A NOTE ON EXPONENTIALLY NASH G MANIFOLDS AND VECTOR BUNDLES

Tomohiro Kawakami

Department of Liberal Arts, Osaka Prefectural
College of Technology, Neyagawa Osaka 572, Japan

1. Introduction.

Nash manifolds have been studied for a long time and there are many brilliant works (e.g. [2], [3], [10], [19], [20], [21], [22], [23]).

The semialgebraic subsets of $\mathbb{R}^n$ are just the subsets of $\mathbb{R}^n$ definable in the standard structure $\mathbf{R}_{stan} := (\mathbb{R}, <, +, \cdot, 0, 1)$ of the field $\mathbb{R}$ of real numbers [24]. However any non-polynomially bounded function is not definable in $\mathbf{R}_{stan}$, where a polynomially bounded function means a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ admitting an integer $N \in \mathbb{N}$ and a real number $x_0 \in \mathbb{R}$ with $|f(x)| \leq x^N$, $x > x_0$. C. Miller [17] proved that if there exists a non-polynomially bounded function definable in an o-minimal expansion $(\mathbb{R}, <, +, \cdot, 0, 1, ...)$ of $\mathbf{R}_{stan}$, then the exponential function $exp : \mathbb{R} \longrightarrow \mathbb{R}$ is definable in this structure. Hence $\mathbf{R}_{exp} := (\mathbb{R}, <, +, \cdot, exp, 0, 1)$ is a natural expansion of $\mathbf{R}_{stan}$. There are a number of results on $\mathbf{R}_{exp}$ (e.g. [11], [12], [13], [14], [26]). Note that there are other structures with properties similar to those of $\mathbf{R}_{exp}$ ([5], [6], [25]).

We say that a C^r manifold $(0 \leq r \leq \omega)$ is *an exponentially C^r Nash manifold* if it is definable in $\mathbf{R}_{exp}$ (See Definition 2.5). Equivariant such manifolds are defined in a similar way (See Definition 2.6).

In this note we are concerned with exponentially C^r Nash manifolds and equivariant exponentially C^r Nash manifolds.

Theorem 1.1. *Any compact exponentially C^r Nash manifold $(0 \leq r < \infty)$ admits an exponentially C^r Nash imbedding into some Euclidean space.*

Note that there exists an exponentially C^ω Nash manifold which does not admit any exponentially C^ω imbedding into any Euclidean space [8]. Hence an exponentially C^ω Nash manifold is called *affine* if it admits an exponentially C^ω Nash imbedding into some Euclidean space (See Definition 2.5). In the usual Nash category, Theorem 1.1 is a fundamental theorem and it holds true without assuming compactness of the Nash manifold [19].

Equivariant exponentially Nash vector bundles are defined as well as Nash ones (See Definition 2.8).

1991 *Mathematics Subject Classification.* 14P10, 14P15, 14P20, 57S05, 57S15, 58A07.

Theorem 1.2. *Let G be a compact affine Nash group and let X be a compact affine exponentially C^ω Nash G manifold with* $\dim X^G \geq 2$*. Then for any $C^\infty G$ vector bundle η of positive rank over X, there exist two exponentially C^ω Nash G vector bundle structures of η such that they are exponentially C^∞ Nash G vector bundle isomorphic but not exponentially C^ω Nash G vector bundle isomorphic.*

Theorem 1.3. *Let G be a compact affine exponentially Nash group and let X be a compact $C^\infty G$ manifold. If* $\dim X \geq 3$ *and* $\dim X^G \geq 2$*, then X admits two exponentially C^ω Nash G manifold structures which are exponentially C^∞ Nash G diffeomorphic but not exponentially C^ω Nash G diffeomorphic.*

In the usual equivariant Nash category, any C^∞ Nash G vector bundle isomorphism is a C^ω Nash G one, and moreover every C^∞ Nash G diffeomorphism is a C^ω Nash G one. Note that Nash structures of $C^\infty G$ manifolds and $C^\infty G$ vector bundles are studied in [9] and [7], respectively.

In this note, all exponentially Nash G manifolds and exponentially Nash G vector bundles are of class C^ω and manifolds are closed unless otherwise stated.

2. Exponentially Nash G manifolds and exponentially Nash G vector bundles.

Recall the definition of exponentially Nash G manifolds and exponentially Nash G vector bundles [8] and basic properties of exponentially definable sets and exponentially Nash manifolds [8].

Definition 2.1. (1) An $\mathbf{R}_{exp}$-*term* is a finite string of symbols obtained by repeated applications of the following two rules:
[1] Constants and variables are $\mathbf{R}_{exp}$-terms.
[2] If f is an m-place function symbol of $\mathbf{R}_{exp}$ and $t_1, \dots, t_m$ are $\mathbf{R}_{exp}$-terms, then the concatenated string $f(t_1, \dots, t_m)$ is an $\mathbf{R}_{exp}$-term.
(2) An $\mathbf{R}_{exp}$-$formula$ is a finite string of $\mathbf{R}_{exp}$-terms satisfying the following three rules:
[1] For any two $\mathbf{R}_{exp}$-terms t_1 and t_2, $t_1 = t_2$ and $t_1 > t_2$ are $\mathbf{R}_{exp}$-formulas.
[2] If ϕ and ψ are $\mathbf{R}_{exp}$-formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are $\mathbf{R}_{exp}$-formulas.
[3] If ϕ is an $\mathbf{R}_{exp}$-formula and v is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are $\mathbf{R}_{exp}$-formulas.
(3) *An* $exponentially\ definable\ set$ $X \subset \mathbb{R}^n$ is the set defined by an $\mathbf{R}_{exp}$-formula (with parameters).
(4) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be exponentially definable sets. A map $f : X \longrightarrow Y$ is called $exponentially\ definable$ if the graph of $f \subset \mathbb{R}^n \times \mathbb{R}^m$ is exponentially definable.

