POINCARÉ SHEAVES ON TOPOLOGICAL SPACES

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ABSTRACT. We define the notion of a Poincaré complex of sheaves over a topological space. Global surgery invariants of spaces and maps are expressed as the assembly of locally defined surgery invariants. For simplicial complexes we recover the theory of Poincaré cycles of Ranicki. For more general spaces we obtain local surgery invariants in the absence of triangulations.

1. Introduction.

The applications of surgery theory to areas such as controlled topology, homology manifolds and stratified spaces suggest the need for local surgery invariants. Such a local theory was constructed in the simplicial category by Ranicki [10]. In this paper we develop an analogous theory in the topological category of locally compact topological spaces using the methods of sheaf theory.

The main objects of study are complexes of sheaves with a Poincaré duality, that is, Poincaré sheaves. For example, the singular cochain sheaf of a manifold admits a natural Poincaré duality. Taking global sections recovers the chain level cup-product as well as the higher relations determining the Steenrod squares. The advantage of having a sheaf is that we may also take local sections over an open subset, thus obtaining local invariants. Typically we might consider sections over an open neighbourhood of a submanifold thus obtaining invariants of the germ neighbourhood of the submanifold, (cf. [2]). Another important example is the intersection cohomology sheaf of a pseudo-manifold which (at least at the prime 2) admits a natural Poincaré duality. This was used in [4] to introduce Steenrod operations in the intersection cohomology groups.

We show other manifold invariants may be sheafified. For example, if $f : M' \to M$ is a (proper) normal map of manifolds then there is determined a natural Poincaré sheaf on M. (Unlike the above examples whose underlying sheaves were built from singular chains this sheaf is built from the the Godement canonical sheaf of the associated spaces. This sheaf has better homological properties as well as admitting a canonical symmetric structure rather than just a canonical class as in the singular theory.) Taking global sections of this sheaf recovers the surgery obstruction of f. Taking sections over open subsets determines local surgery invariants. In the case that f is a homotopy equivalence the sheaf carries the splitting invariants of submanifolds. Similarly there is a sheaf theoretic version of the total

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surgery obstruction for a compact ANR Poincaré space. Together with the splitting results below we recover the Quinn invariant of a homology manifold. We shall explore applications of this approach in a further paper concerning exotic homology manifolds.

Poincaré sheaves have good transversality properties. Any Poincaré sheaf over $X \times \mathbb{R}$ is cobordant (over an arbitrarily small neighbourhood of X) to a Poincaré sheaf that splits over X. It follows that the cobordism groups of Poincaré sheaves form a homology theory. For a locally finite simplicial complex we recover the L-homology groups of Ranicki [10]. For an arbitrary space we obtain a topologically invariant homology theory. This leads to a quick proof of the topological invariance of the rational Pontrjagin classes of a PL-manifold, without using the Bass-Heller-Swan calculation $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}^i]) = 0$, $i \geq 0$.

Using sheaves gives direct construction of invariants which previously required triangulation for their definition. For example, the Cappell-Weinberger geometric construction of Siebenmann periodicity may be performed in the topological category and described algebraically using associated sheaves, [5]

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2. Perfect Sheaves and Weak Chain Duality.

Let R be a commutative Noetherian ring with unit of finite cohomological dimension and X a locally compact space satisfying $\dim_R X < \infty$ (this is cohomological dimension with respect to compact supports [1;II,15]). Let $Sh(X)^{\cdot}$ denote the category of perfect complexes of sheaves on X, [11]. There is a dualizing complex of sheaves $D_X^{\cdot}$ on X inducing a weak chain duality on $Sh(X)^{\cdot}$. We recall the construction of $D_X^{\cdot}$ in this section.

