CHARACTERIZATIONS OF INFINITE-DIMENSIONAL MANIFOLD TUPLES AND THEIR APPLICATIONS TO HOMEOMORPHISM GROUPS

TATSUHIKO YAGASAKI

Chapter 1. Characterization of $(s, S_1, \cdots, S_l)$-manifolds

The purpose of this article is to survey tuples of infinite-dimensional topological manifolds and their application to the homeomorphism groups of manifolds.

1. $(s, S_1, \cdots, S_l)$-MANIFOLDS

A topological E-manifold is a space which is locally homeomorphic to a space E. In this article all spaces are assumed to be separable and metrizable. In infinite-dimensional topological manifold theory, we are mainly concerned with the following model spaces E:
(i) (the compact model) the Hilbert cube: $Q = [-\infty, \infty]^\infty$.
(ii) (the complete linear model) the Hilbert space: ℓ^2.

The Hilbert space ℓ^2, or more generally any separable Frechet space is homeomorphic to $s \equiv (-\infty, \infty)^\infty$ ([1]). If we regard s as a linear space of sequences of real numbers, then it contains several natural (incomplete) linear subspaces:
(iii) the big sigma: $\Sigma = \{(x_n) \in s \,:\, \sup_n |x_n| < \infty\}$ (the subspace of bounded sequences).
(iv) the small sigma: $\sigma = \{(x_n) \in s \,:\, x_n = 0$ for almost all $n\}$ (the subspace of finite sequences).

The main sources of infinite-dimensional manifolds are various spaces of functions, embeddings and homeomorphisms. In Chapter 2 we shall consider the group of homeomorphisms of a manifold. When M is a PL-manifold, the homeomorphism group $\mathcal{H}(M)$ contains the subgroup $\mathcal{H}^{\mathrm{PL}}(M)$ consisting of PL-homeomorphisms of M, and we can ask the natural question: How is $\mathcal{H}^{\mathrm{PL}}(M)$ sited in the ambient group $\mathcal{H}(M)$? This sort of question leads to the following general definition. An $(l+1)$-tuple of spaces means a tuple $(X, X_1, \cdots, X_l)$ consisting of an ambient space X and l subspaces $X_1 \supset \cdots \supset X_l$.

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Definition. A tuple $(X, X_1, \cdots, X_l)$ is said to be an $(E, E_1, \cdots, E_l)$-manifold if for every point $x \in X$ there exist an open neighborhood U of x in X and an open set V of E such that $(U, U \cap X_1, \cdots, U \cap X_l) \cong (V, V \cap E_1, \cdots, V \cap E_l)$.

In this article we shall consider the general model tuple of the form: $(s, S_1, \cdots, S_l)$, where $S_1 \supset \cdots \supset S_l$ are linear subspaces of s. Some typical examples are:
(v) the pairs: (s, Σ), (s, σ),
(vi) the triples: (s, Σ, σ), $(s^2, s \times \sigma, \sigma^2)$, $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$, and $(s^\infty, \sigma^\infty, \sigma_f^\infty)$.
where (a) s^∞, Σ^∞, σ^∞ are the countable product of s, Σ and σ respectively,
(b) $\Sigma_f^\infty = \{(x_n) \in \Sigma^\infty : x_n = 0$ for almost all $n\}$, $\sigma_f^\infty = \{(x_n) \in \sigma^\infty : x_n = 0$ for almost all $n\}$.
Note that (1) $(s^\infty, \sigma_f^\infty) \cong (s, \sigma)$, (2) $(s^\infty, \Sigma_f^\infty, \sigma_f^\infty) \cong (s, \Sigma, \sigma)$ and (3) $(s^\infty, \Sigma^\infty, \sigma_f^\infty) \cong (s^\infty, \sigma^\infty, \sigma_f^\infty)$. The statements (2) and (3) follow from the characterizations of manifolds modeled on these triples (§4.2.2, Theorems 3.11 - 3.14). In Section 4.2.1 we shall give a general characterization of $(s, S_1, \cdots, S_l)$-manifold under some natural conditions on the model $(s, S_1, \cdots, S_l)$.

2. Basic properties of infinite-dimensional manifolds

In this section we will list up some fundamental properties of infinite-dimensional manifolds. We refer to [10, 11, 24] for general references in infinite-dimensional manifold theory.

2.1. **Stability.**

Since s is a countable product of the interval $(-\infty, \infty)$, it is directly seen that $s^2 \cong s$. Applying this argument locally, it follows that $X \times s \cong X$ for every s-manifold X (cf.[25]). More generally, it has been shown that if (X, X_1, X_2) is an (s, Σ, σ)-manifold, then $(X \times s, X_1 \times \Sigma, X_2 \times \sigma) \cong (X, X_1, X_2)$ [27]. This property is one of characteristic properties of infinite-dimensional manifolds. To simplify the notation we shall use the following terminology:

Definition. We say that $(X, X_1, \cdots, X_l)$ is $(E, E_1, \cdots, E_l)$-stable if $(X \times E, X_1 \times E_1, \cdots, X_l \times E_l) \cong (X, X_1, \cdots, X_l)$.