On the other hand, using [12] any exponentially definable subset of $\mathbb{R}^n$ is the image of an $\mathfrak{R}_{n+m}$-semianalytic set by the natural projection $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ for some m. Here a subset X of $\mathbb{R}^n$ is called $\mathfrak{R}_n$-*semianalytic* if X is a finite union of sets of the following form:

$$\{x \in \mathbb{R}^n | f_i(x) = 0, g_j(x) > 0, 1 \leq i \leq k, 1 \leq j \leq l\},$$

where $f_i, g_j \in \mathbb{R}[x_1, \dots, x_n, exp(x_1), \dots, exp(x_n)]$.

The following is a collections of properties of exponentially definable sets (cf. [8]).

Proposition 2.2 (cf. [8]). *(1) Any exponentially definable set consists of only finitely many connected components.*

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be exponentially definable sets.
(2)The closure $Cl(X)$ and the interior $Int(X)$ of X are exponentially definable.
(3) The distance function $d(x, X)$ from x to X defined by $d(x, X) = \inf\{||x-y|| | y \in X\}$ is a continuous exponentially definable function, where $||\cdot||$ denotes the standard norm of $\mathbb{R}^n$.
(4) Let $f : X \longrightarrow Y$ be an exponentially definable map. If a subset A of X is exponentially definable then so is $f(A)$, and if $B \subset Y$ is exponentially definable then so is $f^{-1}(B)$.
(5) Let $Z \subset \mathbb{R}^l$ be an exponentially definable set and let $f : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ be exponentially definable maps. Then the composition $h \circ f : X \longrightarrow Z$ is also exponentially definable. In particular for any two polynomial functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, the function $h : \mathbb{R} - \{f = 0\} \longrightarrow \mathbb{R}$ defined by $h(x) = e^{g(x)/f(x)}$ is exponentially definable.
(6) The set of exponentially definable functions on X forms a ring.
(7) Any two disjoint closed exponentially definable sets X and $Y \subset \mathbb{R}^n$ can be separated by a continuous exponentially definable function. $\square$

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open exponentially definable sets. A C^r $(0 \leq r \leq \omega)$ map $f : U \longrightarrow V$ is called *an exponentially C^r $Nash\ map$* if it is exponentially definable. An exponentially C^r Nash map $g : U \longrightarrow V$ is called *an exponentially C^r $Nash\ diffeomorphism$* if there exists an exponentially C^r Nash map $h : V \longrightarrow U$ such that $g \circ h = id$ and $h \circ g = id$. Note that the graph of an exponentially C^r Nash map may be defined by an $\mathbf{R}_{exp}$-formula with quantifiers.

Theorem 2.3 [14]. *Let $S_1, \dots, S_k \subset \mathbb{R}^n$ be exponentially definable sets. Then there exists a finite family $\mathfrak{W} = \{\Gamma^d_\alpha\}$ of subsets of $\mathbb{R}^n$ satisfying the following four conditions:*
(1) Γ^d_α are disjoint, $\mathbb{R}^n = \cup_{\alpha,d}\Gamma^d_\alpha$ and $S_i = \cup\{\Gamma^d_\alpha | \Gamma^d_\alpha \cap S_i \neq \emptyset\}$ for $1 \leq i \leq k$.
(2) Each Γ^d_α is an analytic cell of dimension d.
(3) $\overline{\Gamma^d_\alpha} - \Gamma^d_\alpha$ is a union of some cells Γ^e_β with $e < d$.
(4) If $\Gamma^d_\alpha, \Gamma^e_\beta \in \mathfrak{W}, \Gamma^e_\beta \subset \overline{\Gamma^d_\alpha} - \Gamma^d_\alpha$ then $(\Gamma^d_\alpha, \Gamma^e_\beta)$ satisfies Whitney's conditions (a) and (b) at all points of Γ^e_β. $\square$

Theorem 2.3 allows us to define the *dimension* of an exponentially definable set E by

$$\dim E = \max\{\dim \Gamma | \Gamma \text{ is an analytic submanifold contained in } E\}.$$

Example 2.4. (1) The C^∞ function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\lambda(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ e^{(-1/x)} & \text{if} \quad x > 0 \end{cases}$$

is exponentially definable but not exponentially Nash. This example shows that an exponentially definable C^∞ map is not always analytic. This phenomenon does not occur in the usual Nash category. We will use this function in section 3.