Let $\mathcal{C}_X^{\cdot}$ denote the canonical Godement resolution of the constant sheaf R_X on X [1; II.2]

$$0 \to R_X \xrightarrow{\epsilon} \mathcal{C}_X^0 \xrightarrow{d} \mathcal{C}_X^1 \to \cdots \to \mathcal{C}_X^n \to \cdots$$

which is a resolution of R_X by flabby sheaves. By definition $\mathcal{C}_X^0(U)$ is the R-module of serrations (ie. not necessarily continuous sections) of R_X over the open set $U \subset X$. In other words

$$\mathcal{C}_X^0(U) = \prod_{x \in U} (R_X)_x = \prod_{x \in U} R.$$

Then $R_X \subset \mathcal{C}_X^0$ as the subsheaf of germs of sections. To construct $\mathcal{C}_X^1$ form the short exact sequence

$$0 \to R_X \to \mathcal{C}_X^0 \to \mathcal{L}_X^1 \to 0.$$

Then $\mathcal{C}_X^1 = \mathcal{C}^0(\mathcal{L}_X^1)$ is the sheaf of germs of serrations of $\mathcal{L}_X^1$. Continuing gives the canonical resolution.

The canonical resolution is naturally pointwise contractible to R_X. For $x \in X$ define maps

$$\eta_x : (\mathcal{C}_X^i)_x \to (\mathcal{L}_X^i)_x \,;\, s \mapsto s(x),$$

evaluating the germ of a serration at x, and

$$\nu_x : (\mathcal{L}_X^{i+1})_x \to (\mathcal{C}_X^i)_x$$

by the relation $\nu_x \cdot d = 1 - \epsilon \cdot \eta_x$. Then

$$D_x = \nu_x \cdot \eta_x : (\mathcal{C}_X^i)_x \to (\mathcal{C}_X^{i-1})_x, \quad i \geq 1,$$

is a pointwise contraction of $\mathcal{C}_X^{\cdot}$. The contraction D allows the construction of a natural product on $\mathcal{C}_X^{\cdot}$ extending that on R_X and which is commutative up to higher homotopies, ie. there is a system of maps

$$h_s : \mathcal{C}_X^{\cdot} \otimes \mathcal{C}_X^{\cdot} \longrightarrow \mathcal{C}_X^{\cdot};\ s \geq 0,\ \deg(h_s) = -s,$$

satisfying the relations

$$h_s + (-)^{s+1} h_s \cdot T - h_{s+1} \cdot d + (-)^s d \cdot h_{s+1} = 0.$$

(See [1;II.21] for details.)

Let ϕ be a family of supports on X [1;1.6] and Γ_ϕ the functor taking a sheaf A to the R-module $\Gamma_\phi A$ of sections of A on X with support in ϕ. For the family of compact supports we write $\phi = c$. If no support is specified then it is understood to be the family of closed supports. The sheaf cohomology of X with constant R-coefficients and support in ϕ is given by

$$H_\phi^i(X; A) = H^i(\Gamma_\phi(\mathcal{C}_X^{\cdot})), \quad i \geq 0.$$

The system of maps h above determines cup-products and Steenrod squares on the sheaf cohomology.

Remark There is a natural homomorphism

$$H_\phi^*(X; R) \longrightarrow {}_\Delta H_\phi^*(X; R)$$

from singular cohomology to sheaf cohomology, which in the case X is HLC is an isomorphism. (X is HLC if for each $x \in X$ and open $V \ni x$ there is an open $U \ni x$ contained in V such that the homomorphism of singular homology groups ${}_\Delta\tilde{H}_*(U) \to {}_\Delta\tilde{H}_*(V)$ is zero. This is equivalent to the singular cochain sheaf $\mathcal{S}_X^{\cdot}$ being a resolution of R_X, [1;III.1])

Let $R \to I^{\cdot}$ be a (fixed) finite resolution of R by injective R-modules. Define a presheaf $D_X^{\cdot}$ on X by

$$D_X^{\cdot}(U) = \mathrm{Hom}^{\cdot}(\Gamma_c(\mathcal{C}_X^{\cdot}), I^{\cdot}), \quad U \subset X \text{ open}.$$