2.2. **Homotopy negligibility.**

Definition. A subset B of Y is said to be homotopy negligible (h.n.) in Y if there exists a homotopy $\phi_t : Y \to Y$ such that $\phi_0 = id$ and $\phi_t(Y) \subset Y \setminus B$ $(0 < t \leq 1)$. In this case, we say that $Y \setminus B$ has the homotopy negligible (h.n.) complement in Y.

When Y is an ANR, B is homotopy negligible in Y iff for every open set U of Y, the inclusion $U \setminus B \subset U$ is a weak homotopy equivalence. Again using the infinite coordinates of s, we can easily verify that σ has the h.n. complement in

s. Therefore, it follows that if (X, X_1) is an (s, σ)-manifold, then X_1 has the h.n. complement in X.

2.3. **General position property – Strong universality.**

2.3.1. *Z-embedding approximation in s-manifolds.*

The most basic notion in infinite-dimensional manifolds is the notion of Z-sets:

Definition. A closed set Z of X is said to be a Z-set (a strong Z-set) of X if for every open cover $\mathcal{U}$ of X there is a map $f : X \to X$ such that $f(X) \cap Z = \emptyset$ ($cl\, f(X) \cap Z = \emptyset$) and $(f, id_X) \leq \mathcal{U}$.

Here, for an open cover $\mathcal{V}$ of Y, two map f, $g : X \to Y$ are said to be $\mathcal{V}$-close and written as $(f, g) \leq \mathcal{V}$ if for every $x \in X$ there exists a $V \in \mathcal{V}$ with $f(x)$, $g(x) \in V$. Using the infinite coordinates of s, we can show the following general position property of s-manifolds:

Facts 2.1. Suppose Y is an s-manifold. Then for every map $f : X \to Y$ from a separable completely metrizable space X and for every open cover $\mathcal{U}$ of Y, there exists a Z-embedding $g : X \to Y$ with $(f, g) \leq \mathcal{U}$. Furthermore, if K is a closed subset of X and $f|_K : K \to Y$ is a Z-embedding, then we can take g so that $g|_K = f|_K$.

2.3.2. *Strong universality.*

To treat various incomplete submanifolds of s-manifolds (σ-manifolds, Σ-manifolds, etc.), we need to restrict the class of domain X in the above statement. Let $\mathcal{C}$ be a class of spaces.

Definition. (M. Bestvina - J. Mogilski [5], et. al.)
A space Y is said to be strongly $\mathcal{C}$-universal if for every $X \in \mathcal{C}$, every closed subset K of X, every map $f : X \to Y$ such that $f|_K : K \to Y$ is a Z-embedding and for every open cover $\mathcal{U}$ of Y, there exists a Z-embedding $g : X \to Y$ such that $g|_K = f|_K$ and $(f, g) \leq \mathcal{U}$.

In some cases, the above embedding approximation conditions can be replaced by the following disjoint approximation conditions.

Definition. We say that a space X has the strong discrete approximation property (or the disjoint discrete cells property) if for every map $f : \oplus_{i \geq 1} Q_i \to X$ of a countable disjoint union of Hilbert cubes into X and for every open cover $\mathcal{U}$ of X there exists a map $g : \oplus_{i \geq 1} Q_i \to X$ such that $(f, g) \leq \mathcal{U}$ and $\{g(Q_i)\}_i$ is discrete in X.

2.3.3. *Strong universality of tuples.* (R. Cauty [6], J. Baars-H. Gladdines-J. van Mill [3], et. al.)

A map of tuples $f : (X, X_1, \cdots, X_l) \to (Y, Y_1, \cdots, Y_l)$ is said to be layer preserving if $f(X_{i-1} \setminus X_i) \subset Y_{i-1} \setminus Y_i$ for every $i = 1, \cdots, l+1$, where $X_0 = X$, $X_{l+1} = \emptyset$. Let $\mathcal{M}$ be a class of $(l+1)$-tuples of spaces.

Definition. An $(l+1)$-tuple $(Y, Y_1, \cdots, Y_l)$ is said to be strongly $\mathcal{M}$-universal if it satisfies the following condition:

$(*)$ for every tuple $(X, X_1, \cdots X_l) \in \mathcal{M}$, every closed subset K of X, every map $f : X \to Y$ such that $f|_K : (K, K \cap X_1, \cdots K \cap X_l) \to (Y, Y_1, \cdots, Y_l)$ is a layer preserving Z-embedding, and every open cover $\mathcal{U}$ of Y, there exists a layer preserving Z-embedding $g : (X, X_1, \cdots X_l) \to (Y, Y_1, \cdots, Y_l)$ such that $g|_K = f|_K$ and $(f, g) \leq \mathcal{U}$.

In Section 4.2.1 we shall see that the stability + h.n. complement implies the strong universality.

2.4. **Uniqueness properties of absorbing sets.**

The notion of h.n. complement can be regarded as a homotopical absorbing property of a subspace in an ambient space. The notion of strong universality of tuples also can be regarded as a sort of absorption property combined with the general position property. Roughly speaking, for a class $\mathcal{M}$, an $\mathcal{M}$-absorbing set of an s-manifold X is a subspace A of X such that (i) A has an absorption property in X for the class $\mathcal{M}$, (ii) A has a general position property for $\mathcal{M}$ and (iii) A "belongs" to the class $\mathcal{M}$. The notion of strong universality of tuples realizes the conditions (i) and (ii) simultaneously. The condition (iii) usually appears in the form: A is a countable union of Z-sets of A which belong to $\mathcal{M}$. The most important property of absorbing sets is the uniqueness property. This property will play a key role in the characterizations of tuples of infinite-dimensional manifolds.