(2) The Zariski closure of the graph of the exponential function $exp : \mathbb{R} \longrightarrow \mathbb{R}$ in $\mathbb{R}^2$ is the whole space $\mathbb{R}^2$. Hence the dimension of the graph of exp is smaller than that of its Zariski closure.
(3) The continuous function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$h(x) = \begin{cases} e^{x-n} & \text{if} \quad n \leq x \leq n+1 \\ e^{n+2-x} & \text{if} \quad n+1 \leq x \leq n+2 \end{cases}, \text{ for } n \in 2\mathbb{Z},$$

is not exponentially definable, but the restriction of h on any bounded exponentially definable set is exponentially definable. $\square$

Definition 2.5. Let r be a non-negative integer, ∞ or ω.
(1) *An exponentially C^r Nash manifold X of dimension d* is a C^r manifold admitting a finite system of charts $\{\phi_i : U_i \longrightarrow \mathbb{R}^d\}$ such that for each i and j $\phi_i(U_i \cap U_j)$ is an open exponentially definable subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1}|\phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is an exponentially C^r Nash diffeomorphism. We call these charts *exponentially C^r Nash*. A subset M of X is called *exponentially definable* if every $\phi_i(U_i \cap M)$ is exponentially definable.
(2) An exponentially definable subset of $\mathbb{R}^n$ is called *an exponentially C^r Nash submanifold of dimension d* if it is a C^r submanifold of dimension d of $\mathbb{R}^n$. An exponentially C^r $(r > 0)$ Nash submanifold is of course an exponentially C^r Nash manifold [8].
(3) Let X (resp. Y) be an exponentially C^r Nash manifold with exponentially C^r Nash charts $\{\phi_i : U_i \longrightarrow \mathbb{R}^n\}_i$ (resp. $\{\psi_j : V_j \longrightarrow \mathbb{R}^m\}_j$). A C^r map $f : X \longrightarrow Y$ is said to be *an exponentially C^r Nash map* if for any i and j $\phi_i(f^{-1}(V_j) \cap U_i)$ is open and exponentially definable in $\mathbb{R}^n$, and that the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \longrightarrow \mathbb{R}^m$ is an exponentially C^r Nash map.
(4) Let X and Y be exponentially C^r Nash manifolds. We say that X is *exponentially C^r Nash diffeomorphic to Y* if one can find exponentially C^r Nash maps $f : X \longrightarrow Y$ and $h : Y \longrightarrow X$ such that $f \circ h = id$ and $h \circ f = id$.
(5) An exponentially C^r Nash manifold is said to be *affine* if it is exponentially C^r Nash diffeomorphic to some exponentially C^r Nash submanifold of $\mathbb{R}^l$.
(6) A group G is called *an exponentially Nash group* (resp. *an affine exponentially Nash group*) if G is an exponentially Nash manifold (resp. an affine exponentially Nash manifold) and that the multiplication $G \times G \longrightarrow G$ and the inversion $G \longrightarrow G$ are exponentially Nash maps.

Definition 2.6. Let G be an exponentially Nash group and let $0 \leq r \leq \omega$.
(1) An exponentially C^r Nash submanifold in a representation of G is called *an exponentially C^r Nash G submanifold* if it is G invariant.
(2) An exponentially C^r Nash manifold X is said to be *an exponentially C^r Nash G manifold* if X admits a G action whose action map $G \times X \longrightarrow X$ is exponentially C^r Nash.
(3) Let X and Y be exponentially C^r Nash G manifolds. An exponentially C^r Nash map $f : X \longrightarrow Y$ is called *an exponentially C^r Nash G map* if it is a G map. An exponentially C^r Nash G map $g : X \longrightarrow Y$ is said to be *an exponentially C^r Nash G diffeomorphism* if there exists an exponentially C^r Nash G map $h : Y \longrightarrow X$ such that $g \circ h = id$ and $h \circ g = id$.

(4) We say that an exponentially C^r Nash G manifold is *affine* if it is exponentially C^r Nash G diffeomorphic to an exponentially C^r Nash G submanifold of some representation of G.

We have the following implications on groups:

an algebraic group$\Longrightarrow$an affine Nash group$\Longrightarrow$an affine exponentially Nash group $\Longrightarrow$ an exponentially Nash group $\Longrightarrow$ a Lie group .

Let G be an algebraic group. Then we obtain the following implications on G manifolds:

a nonsingular algebraic G set $\Longrightarrow$ an affine Nash G manifold $\Longrightarrow$ an affine exponentially Nash G manifold $\Longrightarrow$ an exponentially Nash G manifold $\Longrightarrow$ a $C^\infty G$ manifold .

Moreover, notice that a Nash G manifold is not always an affine exponentially Nash G manifold.

In the equivariant exponentially Nash category, the equivariant tubular neighborhood result holds true [8].

Proposition 2.7 [8]. *Let G be a compact affine exponentially Nash group and let X be an affine exponentially Nash G submanifold possibly with boundary in a representation Ω of G. Then there exists an exponentially Nash G tubular neighborhood (U, p) of X in Ω, namely U is an affine exponentially Nash G submanifold in Ω and the orthogonal projection $p : U \longrightarrow X$ is an exponentially Nash G map.* $\square$

Definition 2.8. Let G be an exponentially Nash group and let $0 \le r \le \omega$.
(1) A C^rG vector bundle (E, p, X) of rank k is said to be *an exponentially C^r Nash G vector bundle* if the following three conditions are satisfied:

- (a) The total space E and the base space X are exponentially C^r Nash G manifolds.
- (b) The projection p is an exponentially C^r Nash G map.
- (c) There exists a family of finitely many local trivializations $\{U_i, \phi_i : U_i \times \mathbb{R}^k \longrightarrow p^{-1}(U_i)\}_i$ such that $\{U_i\}_i$ is an open exponentially definable covering of X and that for any i and j the map $\phi_i^{-1} \circ \phi_j | (U_i \cap U_j) \times \mathbb{R}^k : (U_i \cap U_j) \times \mathbb{R}^k \longrightarrow (U_i \cap U_j) \times \mathbb{R}^k$ is an exponentially C^r Nash map.