Then $D_X^{\cdot}$ is a complex of injective sheaves on X, called the dualizing complex of X (over the ring R) [11], [1;V.3],[6;III]. In the case that $\dim_R X = n < \infty$ the canonical resolution $\mathcal{C}_X^{\cdot}$ may be replaced by the truncated c-soft resolution

$$0 \to R_X \to \mathcal{C}_X^0 \to \mathcal{C}_X^1 \to \cdots \to \mathcal{C}_X^{n-1} \to \mathcal{L}_X^n \to 0$$

in the construction of $D_X^{\cdot}$, where $\mathcal{L}_X^n = \ker(\mathcal{C}_X^n \to \mathcal{C}_X^{n+1})$. For such spaces we may then assume $D_X^{\cdot}$ is a bounded complex. The Borel-Moore homology groups of X with constant R-coefficients and support in ϕ are given by

$$H_i^{\phi}(X;R) = H^{-i}(\Gamma_{\phi} D_X^{\cdot}).$$

For X HLC the Borel-Moore homology of X is naturally isomorphic to the locally finite singular homology. Let $U \subset X$ be open then clearly $(D_X^{\cdot})_{|U} = D_U^{\cdot}$ so that restriction determines homomorphisms

$$H_*^{\phi}(X;R) \to H_*^{\phi \cap U}(U;R),$$

where $\phi \cap U$ is the family $\{K \cap U \mid K \in \rho\}$. The graded derived sheaf $\mathcal{H}D_X^{\cdot}$ is called the sheaf of local homology groups and is generated by the presheaf $U \mapsto H_*(U;R)$.

Now suppose $A^{\cdot} \in \mathrm{Sh}(X)^{\cdot}$. Define the dual sheaf $DA^{\cdot}$ to be the (perfect) complex of sheaves

$$DA^{\cdot} = Hom^{\cdot}(A^{\cdot}, D_X^{\cdot}).$$

There is a natural transformation

$$e(A^{\cdot}) : A^{\cdot} \longrightarrow D^2 A^{\cdot} = Hom^{\cdot}(Hom^{\cdot}(A^{\cdot}, D_X^{\cdot}), D_X^{\cdot})$$

which for perfect sheaves is a quasi-isomorphism [11] and which satisfies

$$D(e(A^{\cdot})) \cdot e(DA^{\cdot}) = 1_{DA^{\cdot}} : DA^{\cdot} \to D^3(A^{\cdot}) \to DA^{\cdot}.$$

Thus we have (see appendix)

Proposition 2.1. *The triple $(Sh(X)^{\cdot}, D, e)$ is a category with weak chain duality.*

This means that the notion of Poincaré duality (as in [10; 1.6]) may be applied to perfect complexes of sheaves of R-modules on X. We will without further mention assume all complexes of sheaves to be perfect. In particular we will only consider spaces whose canonical sheaf $\mathcal{C}_X^{\cdot}$ is perfect. Even though the definition of a Poincaré complex of sheaves on X follows formally from [10], for the sake of completeness we give it here.

Definition 2.2. *An n-dimensional symmetric $\{$quadratic$\}$ Poincaré complex of sheaves $(A^{\cdot}, \phi)$ $\{(A^{\cdot}, \psi)\}$ on X consists of a complex of sheaves $A^{\cdot}$ on X together with a collection of maps ϕ $\{\psi\}$*

$$\begin{aligned} \phi^s :& A^{\cdot} \to DA^{\cdot}, \quad s \geq 0, \quad \deg(\phi^s) = n - s, \\ \{ \quad \psi^s :& A^{\cdot} \to DA^{\cdot}, \quad s \geq 0, \quad \deg(\psi^s) = n + s \quad \} \end{aligned}$$

satisfying the relations

$$\begin{gathered} d\phi^s + (-)^{n+s+1}\phi^s d + (-)^{n+1}\phi^{s-1} + (-)^{n+s} T\phi^{s-1} = 0, \quad s \geq 0,\ \phi^{-1} = 0, \\ \{ \quad d\psi^s + (-)^{n+1}\psi^s d + (-)^s \psi^{s+1} - T\psi^{s+1} = 0, \quad s \geq 0 \quad \} \end{gathered}$$

and such that the chain map

$$\phi^0 : A^{\cdot} \to \Sigma^n DA^{\cdot}$$
$$\{ \quad (1+T)\,\psi^0 : A^{\cdot} \to \Sigma^n DA^{\cdot} \quad \}$$

is a quasi-isomorphism (ie. the induced map of derived sheaves is an isomorphism.)