2.4.1. *Capsets and fd capsets.* (R.D. Anderson and T.A. Chapman [9])

The most basic absorbing sets are capsets and fd capsets. A space is said to be σ-compact (σ-fd-compact) if it is a countable union of compact (finite-dimensional compact) subsets.

Definition. Suppose X is a Q-manifold or an s-manifold. A subset A of X is said to be a (fd) capset of X if A is a union of (fd) compact Z-sets A_n $(n \geq 1)$ which satisfy the following condition: for every $\varepsilon > 0$, every (fd) compact subset K of X and every $n \geq 1$ there exist an $m \geq n$ and an embedding $h : K \to A_m$ such that (i) $d(h, id_K) < \varepsilon$ and (ii) $h = id$ on $A_n \cap K$.

For example Σ is a capset of s and σ is fd capset of s. The (fd) capsets have the following uniqueness property:

Theorem 2.1. *If A and B are (fd) capsets of X, then for every open cover $\mathcal{U}$ of X there exists a homeomorphism $f : (X, A) \to (X, B)$ with $(f, id_X) \leq \mathcal{U}$.*

2.4.2. *Absorbing sets in s-manifolds.*

The notion of (fd) capsets works only for the class of σ-(fd-)compact subsets. To treat other classes of subsets we need to extend this notion.

Definition. A class $\mathcal{C}$ of spaces is said to be
(i) topological if $D \cong C \in \mathcal{C}$ implies $D \in \mathcal{C}$.
(ii) additive if $C \in \mathcal{C}$ whenever $C = A \cup B$, A and B are closed subsets of C, and A, $B \in \mathcal{C}$.
(iii) closed hereditary if $D \in \mathcal{C}$ whenever D is a closed subset of a space $C \in \mathcal{C}$.

[1] The non-ambient case: (M. Bestvina - J. Mogilski [5])

Let $\mathcal{C}$ be a class of spaces.

Definition. A subset A of an s-manifold X is said to be a $\mathcal{C}$-absorbing set of X if
(i) A has the h.n. complement in X,
(ii) A is strongly $\mathcal{C}$-universal,
(iii) $A = \cup_{n=1}^{\infty} A_n$, where each A_n is a Z-set of A and $A_n \in \mathcal{C}$.

Theorem 2.2. *Suppose a class $\mathcal{C}$ is topological, additive and closed hereditary. If A and B are two $\mathcal{C}$-absorbing sets in an s-manifold X, then every open cover $\mathcal{U}$ of X there exists a homeomorphism $h : X \to Y$ which is $\mathcal{U}$-close to the inclusion $A \subset X$.*

In general, h cannot be extended to any ambient homeomorphism of X.

[2] The ambient case: (J. Baars-H. Gladdines-J. van Mill [3], R. Cauty [6], T. Yagasaki [32], et.al.)

Let $\mathcal{M}$ be a class of $(l+1)$-tuples. We assume that $\mathcal{M}$ is topological, additive and closed hereditary. We consider the following condition (I):

The condition (I)
(I-1) $(X, X_1, \cdots, X_l)$ is strongly $\mathcal{M}$-universal,
(I-2) there exist Z-sets Z_n $(n \geq 1)$ of X such that
 (i) $X_1 \subset \cup_n Z_n$ and (ii) $(Z_n, Z_n \cap X_1, \cdots, Z_n \cap X_l) \in \mathcal{M}$ $(n \geq 1)$.

In this case we have ambient homeomorphisms:

Theorem 2.3. ([6, 32]) *Suppose E is an s-manifold and $(l+1)$-tuples $(E, X_1, \cdots, X_l)$ and $(E, Y_1, \cdots, Y_l)$ satisfy the condition* (I). *Then for any open cover $\mathcal{U}$ of E there exists a homeomorphism $f : (E, X_1, \cdots, X_l) \to (E, Y_1, \cdots, Y_l)$ with $(f, id_E) \leq \mathcal{U}$.*

2.5. **Homotopy invariance.**

Classification of infinite-dimensional manifolds is rather simple. Q-manifolds are classified by simple homotopy equivalence (T.A. Chapman [10]) and s-manifolds are classified by homotopy equivalence (D. W. Henderson and R. M. Schori [18]).

Theorem 2.4. *Suppose X and Y are s-manifolds. Then $X \cong Y$ iff $X \simeq Y$ (homotopy equivalence).*

3. Characterization of infinite-dimensional manifolds in term of general position property and stability

3.1. **Edwards' program.**

There is a general method, called as Edwards' program, of detecting topological E-manifolds. For infinite-dimensional topological manifolds, it takes the following form: Let X be an ANR.

(i) Construct a fine homotopy equivalence from an E-manifold to the target X.

(ii) Show that f can be approximated by homeomorphisms under some general position property of X.

This program yields basic characterizations of Q-manifolds, s-manifolds and other incomplete manifolds.