We call these local trivializations *exponentially C^r Nash*.
(2) Let $\eta = (E, p, X)$ (resp. $\zeta = (F, q, X)$) be an exponentially C^r Nash G vector bundle of rank n (resp. m). Let $\{U_i, \phi_i : U_i \times \mathbb{R}^n \longrightarrow p^{-1}(U_i)\}_i$ (resp. $\{V_j, \psi_j : V_j \times \mathbb{R}^m \longrightarrow q^{-1}(V_j)\}_j$) be exponentially C^r Nash local trivializations of η (resp. ζ). A C^rG vector bundle map $f : \eta \longrightarrow \zeta$ is said to be *an exponentially C^r Nash G vector bundle map* if for any i and j the map $(\psi_j)^{-1} \circ f \circ \phi_i | (U_i \cap V_j) \times \mathbb{R}^n : (U_i \cap V_j) \times \mathbb{R}^n \longrightarrow (U_i \cap V_j) \times \mathbb{R}^m$ is an exponentially C^r Nash map. A C^rG section s of η is called *exponentially C^r Nash* if each $\phi_i^{-1} \circ s | U_i : U_i \longrightarrow U_i \times \mathbb{R}^n$ is exponentially C^r Nash.
(3) Two exponentially C^r Nash G vector bundles η and ζ are said to be *exponen-*

tially C^r Nash G vector bundle isomorphic if there exist exponentially C^r Nash G vector bundle maps $f : \eta \longrightarrow \zeta$ and $h : \zeta \longrightarrow \eta$ such that $f \circ h = id$ and $h \circ f = id$.

Recall universal G vector bundles (cf. [7]).

Definition 2.9. Let Ω be an n-dimensional representation of G and B the representation map $G \longrightarrow GL_n(\mathbb{R})$ of Ω. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \longrightarrow B(g)^{-1}AB(g) \in M(\Omega)$. For any positive integer k, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = k\},$$

$$E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\},$$

$$u : E(\Omega, k) \longrightarrow G(\Omega, k) : u((A, v)) = A,$$

where A' denotes the transposed matrix of A and TrA stands for the trace of A. Then $\gamma(\Omega, k)$ is an algebraic set. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with Ω and k.* Since $G(\Omega, k)$ and $E(\Omega, k)$ are nonsingular, $\gamma(\Omega, k)$ is a Nash G vector bundle, hence it is an exponentially Nash one.

Definition 2.10. An exponentially C^r Nash G vector bundle $\eta = (E, p, X)$ of rank k is said to be *strongly exponentially C^r Nash* if the base space X is affine and that there exist some representation Ω of G and an exponentially C^r Nash G map $f : X \longrightarrow G(\Omega, k)$ such that η is exponentially C^r Nash G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$.

Let G be a Nash group. Then we have the following implications on G vector bundles over an affine Nash G manifold:

a Nash G vector bundle $\Longrightarrow$ an exponentially Nash G vector bundle $\Longrightarrow$ a $C^\omega G$ vector bundle, and

a strongly Nash G vector bundle $\Longrightarrow$ a strongly exponentially Nash G vector bundle $\Longrightarrow$ an exponentially Nash G vector bundle.

3. Proof of results.

A subset of $\mathbb{R}^n$ is called *locally closed* if it is the intersection of an open set $\subset \mathbb{R}^n$ and a closed set $\subset \mathbb{R}^n$.

To prove Theorem 1.1, we recall the following.

Proposition 3.1 [8]. *Let $X \subset \mathbb{R}^n$ be a locally closed exponentially definable set and let f and g be continuous exponentially definable functions on X with $f^{-1}(0) \subset g^{-1}(0)$. Then there exist an integer N and a continuous exponentially definable function $h : X \longrightarrow \mathbb{R}$ such that $g^N = hf$ on X. In particular, for any compact subset K of X, there exists a positive constant c such that $|g^N| \le c|f|$ on K* $\square$

Proof of Theorem 1.1. Let X be an exponentially C^r Nash manifold. If dim $X = 0$ then X consists of finitely many points. Thus the result holds true.

Assume that dim $X \ge 1$. Let $\{\phi_i : U_i \longrightarrow \mathbb{R}^m\}_{i=1}^l$ be exponentially C^r Nash charts of X. Since X is compact, shrinking U_i, if necessarily, we may assume that every $\phi_i(U_i)$ is the open unit ball of $\mathbb{R}^m$ whose center is the origin. Let f be the function on $\mathbb{R}^m$ defined by $f(x) = ||x|| - 1$. Then $f^{-1}(0) = \overline{\phi_i(U_i)} - \phi_i(U_i)$. Hence replacing the graph of $1/f$ on $\phi_i(U_i)$ by $\phi_i(U_i)$, each $\phi_i(U_i)$ is closed in $\mathbb{R}^m$. Consider the stereographic projection $s : \mathbb{R}^m \longrightarrow S^m \subset \mathbb{R}^m \times \mathbb{R}$. Composing ϕ_i and s, we have an exponentially C^ω Nash imbedding $\phi'_i : \phi_i(U_i) \longrightarrow \mathbb{R}^{m'}$ such that the image is bounded in $\mathbb{R}^{m'}$ and

$$\overline{\phi'_i \circ \phi_i(U_i)} - \phi'_i \circ \phi_i(U_i)$$

consists of one point, say 0. Set

$$\eta : \mathbb{R}^{m'} \longrightarrow \mathbb{R}^{m'}, \eta(x_1, \ldots, x_{m'}) = (\sum_{j=1}^{m'} x_j^{2k} x_1, \ldots, \sum_{j=1}^{m'} x_j^{2k} x_{m'}),$$