The relations are equivalent to the requirement that ϕ $\{\psi\}$ is a $(-n)$-cycle in

$$W^{\%}A^{\cdot} = \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(A^{\cdot}, DA^{\cdot}) = \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(A^{\cdot} \otimes A^{\cdot}, D^{\cdot}_X))$$
$$\{ \quad W_{\%}A^{\cdot} = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}^{\cdot}(A^{\cdot}, DA^{\cdot}) = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}^{\cdot}(A^{\cdot} \otimes A^{\cdot}, D^{\cdot}_X) \quad \}.$$

The definition is simply that of a Poincaré complex in the category with weak chain duality $\mathrm{Sh}(X)^{\cdot}$. If the quasi-isomorphism condition is dropped then we have merely a symmetric {quadratic} complex of sheaves on X. (See appendix for details.) For $U \subset X$ open there are restriction maps

$$W^{\%}A^{\cdot} \to W^{\%}A^{\cdot}_{|U},$$
$$\{ \quad W_{\%}A^{\cdot} \to W_{\%}A^{\cdot}_{|U}. \quad \}$$

The chain map $\phi^0 : A^{\cdot} \to \Sigma^n DA^{\cdot}$ induces an isomorphism on homology

$$\phi^0_* : H^*(X; A^{\cdot}) \longrightarrow \Sigma^n \mathrm{Ext}^{\cdot}(\Gamma_c A^{\cdot}, R),$$

(where we assume $A^{\cdot}$ is c-soft or else resolve.) For example, for R a field $A^{\cdot}$ admits a Poincaré duality isomorphism

$$\phi^0_* : H^i(X; A^{\cdot}) \longrightarrow \mathrm{Hom}(H^{n-i}_c(X; A^{\cdot}), R).$$

Let $f : X \to Y$ and $A^{\cdot}$, $B^{\cdot}$ be sheaves on X, Y respectively. Let $f^*B^{\cdot}$ be the pullback sheaf on X and $f_*A^{\cdot}$ be the direct image sheaf on Y. (Sometimes we write $fA^{\cdot}$ for the direct image.) Then f^* and f_* are adjoint functors. In particular the identity $f^*R_Y = R_X$ has adjoint a sheaf map $R_Y \to f_*R_X$ which extends to a map $h(f) : \mathcal{C}^{\cdot}_Y \to f_*\mathcal{C}^{\cdot}_X$, inducing a homomorphism

$$f^* : H^*(Y; R) \to H^*(X; R).$$

If in addition f is proper there is defined in an obvious way a chain map $Dh(f) : f_*D^{\cdot}_X \to D^{\cdot}_Y$ of complexes of sheaves on Y. Given an an n-dimensional symmetric complex $(A^{\cdot}, \phi)$ on X define the direct image of $(A^{\cdot}, \phi)$ under the (proper) map f to be the n-dimensional symmetric complex $(f_*A^{\cdot}, f_*\phi)$ on Y with $f_*\phi$ given by

$$(f_*\phi)^s : f_*A^{\cdot} \otimes f_*A^{\cdot} \xrightarrow{f_*\phi^s} \Sigma^n f_*D^{\cdot}_X \xrightarrow{Dh(f)} \Sigma^n D^{\cdot}_Y, \quad s \geq 0,$$

or adjointly

$$(f_*\phi)^s : f_*A^{\cdot} \xrightarrow{f_*\phi^s} f_*Hom^{\cdot}(A^{\cdot}, D^{\cdot}_X) \xrightarrow{F} Hom^{\cdot}(f_*A^{\cdot}, f_*D^{\cdot}_X) \xrightarrow{Dh(f)_*} Hom^{\cdot}(f_*A^{\cdot}, D^{\cdot}_Y).$$

where F is the natural transformation. Similarly for quadratic.