3.2. **The complete cases:**

(1) Q-manifolds:

Theorem 3.1. ([10]) *A space X is an Q-manifold iff*
(i) *X is a locally compact separable metrizable ANR*
(ii) *X has the disjoint cells property.*

(2) s-manifolds:

Theorem 3.2. ([30]) *A space X is an s-manifold iff*
(i) *X is a separable completely metrizable ANR*
(ii) *X has the strong discrete approximation property.*

Since the Q-stability implies the disjoint cells property and the s-stability implies the strong discrete approximation property, we can replace the condition (ii) by
(ii$'$) X is Q-stable (respectively s-stable)

3.3. **The incomplete cases:**

M. Bestvina-J. Mogilski [5] has shown that in the incomplete case the above program is formulated in the following form:

Theorem 3.3. (M. Bestvina-J. Mogilski [5])
Suppose $\mathcal{C}$ is a class of spaces which is topological, additive and closed hereditary.
(i) *For every ANR X there exists an s-manifold M such that for every $\mathcal{C}$-absorbing set Ω in M there exists a fine homotopy equivalence $f : \Omega \to X$.*
(ii) *Suppose* (a) *X is a strongly $\mathcal{C}$-universal ANR and* (b) *$X = \cup_{i=1}^{\infty} X_i$, where each X_i is a strong Z-set in X and $X_i \in \mathcal{C}$. Then every fine homotopy equivalence $f : \Omega \to X$ from any $\mathcal{C}$-absorbing set Ω in an s-manifold can be approximated by homeomorphisms.*

Example: Σ-manifolds and σ-manifolds

Let $\mathcal{C}_c$ ($\mathcal{C}_{fdc}$) denote the class of all (finite dimensional) compacta.

Theorem 3.4. (M. Bestvina-J. Mogilski [5, 23])
A space X is a Σ-manifold (σ-manifold) iff
(i) *X is a separable ANR and σ-compact (σ-fd compact),*
(ii) *X is strongly $\mathcal{C}_c$-universal (strongly $\mathcal{C}_{fdc}$-universal),*
(iii) *$X = \cup_{n=1}^{\infty} X_n$, where each X_n is a strong Z-set in X.*

The condition (iii) can be replaced by
(iii′) X satisfies the strong discrete approximation property.

In [28] H. Toruńczyk has obtained a characterization of σ-manifolds in term of stability.

Theorem 3.5. ([29])

X is a σ-manifold iff X is (i) *a separable ANR,* (ii) *σ-fd-compact and* (iii) *σ-stable.*

4. Characterizations of $(s, S_1, \cdots, S_l)$-manifolds

In this section we will investigate the problem of detecting $(s, S_1, \cdots, S_l)$-manifolds. Since we have obtained a characterization of s-manifolds (Theorem 3.2), the remaining problem is how to compare a tuple $(X, X_1, \cdots, X_l)$ locally with $(s, S_1, \cdots, S_l)$ when X is an s-manifold. For this purpose we will use the uniqueness property of absorbing sets in s-manifolds (§2.4). Since s-manifolds are homotopy invariant (Theorem 2.4), at the same time we can show the homotopy invariance of $(s, S_1, \cdots, S_l)$-manifolds.

4.1. **Characterizations of manifold tuples in term of the absorbing sets.**

4.1.1. *Characterizations in term of capsets and fd-capsets.*

Theorem 4.1. (T.A. Chapman [9])
(1) *(X, X_1) is an (s, Σ)-manifold ((s, σ)-manifold) iff*
(i) *X is an s-manifold,*
(ii) *X_1 is a capset (a fd capset).*

(2) *Suppose* (X, X_1) *and* (Y, Y_1) *are* (s, Σ)*-manifolds (* (s, σ)*-manifolds). Then* $(X, X_1) \cong (Y, Y_1)$ *iff* $X \simeq Y$.

Theorem 4.2. (K. Sakai-R.Y. Wong [27])
(1) (X, X_1, X_2) *is an* (s, Σ, σ)*-manifold iff*
(i) X *is an s-manifold,*
(ii) (X_1, X_2) *is a (cap, fd cap)-pair in* X.
(2) *Suppose* (X, X_1, X_2) *and* (Y, Y_1, Y_2) *are* (s, Σ, σ)*-manifolds. Then* $(X, X_1, X_2) \cong (Y, Y_1, Y_2)$ *iff* $X \simeq Y$.

4.1.2. *Characterizations in term of strong universality.*

We assume that $\underline{(s, S_1, \cdots, S_l) \text{ satisfies the condition (I)}}$ in Section 2.4.2.[2].

Theorem 4.3. (1) $(X, X_1, \cdots, X_l)$ *is an* $(s, S_1, \cdots, S_l)$*-manifold iff*
(i) X *is an s-manifold,*
(ii) $(X, X_1, \cdots, X_l)$ *satisfies the condition* (I).
(2) *Suppose* $(X, X_1, \cdots, X_l)$ *and* $(Y, Y_1, \cdots, Y_l)$ *are* $(s, S_1, \cdots, S_l)$*-manifolds. Then* $(X, X_1, \cdots, X_l) \cong (Y, Y_1, \cdots, Y_l)$ *iff* $X \simeq Y$.