$$g_i : U_i \longrightarrow \mathbb{R}^{m'}, \eta \circ \phi'_i \circ \phi_i,$$

for a sufficiently large integer k. Then g_i is an exponentially C^r Nash imbedding of U_i into $\mathbb{R}^{m'}$. Moreover the extension $\tilde{g_i} : X \longrightarrow \mathbb{R}$ of g_i defined by $\tilde{g_i} = 0$ on $X - U_i$. We now prove that $\tilde{g_i}$ is of class exponentially C^r Nash. It is sufficient to see this on each exponentially C^r Nash coordinate neighborhood of X. Hence we may assume that X is open in $\mathbb{R}^m$. We only have to prove that for any sequence $\{a_j\}_{j=1}^\infty$ in U_i convergent to a point of $X - U_i$ and for any $\alpha \in \mathbb{N}^m$ with $|\alpha| < r$, $\{D^\alpha g_i(a_j)\}_{j=1}^\infty$ converges to 0. On the other hand $g_i = (\sum_{j=1}^{m'} \phi_{ij}^{2k} \phi_{i1}, \cdots \sum_{j=1}^{m'} \phi_{ij}^{2k} \phi_{im'})$, where $\phi'_i \circ \phi_i = (\phi_{i1}, \ldots, \phi_{im'})$. Each ϕ_{ij} is bounded, and every $\{\phi_{ij}(a_i)\}_{i=1}^\infty$ converges to zero, and

$$|D^\alpha(\phi_{ij}^{2k}\phi_{is})| = |\sum_{\beta+\gamma=\alpha} (\alpha!/(\beta!\gamma!)) D^\beta \phi_{ij}^{2k} D^\gamma \phi_{is}| \leq$$

$$C \sum_{\beta_1+\cdots+\beta_{l'}+\gamma=\alpha, \beta_i \neq 0} |\phi_{ij}^{2k-l'} D^{\beta_1}\phi_{ij} \cdots D^{\beta_{l'}}\phi_{ij} D^\gamma \phi_{is}| \leq C'|\phi_{ij}^{2k-\gamma}|\psi,$$

where C, C' are constants, and ψ is the positive continuous exponentially definable function defined by

$$\psi(x) = \max\{1, \sum_{\beta_1+\cdots+\beta_{l'}+\gamma=\alpha} |D^{\beta_1}\phi_{ij}(x) \cdots D^{\beta_{l'}}\phi_{ij}(x) D^\gamma \phi_{is}(x)|\}.$$

Define

$$\theta_{ij}(x) = \begin{cases} \min\{|\phi_{ij}(x)|, 1/\psi(x)\} & \text{on } U_i \\ 0 & \text{on } X - U_i, \end{cases} \quad \tilde{\phi_{ij}} = \begin{cases} \phi_{ij} & \text{on } U_i \\ 0 & \text{on } X - U_i. \end{cases}$$

Then θ_{ij} and $\tilde{\phi_{ij}}$ are continuous exponentially definable functions on X such that

$$X - U_i \subset \theta_{ij}^{-1}(0) = \tilde{\phi_{ij}}^{-1}(0).$$

Hence by Proposition 3.1 we have $|\tilde{\phi_{ij}}^{l''}| \leq d\theta_{ij}$ on some open exponentially definable neighborhood V of $X - U_i$ in X for some integer l'', where d is a constant.

On the other hand, by the definition of θ_{ij} $|\psi\theta_{ij}| \leq 1$. Hence the above argument proves that

$$|D^\alpha(\phi_{ij}^{2k}\phi_{is})| \leq c'|\phi_{ij}^{2k-r-l''}|$$

on $U_i \cap V$, where c' is a constant and we take k such that $2k \geq r + l'' + 1$. Hence each $\tilde{g}_i$ is of class exponentially C^r Nash. It is easy to see that

$$\prod_{i=1}^{l} \tilde{g}_i : X \longrightarrow \mathbb{R}^{lm'}$$

is an exponentially C^r Nash imbedding. □

By the similar method of [7], we have the following.

Theorem 3.2 [8]. *Let G be a compact affine exponentially Nash group and let X be a compact affine exponentially Nash G manifold.*
(1) For every $C^\infty G$ vector bundle η over X, there exists a strongly exponentially Nash G vector bundle ζ which is $C^\infty G$ vector bundle isomorphic to η.
(2) For any two strongly exponentially Nash G vector bundles over X, they are exponentially Nash G vector bundle isomorphic if and only if they are C^0G vector bundle isomorphic. □

We prepare the following results to prove Theorem 1.2.

Proposition 3.3 [8]. *Let M be an affine exponentially Nash G manifold in a representation Ω of G.*
(1) The normal bundle (L, q, M) in Ω realized by

$$L = \{(x, y) \in M \times \Omega | y \text{ is orthogonal to } T_xM\}, q : L \longrightarrow M, q(x, y) = x$$

is an exponentially Nash G vector bundle.
(2) If M is compact, then some exponentially Nash G tubular neighborhood U of M in Ω obtained by Proposition 2.7 is exponentially Nash G diffeomorphic to L. □

Proposition 3.4 [8]. *Let G be a compact affine exponentially Nash group and let $\eta = (E, p, Y)$ be an exponentially Nash G vector bundle of rank k over an affine exponentially Nash G manifold Y. Then η is strongly exponentially Nash if and only if E is affine.* □

Lemma 3.5. *Let D_1 and D_2 be open balls of $\mathbb{R}^n$ which have the same center x_0, and let a (resp. b) be the radius of D_1 (resp. D_2) with $a < b$. Suppose that A and B are two real numbers. Then there exists a C^∞ exponentially definable function f on $\mathbb{R}^n$ such that $f = A$ on D_1 and $f = B$ on $\mathbb{R}^n - \overline{D_2}$.*

Proof. We can assume that $A = 1$, $B = 0$ and $x_0 = 0$.