Suppose $A^{\cdot}$ is a complex of c-soft sheaves (or else resolve by such a resolution).

Lemma 2.3. *If $(A^\cdot, \phi)$ is an n-dimensional symmetric Poincaré complex on X then the direct image $(f_* A^\cdot, f_*\phi)$ under the proper map $f : X \to Y$ is an n-dimensional symmetric Poincaré complex on Y. Similarly for quadratic.*

Proof. The Poincaré-Verdier duality theorem [12],[6;3.1.10] may be expressed as a quasi-isomorphism of complexes of sheaves on Y

$$\mathrm{PV} : f_* Hom^\cdot(A^\cdot, f^! D^\cdot_Y) \longrightarrow\ Hom^\cdot(f_! A^\cdot, D^\cdot_Y)$$

where $f_!$ is the direct image with proper support functor [6;2.5] and $f^!$ is its right adjoint. Since f is assumed proper the inclusion $f_! \to f_*$ is the identity. The map PV factors as

$$f_* Hom^\cdot(A^\cdot, f^! D^\cdot_Y) \to f_* Hom^\cdot(A^\cdot, D^\cdot_X) \xrightarrow{\mathrm{F}}\ Hom^\cdot(f_* A^\cdot, f_* D^\cdot_{)} \stackrel{Dh(f)_*}{\to}\ Hom^\cdot(f_* A^\cdot, D^\cdot_Y),$$

where the first map is induced by the isomorphism $f^! D^\cdot_Y = D^\cdot_X$, [6;3.1.16]. Thus $Dh(f)_* \cdot F$ is a quasi-isomorphism. But $\phi^0 : A^\cdot \to Hom^\cdot(A^\cdot, D^\cdot_X)$ is a map of c-soft sheaves so that $f_* A^\cdot \to f_* Hom^\cdot(A^\cdot, D^\cdot_X)$ is a quasi-isomorphism and the lemma follows. ∎

In the case $f : X \to Y$ is not proper an n-dimensional symmetric complex of sheaves $(f_! A^\cdot f_! \phi)$ on Y may still be constructed. Here $f_! A^\cdot$ is the direct image of $A^\cdot$ with proper support and

$$(f_!\phi)^s : f_! A^\cdot \otimes f_! A^\cdot \stackrel{f_!\phi^s}{\to} \Sigma^n f_! D^\cdot_X \to \Sigma^n D^\cdot_Y$$

where $f_! D^\cdot_X \to D^\cdot_Y$ is the adjoint of $f^! D^\cdot_Y = D^\cdot_X$. However it will not in general be Poincaré.

Definition 2.4. For a space X define the symmetric {quadratic} L-groups (with coefficients in R) of X, $L^n(X;R)$ $\{L_n(X;R)\}$, to be the abelian group of cobordism classes of n-dimensional symmetric {quadratic} Poincaré complexes of (perfect) sheaves of R-modules on X. Then $L^n(-;R)$ $\{L_n(-;R)\}$ is a covariant functor on the category of locally compact spaces and proper maps taking values in the category of abelian groups.

The simply-connected assembly of an n-dimensional symmetric complex $(A^\cdot, \phi)$ on a compact space X is defined to be the n-dimensional symmetric complex of R-modules $(p_* A^\cdot, p_*\phi)$ where $p : X \to \mathrm{pt}$ maps X to a point. Similarly for quadratic. There are then defined simply-connected assembly functors

$$\begin{array}{c} L^n(X;R) \longrightarrow L^n_{\mathrm{p}}(R) \\ \{\quad L_n(X;R) \longrightarrow L^{\mathrm{p}}_n(R) \quad\} \end{array}$$

from the L-groups of X to the projective L-groups of R. Universal assembly (taking into account the fundamental group of X) is defined as follows. Let X be a compact space and $(A^\cdot, \phi)$ an n-dimensional symmetric Poincaré complex of sheaves on X.