4.2. **Characterization in term of stability and homotopy negligible complement.**

4.2.1. *General characterization theorem.*

We can show that the stability + h.n. complement implies the strong universality. This leads to a characterization based upon the stability condition. We consider the following condition (II).

The condition (II):
(II-1) S_1 is contained in a countable union of Z-sets of s,
(II-2) S_l has the h.n. complement in s,
(II-3) (Infinite coordinates) There exists a sequence of disjoint infinite subsets $A_n \subset \mathbb{N}$ $(n \geq 1)$ such that for each $i = 1, \cdots, l$ and $n \geq 1$, (a) $S_i = \pi_{A_n}(S_i) \times \pi_{\mathbb{N}\setminus A_n}(S_i)$ and (b) $(\pi_{A_n}(s), \pi_{A_n}(S_1), \cdots, \pi_{A_n}(S_l)) \cong (s, S_1, \cdots, S_l)$.

Here for a subset A of $\mathbb{N}$, $\pi_A : s \to \prod_{k\in A}(-\infty, \infty)$ denotes the projection onto the A-factor of s.

Assumption. We assume that $\underline{(s, S_1, \cdots, S_l) \text{ satisfies the condition (II)}}$:

Notation. Let $\mathcal{M} \equiv \mathcal{M}(s, S_1, \cdots, S_l)$ denote the class of $(l+1)$-tuples $(X, X_1, \cdots, X_l)$ which admits a layer preserving closed embedding $h : (X, X_1, \cdots, X_l) \to (s, S_1, \cdots, S_l)$

Theorem 4.4. (T.Yagasaki [32], R.Cauty, et. al.)
Suppose $(Y, Y_1, \cdots, Y_l)$ *satisfies the following conditions:*
(i) Y *is a completely metrizable ANR,*

(ii) *Y_l has the h.n. complement in Y*
(iii) *$(Y, Y_1, \cdots, Y_l)$ is $(s, S_1, \cdots, S_l)$-stable.*
Then $(Y, Y_1, \cdots, Y_l)$ is strongly $\mathcal{M}(s, S_1, \cdots, S_l)$-universal.

From Theorems 4.3, 4.4 we have:

Theorem 4.5. (1) *$(X, X_1, \cdots, X_l)$ is an $(s, S_1, \cdots, S_l)$-manifold iff*

(i) *X is a completely metrizable ANR,*
(ii) *$\underline{(X, X_1, \cdots, X_l) \in \mathcal{M}(s, S_1, \cdots, S_l)}$,*
(iii) *X_l has the h.n. complement in X,*
(iv) *$(X, X_1, \cdots, X_l)$ is $(s, S_1, \cdots, S_l)$-stable.*

(2) *Suppose $(X, X_1, \cdots, X_l)$ and $(Y, Y_1, \cdots, Y_l)$ are $(s, S_1, \cdots, S_l)$-manifolds. Then $(X, X_1, \cdots, X_l) \cong (Y, Y_1, \cdots, Y_l)$ iff $X \simeq Y$.*

4.2.2. *Examples.*

To apply Theorem 4.5 we must distinguish the class $\mathcal{M}(s, S_1, \cdots, S_l)$. This can be done for the triples: (s, Σ, σ), $(s^2, s \times \sigma, \sigma^2)$, $(s^\infty, \sigma^\infty, \sigma_f^\infty)$, and $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$. This leads to the practical characterizations of manifolds modeled on these triples.

[1] (s, Σ, σ):

$\mathcal{M}(s, \Sigma, \sigma) =$ the class of triples (X, X_1, X_2) such that

(a) X is completely metrizable, (b) X_1 is σ-compact, and (c) X_2 is σ-fd-compact.

Theorem 4.6.
(X, X_1, X_2) is an (s, Σ, σ)-manifold iff

(i) *X is a separable completely metrizable ANR,*
(ii) *X_1 is $\underline{\sigma\text{-compact}}$, X_2 is $\underline{\sigma\text{-fd-compact}}$,*
(iii) *X_2 has the h.n. complement in X,*
(iv) *(X, X_1, X_2) is $\underline{(s, \Sigma, \sigma)}$-stable.*

[2] $(s^2, s \times \sigma, \sigma^2)$:

$\mathcal{M}(s^2, s \times \sigma, \sigma^2) =$ the class of triples (X, X_1, X_2) such that

(a) X is completely metrizable, (b) X_1 is F_σ in X, (c) X_2 is σ-fd-compact.

Theorem 4.7.
(X, X_1, X_2) is an $(s^2, s \times \sigma, \sigma^2)$-manifold iff

(i) *X is a separable completely metrizable ANR,*
(ii) *X_1 is an $\underline{F_\sigma}$-subset of X, X_2 is $\underline{\sigma\text{-fd-compact}}$,*
(iii) *X_2 has the h.n. complement in X,*
(iv) *(X, X_1, X_2) is $\underline{(s^2, s \times \sigma, \sigma^2)}$-stable.*

[3] $(s^\infty, \sigma^\infty, \sigma_f^\infty)$:

$\mathcal{M}(s^\infty, \sigma^\infty, \sigma_f^\infty)$ = the class of triples (X, X_1, X_2) such that

(a) X is completely metrizable, (b) X_1 is $F_{\sigma\delta}$ in X, (c) X_2 is σ-fd-compact.