At first we construct such a function when $n = 1$. Then we may assume that $D_1 = (-a, a)$ and $D_2 = (-b, b)$ be open intervals. Recall the exponentially definable C^∞ function λ defined in Example 2.4. The function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\phi(x) = \lambda(b - x)\lambda(b + x)/(\lambda(b - x)\lambda(b + x) + \lambda(x^2 - a^2))$$

is the desired function. Therefore $f : \mathbb{R}^n \longrightarrow \mathbb{R}, f(x) = \phi(|x|)$ is the required function, where $|x|$ denotes the standard norm of $\mathbb{R}^n$. $\square$

proof of Theorem 1.2. By Theorem 3.2 we may assume that η is a strongly exponentially C^ω Nash G vector bundle. We only have to find an exponentially C^ω Nash G vector bundle ζ which is exponentially C^∞ Nash G vector bundle isomorphic to η but not exponentially C^ω Nash G vector bundle isomorphic to η.

As well as the usual equivariant Nash category, X^G is an exponentially Nash G submanifold of X. Take an open exponentially definable subset U of X such that $\eta|U$ is exponentially C^ω Nash vector bundle isomorphic to the trivial bundle and that $X^G \cap U \neq \emptyset$. Since $\dim X^G \geq 2$, there exists a one-dimensional exponentially Nash G submanifold S of U which is exponentially Nash diffeomorphic to the unit circle S^1 in $\mathbb{R}^2$. Moreover there exist two open G invariant exponentially definable subsets V_1 and V_2 of U such that $V_1 \cup V_2 \supset S$ and $V_1 \cap V_2$ consists of two open balls Z_1 and Z_2. We define the exponentially C^ω Nash G vector bundle $\zeta' := (E, r, V_1 \cup V_2)$ over $V_1 \cup V_2$ to be the bundle obtained by the coordinate transformation

$$g_{12} : V_1 \cap V_2 \longrightarrow GL(\Xi), g_{12} = \begin{cases} I & \text{on } Z_1 \\ (1+\epsilon)I & \text{on } Z_2, \end{cases}$$

where I denotes the unit matrix, $\epsilon > 0$ is sufficiently small and Ξ stands for the fiber of $\eta|U$. This construction is inspired by the proof of 4.2.8 [23].

Let $\phi_i : V_i \times \Xi \longrightarrow p^{-1}(V_i)$, $i = 1, 2$ be exponentially Nash G coordinate functions of ζ'. Consider an extension of the exponentially C^ω Nash section f on $S \cap V_1$ defined by $\phi_1^{-1} \circ f(x) = (x, I)$. If we extend f through Z_1, then the analytic extension $\tilde{f}$ to $S \cap V_2$ satisfies $\phi_2^{-1} \circ \tilde{f} = (x, I), x \in S \cap V_2$. However the analytic extension $\tilde{f}$ to $S \cap V_2$ through Z_2 satisfies $\phi_2^{-1} \circ \tilde{f} = (x, 1/(1+\epsilon)I)$. Thus the smallest analytic set containing the graph of f spins infinitely over S. Hence $\zeta'|S$ is not exponentially C^ω Nash G vector bundle isomorphic to $\eta|S$. By Theorem 3.2 $\zeta'|S$ is not strongly exponentially C^ω Nash. Thus the exponentially C^ω Nash G vector bundle ζ over X obtained by replacing $\eta|V_1 \cup V_2$ by ζ' is not exponentially C^ω Nash G vector bundle isomorphic to η.

On the other hand, by Lemma 3.5 we can construct an exponentially C^∞ Nash G map H from a G invariant exponentially definable neighborhood of $U \cap X^G$ in U to $GL(\Xi)$ such that $H|Z_2 = (1+\epsilon)I$ and $H = I$ outside of some G invariant exponentially definable neighborhood of Z_2. Since ϵ is sufficiently small, using this map, we get an exponentially C^∞ Nash G vector bundle isomorphism $\eta \longrightarrow \zeta$. $\square$

Proof of Theorem 1.3. By the proof of Theorem 1 (1) [9], X is $C^\infty G$ diffeomorphic to some affine exponentially Nash G manifold. Hence we may assume that X is an affine exponentially C^ω Nash G manifold.

Since X^G is an exponentially C^ω Nash G submanifold of X, there exists an exponentially Nash G tubular neighborhood (T, q) of X^G in X by Proposition 2.7. Moreover we may assume that T is exponentially C^ω Nash G diffeomorphic to the total space of the normal bundle η of X^G in X because of Proposition 3.3. Note that η is a strongly exponentially C^ω Nash G vector bundle over X^G and that each fiber is a representation of G. Take an open G invariant exponentially definable subset U of X^G such that $\eta|U$ is exponentially C^ω Nash G vector bundle isomorphic to the trivial bundle $U \times \Xi$, where Ξ denotes the fiber of $\eta|U$.

By the proof of Theorem 1.2, there exists an exponentially C^ω Nash G vector bundle η' over U such that η' is not exponentially C^ω Nash G vector bundle isomorphic to $\eta|U$ and that there exists an exponentially C^∞ Nash G vector bundle isomorphism $H : \eta|U \longrightarrow \eta'$ such that H is the identity outside of some open G invariant exponentially definable set.