Let $p : \tilde{X} \to X$ be the universal cover of X. Since p is a local homeomorphism the complex of sheaves $\tilde{A}^{\cdot} = p^* A^{\cdot}$ on $\tilde{X}$ admits the structure of an n-dimensional symmetric Poincaré complex of sheaves $(\tilde{A}^{\cdot}, \tilde{\phi})$, that is

$$(\tilde{\phi})^s : \tilde{A}^{\cdot} \otimes \tilde{A}^{\cdot} \overset{p^*\phi^s}{\to} \Sigma^n p^* D_X^{\cdot} = \Sigma^n D_{\tilde{X}}^{\cdot}, \quad s \geq 0,$$

or adjointly

$$\tilde{\phi}^s : \tilde{A}^{\cdot} \longrightarrow p^* DA^{\cdot} = Dp^* A^{\cdot} = D\tilde{A}^{\cdot}, \quad s \geq 0.$$

Applying the functor Γ_c we obtain maps

$$\Gamma_c(\tilde{\phi}^s) : \Gamma_c \tilde{A}^{\cdot} \longrightarrow \Gamma_c D\tilde{A}^{\cdot}, \quad s \geq 0,$$

with $\Gamma_c(\tilde{\phi}^0)$ a quasi-isomorphism (as usual assuming $A^{\cdot}$ c-soft). By [11] $\Gamma_c(\tilde{A}^{\cdot})$ is a perfect complex of $R[\pi]$-modules. Furthermore by definition $\Gamma_c(D\tilde{A}^{\cdot})$ is the subcomplex of $D\tilde{A}^{\cdot}(\tilde{X}) = \mathrm{Hom}_R^{\cdot}(\Gamma_c \tilde{A}^{\cdot}, I^{\cdot})$ consisting of maps with compact support. The support of a map $f : \Gamma_c \tilde{A}^i \to I^j$ is the smallest closed subset $K \subset \tilde{X}$ such that $f(s) = 0$ for all $s \in \Gamma_c \tilde{A}^i$ satisfying $|s| \cap K = \emptyset$. Since X is compact this is equivalent to the requirement that for each $s \in \Gamma_C \tilde{A}^i$ there are at most a finite number of $g \in \pi$ such that $f(gs) \neq 0$. The map

$$\Gamma_c D\tilde{A}^{\cdot} \longrightarrow \mathrm{Hom}_{R[\pi]}^{\cdot}(\Gamma_c \tilde{A}^{\cdot}, I^{\cdot}[\pi]) \, ; \, f \mapsto \{f' : s \mapsto \Sigma f(gs) g^{-1}\}$$

is then an isomorphism. Thus the $R[\pi]$-module chain complex $\Gamma_c \tilde{A}^{\cdot}$ admits a system of maps

$$\Gamma_c(\tilde{\phi}^s) : \Gamma_c \tilde{A}^{\cdot} \longrightarrow \mathrm{Hom}_{R[\pi]}^{\cdot}(\Gamma_c \tilde{A}^{\cdot}, I^{\cdot}[\pi])$$

such that $\Gamma_c(\tilde{\phi}^0)$ is a quasi-isomorphism. Since $\Gamma_c \tilde{A}^{\cdot}$ is a perfect complex this determines an n-dimensional symmetric Poincaré complex $\mathrm{As}(A^{\cdot}, \phi)$ in the category of bounded chain complexes of projective R-modules of finite rank. In particular there is defined an assembly functor

$$\mathrm{As} : L^n(X; R) \to L^n_{\mathrm{p}}(R[\pi_1 X]).$$

Similarly for quadratic. Writing $S_{n+1}(X; R)$ for the cobordism group of n-dimensional quadratic Poincaré complexes of sheaves on X with a null-cobordant assembly we obtain the (sheaf theoretic) algebraic surgery long exact sequence (cf. [10;14.6])

$$\cdots \to L_{n+1}(R[\pi_1 X]) \to S_{n+1}(X; R) \to L_n(X; R) \overset{\mathrm{As}}{\to} L_n(R[\pi_1 X]) \to \cdots .$$