Theorem 4.8.
(X, X_1, X_2) is an $(s^\infty, \sigma^\infty, \sigma_f^\infty)$-manifold iff

(i) *X is a separable completely metrizable ANR,*
(ii) *X_1 is an $\underline{F_{\sigma\delta}}$-subset of X, X_2 is $\underline{\sigma\text{-fd-compact}}$,*
(iii) *X_2 has the h.n. complement in X*
(iv) *(X, X_1, X_2) is $\underline{(s^\infty, \sigma^\infty, \sigma_f^\infty)}$-stable.*

[4] $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$:

$\mathcal{M}(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$ = the class of triples (X, X_1, X_2) such that
(a) X is completely metrizable, (b) X_1 is $F_{\sigma\delta}$ in X, (c) X_2 is σ-compact.

Theorem 4.9.
(X, X_1, X_2) is an $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$-manifold iff

(i) *X is a separable completely metrizable ANR,*
(ii) *X_1 is an $\underline{F_{\sigma\delta}}$-subset of X, X_2 is $\underline{\sigma\text{-compact}}$,*
(iii) *X_2 has the h.n. complement in X,*
(iv) *(X, X_1, X_2) is $\underline{(s^\infty, \Sigma^\infty, \Sigma_f^\infty)}$-stable.*

In the next chapter these characterizations will be applied to determine the local topological types of some triples of homeomorphism groups of manifolds.

Chapter 2. Applications to homeomorphism groups of manifolds

5. Main problems

Notation.
(i) $\mathcal{H}(X)$ denotes the homeomorphism group of a space of X with the compact-open topology.
(ii) When X has a fixed metric, $\mathcal{H}^{\mathrm{LIP}}(X)$ denotes the subgroup of locally LIP-homeomorphisms of X.
(iii) When X is a polyhedron, $\mathcal{H}^{\mathrm{PL}}(X)$ denotes the subgroup of PL-homeomorphisms of X.

We shall consider the following problem:

Problem.
Determine the local and global topological types of groups: $\mathcal{H}(M)$, $\mathcal{H}^{\mathrm{LIP}}(M)$, $\mathcal{H}^{\mathrm{PL}}(M)$, etc. and tuples: $(\mathcal{H}(M), \mathcal{H}^{\mathrm{PL}}(M))$, $(\mathcal{H}(M), \mathcal{H}^{\mathrm{LIP}}(M), \mathcal{H}^{\mathrm{PL}}(M))$, etc.

In the analogy with diffeomorphism groups, when X is a topological manifold, we can expect that these groups are topological manifold modeled on some typical infinite-dimensional spaces. In fact, R.D. Anderson showed that:

Facts 5.1. ([2]) :
(i) $\mathcal{H}_+(\mathbb{R}) \cong s$.
(ii) If G is a finite graph, then $\mathcal{H}(G)$ is an s-manifold.

After this result it was conjectured that

Conjecture. $\mathcal{H}(M)$ is an s-manifold for any compact manifold M.

This basic conjecture is still open for $n \geq 3$ and this imposes a large restriction to our work since most results in Chapter 1 works only when ambient spaces are s-manifolds. Thus in the present situation, in order to obtain some results in dimension $n \geq 3$, we must assume that $\mathcal{H}(M)$ is an s-manifold. On the other hand in dimension 1 or 2 we can obtain concrete results due to the following fact:

Theorem 5.1. (R. Luke - W.K. Mason [22], W. Jakobsche [19])
If X is a 1 or 2-dimensional compact polyhedron, then $\mathcal{H}(X)$ is an s-manifold.

Below we shall follow the next conventions: For a pair (X, A), let $\mathcal{H}(X, A) = \{f \in \mathcal{H}(X) : f(A) = A\}$. When (X, A) is a polyhedral pair, let $\mathcal{H}^{\mathrm{PL}}(X, A) = \mathcal{H}(X, A) \cap \mathcal{H}^{\mathrm{PL}}(X)$ and $\mathcal{H}(X; \mathrm{PL}(A)) = \{f \in \mathcal{H}(X, A) : f \text{ is PL on } A\}$. The superscript "c" denotes "compact supports", the subscript "+" means "orientation preserving", and "0" denotes "the identity connected components" of the corresponding groups. An Euclidean PL-manifold means a PL-manifold which is a subpolyhedron of some Euclidean space $\mathbb{R}^n$ and has the standard metric induced from $\mathbb{R}^n$.

6. Stability properties of homeomorphism groups of polyhedra

First we shall summarize the stability properties of various triples of homeomorphism groups of polyhedra. These properties will be used to determine the corresponding model spaces.

(1) Basic cases: (R. Geoghegan [14, 15], J. Keesling-D. Wilson [20, 21], K. Sakai-R.Y. Wong [26])
(i) If X is a topological manifold, then $\mathcal{H}(X)$ is s-stable.
(ii) If X is a locally compact polyhedron, then the pair $(\mathcal{H}(X), \mathcal{H}^{\mathrm{PL}}(X))$ is (s, σ)-stable.
(iii) If X is a Euclidean polyhedron with the standard metric, then the triple $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{PL}}(X))$ is (s, Σ, σ)-stable.
(iv) (T. Yagasaki [32]) If (X, K) is a locally compact polyhedral pair such that $\dim K \geq 1$ and $\dim(X \setminus K) \geq 1$, then $(\mathcal{H}(X, K), \mathcal{H}(X; \mathrm{PL}(K)), \mathcal{H}^{\mathrm{PL}}(X, K))$ is $(s^2, s \times \sigma, \sigma^2)$-stable.

