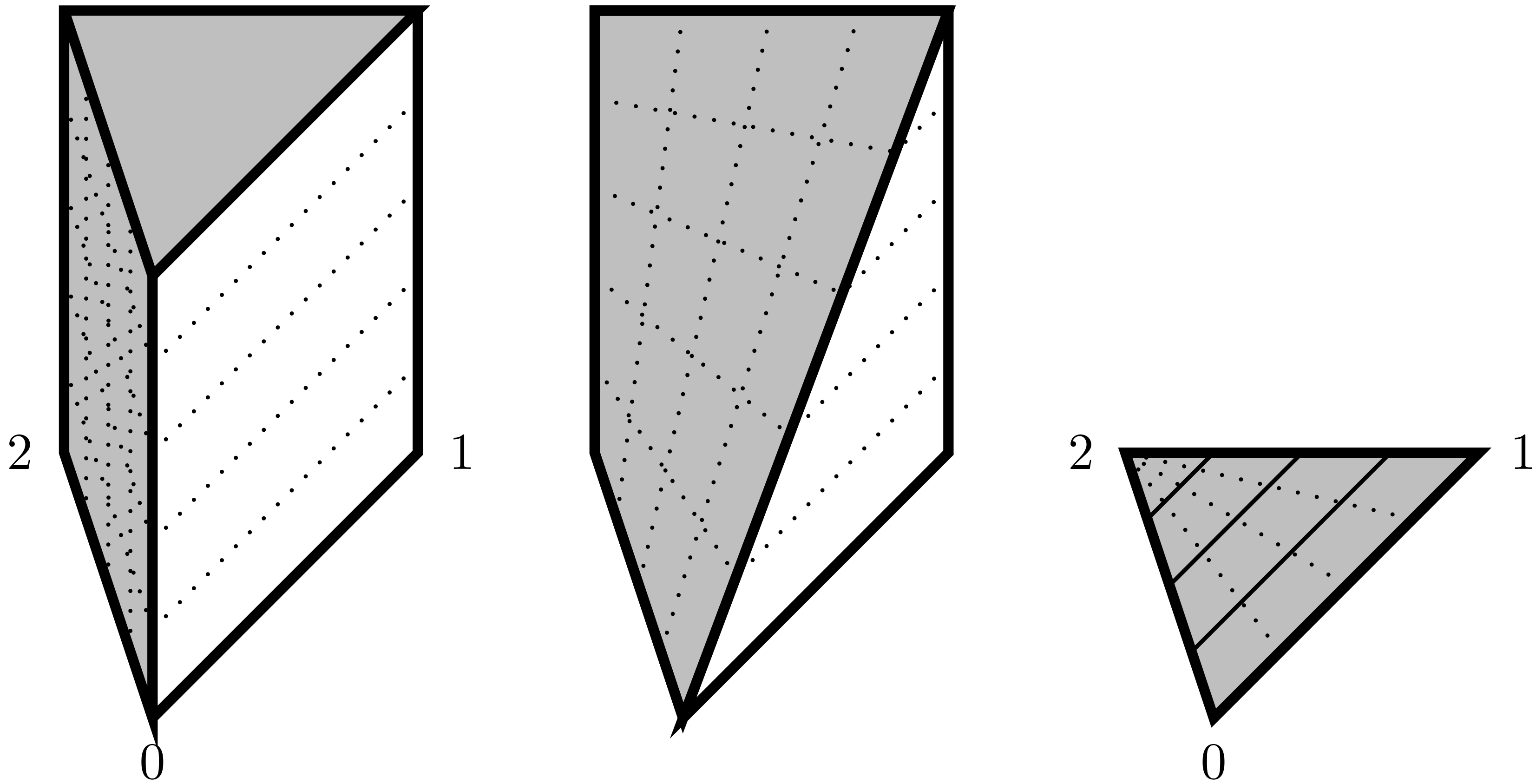
Now consruct a tall tower on F, and apply Eilenberg swindle to the P's except for the top P. Replace the top P by a free cx on A and shrink!



On a 2-simplex, we use two types of shrinking.

(1) $0 \longrightarrow 1$, $1 \longrightarrow 1$, $2 \longrightarrow 2$

The radius in the direction of solid lines are controlled.



The second type finishes the squeezing:

(2) $0 \longrightarrow 2,\ 1 \longrightarrow 2,\ 2 \longrightarrow 2$