Replacing the total space of $\eta|U$ by that of η', we have an exponentially C^ω Nash G manifold Y which is not exponentially C^ω Nash G diffeomorphic to X. Moreover using H, one can find an exponentially C^∞ Nash G diffeomorphism from X to Y. $\square$

Note that Y is not exponentially C^ω Nash G affine but exponentially C^∞ Nash G affine by Proposition 3.4.

4. Remarks.

It is known in [1] that every compact Lie group admits one and exactly one algebraic group structure up to algebraic group isomorphism. Hence it admits an affine Nash group structure. Notice that all connected one-dimensional Nash groups and locally Nash groups are classified by [16] and [22], respectively. In particular, the unit circle S^1 in $\mathbb{R}^2$ admits a nonaffine Nash group structure.

But the analogous result concerning nonaffine exponentially Nash group structures of centerless Lie groups does not hold.

Remark 4.1. *Let G be a compact centerless Lie group. Then G does not admit any nonaffine exponentially Nash group structure.*

Proof. Let G' be an exponentially Nash group which is isomorphic to G as a Lie group. Then the adjoint representation $Ad : G' \longrightarrow Gl_n(\mathbb{R})$ is exponentially definable by the similar method of Lemma 2.2 [15] and it is C^ω, where n denotes the dimension of G. Hence Ad is an exponentially Nash one and its kernel is the center of G'. Therefore the image G" of Ad is an affine exponentially Nash group and Ad is an exponentially Nash group isomorphism from G' to G". $\square$

It is known that any two disjoint closed semialgebraic sets X and Y in $\mathbb{R}^n$ can be separated by a C^ω Nash function on $\mathbb{R}^n$ [18], namely there exists a C^ω Nash function f on $\mathbb{R}^n$ such that

$$f > 0 \text{ on } X \text{ and } f < 0 \text{ on } Y.$$

The following is a weak equivariant version of Nash category and exponentially Nash category.

Remark 4.2. *Let G be a compact affine Nash (resp. a compact affine exponentially Nash) group. Then any two disjoint closed G invariant semialgebraic (resp. disjoint closed G invariant exponentially definable) sets in a representation Ω of G can be separated by a G invariant continuous semialgebraic (resp. a G invariant continuous exponentially definable) function on Ω.*

Proof. By the distance $d(x, X)$ of x between X is semialgebraic (resp. exponentially definable). Since G is compact, $d(x, X)$ is equivariant. Hence $F : \Omega \longrightarrow \mathbb{R}, F(x) = d(x, Y) - d(x, X)$ is the desired one. $\square$

Remark 4.3. *Under the assumption of 4.2, if one of the above two sets is compact, then they are separated by a G invariant entire rational function on Ω, where an entire rational function means a fraction of polynomial functions with nowhere vanishing denominator.*

Proof. Assume that X is compact and Y is noncompact. Let $s : \Omega \longrightarrow S \subset \Omega \times \mathbb{R}$ be the stereographic projection and let $S = \Omega \cup \{\infty\}$. Since X is compact, $s(X)$ and $s(Y) \cup \{\infty\}$ are compact and disjoint. Applying Remark 4.2, we have a G invariant continuous semialgebraic (resp. a G invariant continuous exponentially definable) function f on $\Omega \times \mathbb{R}$. By the classical polynomial approximation theorem and Lemma 4.1 [4], we get a G invariant polynomial F on $\Omega \times \mathbb{R}$ such that $F|S$ is an approximation of f. Since $s(X)$ and $s(Y) \cup \{\infty\}$ are compact, $F \circ s$ is the required one. $\square$

Remark 4.4. *Let $X \subset \mathbb{R}^n$ be an open (resp. a closed) exponentially definable set. Suppose that X is a finite union of sets of the following form:*

$$\{x \in \mathbb{R}^n | f_1(x) = \cdots = f_i(x) = 0, g_1(x) > 0, \ldots, g_j(x) > 0\},$$

$$(\textit{resp. } \{x \in \mathbb{R}^n | f_1(x) = \cdots = f_i(x) = 0, g_1(x) \geq 0, \ldots, g_j(x) \geq 0\},)$$

where $f_1, \ldots, f_i$ and $g_1, \ldots, g_j$ are exponentially Nash functions on $\mathbb{R}^n$. Then X is a finite union of sets of the following form:

$$\{x \in \mathbb{R}^n | h_1(x) > 0, \ldots, h_k(x) > 0\},$$

$$(\textit{resp. } \{x \in \mathbb{R}^n | h_1(x) \geq 0, \ldots, h_k(x) \geq 0\},)$$

where $h_1, \ldots, h_i$ are exponentially Nash functions on $\mathbb{R}^n$.

Note that any exponentially definable set in $\mathbb{R}^n$ can be described as a finite union of sets of the following form [8]:

$$\{x \in \mathbb{R}^n | F_1(x) = \cdots = F_s(x) = 0, G_1(x) > 0, \ldots, G_t(x) > 0\}.$$

Here each of $F_1, \ldots, F_s$ and $G_1, \ldots, G_t$ is an exponentially Nash function defined on some open exponentially definable subset of $\mathbb{R}^n$, however its domain is not always the whole space $\mathbb{R}^n$.