(2) Noncompact cases: (T. Yagasaki [32])
(i) If X is a noncompact, locally compact polyhedron, then the triple $(\mathcal{H}(X), \mathcal{H}^{\mathrm{PL}}(X), \mathcal{H}^{\mathrm{PL,c}}(X))$ is $(s^\infty, \sigma^\infty, \sigma_f^\infty)$-stable.
(ii) If X is a noncompact Euclidean polyhedron with the standard metric, then the triple $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{LIP,c}}(X))$ is $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$-stable.

We can also consider the spaces of embeddings. Suppose X and Y are Euclidean polyhedra. Let $\mathcal{E}(X,Y)$ denote the spaces of embeddings of X into Y with the compact-open topology, and let $\mathcal{E}^{\mathrm{LIP}}(X,Y)$ and $\mathcal{E}^{\mathrm{PL}}(X,Y)$ denote the subspaces of locally Lipschitz embeddings and PL-embeddings respectively.

(3) Embedding case: (K. Sakai-R.Y. Wong [26], cf. [32])
The triple $(\mathcal{E}(X,Y), \mathcal{E}^{\mathrm{LIP}}(X,Y), \mathcal{E}^{\mathrm{PL}}(X,Y))$ is (s, Σ, σ)-stable.

These stability property are verified by using the Morse length of the image of a fixed segment under the homeomorphisms.

7. The triple $(\mathcal{H}(M),\ \mathcal{H}^{\mathrm{LIP}}(X),\ \mathcal{H}^{\mathrm{PL}}(X))$

[1] $\mathcal{H}(M)$

Suppose M^n is a compact n-dimensional manifold. Since $\mathcal{H}(M)$ is s-stable, by the characterization of s-manifold (Theorem 3.2), $\mathcal{H}(M)$ is an s-manifold iff it is an ANR. Here we face with the difficulty of detecting infinite-dimensional ANR's. A.V. Černavskiĭ [8] and R.D. Edwards - R.C. Kirby [12] have shown:

Theorem 7.1. (Local contractibility): $\mathcal{H}(M)$ *is locally contractible.*

[2] $\mathcal{H}^{\mathrm{PL}}(M)$

Suppose M^n is a compact n-dimensional PL-manifold.

Basic Facts.
(1) (J. Keesling-D. Wilson [21]) $(\mathcal{H}(M), \mathcal{H}^{\mathrm{PL}}(M))$ is (s,σ)-stable.
(2) (D. B. Gauld [13]) $\mathcal{H}^{\mathrm{PL}}(M)$ is locally contractible.
(3) (R. Geoghegan [15]) $\mathcal{H}^{\mathrm{PL}}(M)$ is σ-fd-compact.
(4) (W.E. Haver [17]) A countable dimensional metric space is an ANR iff it is locally contractible.

From (2),(3),(4) it follows that $\mathcal{H}^{\mathrm{PL}}(M)$ is always an ANR. Hence by the characterization of σ-manifold (Theorem 3.5), we have:

Main Theorem. (J. Keesling-D. Wilson [21]) $\mathcal{H}^{\mathrm{PL}}(M)$ *is an σ-manifold.*

Let $\mathcal{H}(M)^* = cl\, \mathcal{H}^{\mathrm{PL}}(M)$. Consider the condition:

$$(*) \quad \underline{n \neq 4 \text{ and } \partial M = \emptyset \text{ for } n = 5.}$$

Under this condition $\mathcal{H}(M)^*$ is the union of some components of $\mathcal{H}(M)$.

Theorem 7.2. (R. Geoghegan, W. E. Haver [16])
If $\mathcal{H}(X)$ is an s-manifold and M satisfies $(*)$, then $(\mathcal{H}(X)^*, \mathcal{H}^{\mathrm{PL}}(X))$ is an (s, σ)-manifold.

[3] $\mathcal{H}^{\mathrm{LIP}}(M)$ (K. Sakai-R.Y. Wong [26])

Suppose M^n is a compact n-dimensional Euclidean PL-manifold.

Basic Facts. ([26])
(1) $(\mathcal{H}(M), \mathcal{H}^{\mathrm{LIP}}(M))$ is (s, Σ)-stable.
(2) $\mathcal{H}^{\mathrm{LIP}}(M)$ is σ-compact.

Theorem 7.3. ([26])
If $\mathcal{H}(X)$ is an s-manifold and M satisfies $(*)$, then $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X))$ is an (s, Σ)-manifold.

[4] The triple $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{LIP}}(M))$ (T. Yagasaki [32])

Suppose M^n is a compact n-dimensional Euclidean PL-manifold.

Basic Facts.
(1) (K. Sakai-R. Y. Wong [26]) $(\mathcal{H}(M), \mathcal{H}^{\mathrm{LIP}}(M), \mathcal{H}^{\mathrm{PL}}(M))$ is (s, Σ, σ)-stable.

Let $\mathcal{H}^{\mathrm{LIP}}(X)^* = \mathcal{H}^{\mathrm{LIP}}(X) \cap cl\mathcal{H}^{\mathrm{PL}}(M)$. From Theorem 7.2, Basic Facts and the characterization of (s, Σ, σ)-manifolds (Theorem 4.6) it follows that:

Theorem 7.4. ([32])
(1) *If $\mathcal{H}(X)$ is an s-manifold and M satisfies $(*)$, then $(\mathcal{H}(X)^*, \mathcal{H}^{\mathrm{LIP}}(X)^*, \mathcal{H}^{\mathrm{PL}}(X))$ is an (s, Σ, σ)-manifold.*
(2) *If X is a 1 or 2-dimensional compact Euclidean polyhedron with the standard metric, then $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{PL}}(X))$ is (s, Σ, σ)-manifold.*

8. Other triples

[1] The triple $(\mathcal{H}(X, K), \mathcal{H}(X; \mathrm{PL}(K)), \mathcal{H}^{\mathrm{PL}}(X, K))$ (T. Yagasaki [32])

Theorem 8.1.
(i) *Suppose M^n is a compact PL n-manifold with $\partial M \neq \emptyset$. If $n \geq 2$, $n \neq 4, 5$ and $\mathcal{H}(M)$ is an s-manifold, then $(\mathcal{H}(M), \mathcal{H}(M; \mathrm{PL}(\partial M)), \mathcal{H}^{\mathrm{PL}}(M))$ is an $(s^2, s \times \sigma, \sigma^2)$-manifold.*
(ii) *Suppose (X, K) is a compact polyhedral pair such that $\dim X = 1, 2$, $\dim K \geq 1$ and $\dim (X \setminus K) \geq 1$. Then $(\mathcal{H}(X, K), \mathcal{H}(X; \mathrm{PL}(K)), \mathcal{H}^{\mathrm{PL}}(X, K))$ is an $(s^2, s \times \sigma, \sigma^2)$-manifold.*

[2] The triples $(\mathcal{H}(X), \mathcal{H}^{\mathrm{PL}}(X), \mathcal{H}^{\mathrm{PL,c}}(X))$ and $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{LIP,c}}(X))$ (T. Yagasaki [33])

(1) 1-dim. case: $(\mathcal{H}_+(\mathbb{R}), \mathcal{H}^{\mathrm{PL}}_+(\mathbb{R}), \mathcal{H}^{\mathrm{PL,c}}(\mathbb{R})) \cong (s^\infty, \sigma^\infty, \sigma^\infty_f)$.

(2) 2-dim. case:

Theorem 8.2. *If M is a noncompact connected PL 2-manifold, then $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{PL}}(M)_0, \mathcal{H}^{\mathrm{PL,c}}(M)_0)$ is an $(s^\infty, \sigma^\infty, \sigma_f^\infty)$-manifold.*

Corollary 8.1.
(i) *If $M \cong \mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}, \mathbb{S}^1 \times [0,1), \mathbb{P}^2\mathbb{R} \setminus 1pt$, then $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{PL}}(M)_0, \mathcal{H}^{\mathrm{PL,c}}(M)_0) \cong \mathbb{S}^1 \times (s^\infty, \sigma^\infty, \sigma_f^\infty)$*
(ii) *In the remaining cases, $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{PL}}(M)_0, \mathcal{H}^{\mathrm{PL,c}}(M)_0) \cong (s^\infty, \sigma^\infty, \sigma_f^\infty)$*

(3) There exist a (LIP, Σ)-version of the (PL, σ)-case.

[3] The group of quasiconformal (QC-)homeomorphisms of a Riemann surface (T. Yagasaki [34])

Suppose M is a connected Riemann surface. Let $\mathcal{H}^{\mathrm{QC}}(M)$ denote the subgroup of QC-homeomorphisms of M.

Theorem 8.3.
(i) *If M is compact, then $(\mathcal{H}_+(M), \mathcal{H}^{\mathrm{QC}}(M))$ is an (s, Σ)-manifold.*
(ii) *If M is noncompact, then $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0)$ is an (s, Σ)-manifold*

[4] The space of embeddings (T. Yagasaki [33])

Suppose M is a Euclidean PL 2-manifold.

Theorem 8.4. *If X is a compact subpolyhedron of M, then $(\mathcal{E}(X,M), \mathcal{E}^{\mathrm{LIP}}(X,M), \mathcal{E}^{\mathrm{PL}}(X,M))$ is an (s, Σ, σ)-manifold.*

Example. The case $X = I \equiv [0,1]$:

$$(\mathcal{E}(I,M), \mathcal{E}^{\mathrm{LIP}}(I,M), \mathcal{E}^{\mathrm{PL}}(I,M)) \cong S(TM) \times (s, \Sigma, \sigma)$$

where $S(TM)$ is the sphere bundle of the tangent bundle of M with respect to some Riemannian metric.

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Department of Mathematics, Kyoto Institute of Technology, Matsugasaki, Sakyo-ku, Kyoto 606, Japan
E-mail address: yagasaki@ipc.kit.ac.jp