We define $exp_n(x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$ by $exp_0(x) = x$ and $exp_{n+1}(x) = exp(exp_n(x))$. The following is a bound of the growth of continuous exponentially definable functions

Proposition 4.5 [8]. *Let F be a closed exponentially definable set in $\mathbb{R}^k$ and let $f : F \longrightarrow \mathbb{R}$ be a continuous exponentially definable function. Then there exist $c > 0, n, m \in \mathbb{N}$ such that*

$$|f(x)| \leq c(1 + exp_n(||x||^m)) \textit{ for any } x \in F,$$

where $||\cdot||$ denotes the standard norm of $\mathbb{R}^k$. $\square$

Proof of Remark 4.4. It suffices to prove the result when X is open because the other case follows by taking complements.

Let

$$B = \{x \in \mathbb{R}^n | f_1(x) = \cdots = f_u(x) = 0, g_1(x) > 0, \ldots, g_v(x) > 0\},$$

where all f_i and all g_j are exponentially Nash functions on $\mathbb{R}^n$. Set $f := f_1^2 + \cdots + f_u^2$ and $g(x) := \prod_{i=1}^{v}(|g_i(x)| + g_i(x))$. On $\mathbb{R}^n - X$, $g(x) = 0$ if $f(x) = 0$. By Proposition 3.1 there exists an integer N and a continuous exponentially definable function h on $\mathbb{R}^n - X$ such that $g^N = hf$ on $\mathbb{R}^n - X$. By Proposition 4.5 we have some $c \in \mathbb{R}$ and some $m, n \in \mathbb{N}$ such that $|h(x)| \leq c(1 + exp_n(||x||^m)$ on $\mathbb{R}^n - X$. Define $B_1 = \{x \in \mathbb{R}^n | cf(x)(1 + exp_n(||x||^m)) < (2^m \prod_{i=1}^{m} g_i(x))^N, g_1(x) > 0, \ldots, g_m(x) > 0\}$. Then $B \subset B_1 \subset X$. Therefore replacing B by B_1, we have the required union. $\square$

References

1. C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, Princeton , N. J., 1946.
2. M. Coste and M. Shiota, *Nash triviality in families of Nash manifolds*, Invent. Math. **108** (1992), 349-368.
3. M. Coste and M. Shiota, *Thom's first isotopy lemma: a semialgebraic version, with uniform bound*, Real analytic and algebraic geometry, Proceedings of the International Conference held at Trento (Italy), Walter de Gruyter, Berlin, New York (1995).
4. K.H. Dovermann, M. Masuda, and T. Petrie, *Fixed point free algebraic actions on varieties diffeomorphic to* $\mathbb{R}^n$, Progress in Math. 80, Birkhäuser (1990), 49-80.
5. L. van den Dries, A. Macintyre and D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. of Math. **140** (1994), 183-205.
6. E. Hrushovski and A. Pillay, *Groups definable in local fields and psedo-finite fields*, Israel J. Math. **85** (1994), 203-262.
7. T. Kawakami, *Algebraic G vector bundles and Nash G vector bundles*, Chinese J. Math. **22** (1994), 275-289.
8. T. Kawakami, *Exponentially Nash manifolds and equivariant exponentially Nash manifolds*, preprint.
9. T. Kawakami, *Nash G manifold structures of compact or compactifiable $C^\infty G$ manifolds*, J. Math. Soc. Japan. **48** (1996), 321-331.
10. S.Koike, T.Fukui and, M.Shiota, *Modified Nash triviality of family of zero-sets of real polynomial mappings*, preprint.
11. T.L. Loi, *Analytic cell decomposition of sets definable in the structure* $\mathbb{R}_{exp}$, Ann. Pol. Math. **59** (1994), 255-266.
12. T.L. Loi, *C^k-regular points of sets definable in the structure* $< \mathbb{R}, <, 0, 1, +, -, exp >$, preprint.
13. T.L. Loi, *On the global Lojasiewicz inequalities for the class of analytic logarithmico-exponential functions*, C.R. Acad. Sci. Paris. **318** (1994), 543-548.
14. T.L. Loi, *Whitney stratifications of sets definable in the structure* $\mathbb{R}_{exp}$, to appear.
15. A.N. Nesin and A. Pillay, *Some model theory of compact Lie groups*, Trans. Amer. Math. Soc **326** (1991), 453-463.
16. J.J. Madden and C.H. Stanton, *One-dimensional Nash groups*, Pacific J. Math. **154** (1992), 331-344.
17. C. Miller, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. **122** (1994), 257-259.
18. T. Mostowski, *Some properties of the ring of Nash functions*, Ann. Scu. Norm. Sup. Pisa, 3-2 (1976), 245-266.
19. M. Shiota, *Abstract Nash manifolds*, Proc. Amer. Math. Soc. **96** (1986), 155-162.
20. M. Shiota, *Approximation theorems for Nash mappings and Nash manifolds*, Trans. Amer. Math. Soc. **293** (1986), 319-337.
21. M. Shiota, *Classification of Nash manifolds*, Ann. Inst. Fourier, Grenoble **33** (1983), 209-232.
22. M. Shiota, *Geometry of subanalytic and semialgebraic sets*, to appear.
23. M. Shiota, *Nash manifolds*, Lecture Note in Mathematics vol 1269, Springer-Verlag, 1987.
24. A. Tarski, *A decision method for elementary algebra and geometry, 2nd. ed., revised*, Berkely and Los Angels, 1951.

25. A. Wilkie, *Model completeness results for expansions of the real field 1: restricted Pfaffian functions*, to appear.
26. A. Wilkie, *Model completeness results for expansions of the real field 2: the exponential function*, to appear.