3. Signatures of Spaces and Maps

Let $Q^n(\mathcal{C}_X^{\cdot})$ $\{Q_n(\mathcal{C}_X^{\cdot})\}$ denote the Q-group of equivalence classes of n-dimensional symmetric structures on $\mathcal{C}_X^{\cdot}$, ie.

$$\begin{aligned} Q^n(\mathcal{C}_X^{\cdot}) &= H_{-n}(W^{\%}\mathcal{C}_X^{\cdot}) \\ \{ \quad Q_n(\mathcal{C}_X^{\cdot}) &= H_{-n}(W_{\%}\mathcal{C}_X^{\cdot}) \quad \}. \end{aligned}$$

A proper map induces a homomorphism of Q-groups.

Proposition 3.1. *Let X be a space and $\mathcal{C}^{\cdot}_X$ the canonical resolution of the constant sheaf of R-modules R_X over X. There is defined a symmetric signature homomorphism*

$$\sigma^* : H_n(X;R) \longrightarrow Q^n(\mathcal{C}^{\cdot}_X),$$

commuting with restriction to open sets and with assembly the symmetric signature of [9;II,1.2]. *Furthermore a proper map $f : X \to Y$ determines a commutative diagram*

$$\begin{array}{ccc} H_n(X;R) & \xrightarrow{\sigma^*} & Q^n(\mathcal{C}^{\cdot}_X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H_n(Y;R) & \xrightarrow{\sigma^*} & Q^n(\mathcal{C}^{\cdot}_Y) \end{array}$$

Proof. The system of maps $\{h^s : \mathcal{C}^{\cdot}_X \otimes \mathcal{C}^{\cdot}_X \to \mathcal{C}^{\cdot}_X\}_{s\geq 0}$ may be regarded as a 0-cocycle

$$\Phi_X \in \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W^{\cdot}, \mathrm{Hom}^{\cdot}(\mathcal{C}^{\cdot}_X \otimes \mathcal{C}^{\cdot}_X, \mathcal{C}^{\cdot}_X)).$$

The resolution $\epsilon : R_X \to \mathcal{C}^{\cdot}_X$ determines a quasi-isomorphism of injective sheaves

$$\mathit{Hom}^{\cdot}(\mathcal{C}^{\cdot}_X, D^{\cdot}_X) \longrightarrow \mathit{Hom}^{\cdot}(R_X, D^{\cdot}_X) = D^{\cdot}_X$$

so that any homology class $\alpha \in H_n(X;R)$ may be represented by a section of $\mathit{Hom}^{\cdot}(\mathcal{C}^{\cdot}_X, D^{\cdot}_X)$, ie. a chain map $\alpha : \mathcal{C}^{\cdot}_X \to \Sigma^n D^{\cdot}_X$. Define an n-dimensional symmetric complex $(\mathcal{C}^{\cdot}_X, \phi)$ by

$$\phi^s_\alpha = \alpha_* \cdot h^s \in \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(\mathcal{C}^{\cdot}_X \otimes \mathcal{C}^{\cdot}_X, D^{\cdot}_X)).$$

The symmetric signature map is defined by

$$H_n(X;R) \to Q^n(\mathcal{C}^{\cdot}_X)\,;\, \alpha \mapsto \phi_\alpha.$$

For a proper map $f : X \to Y$ the commutative diagram in the statement of the theorem follows from the naturality of the 0-cocycles Φ_X and Φ_Y. ■

Let $\mathcal{S}^{\cdot}_X$ denote the sheaf of singular cochains on X with constant coefficients in R, [1;I.7,III.1]. If X is HLC then $\mathcal{S}^{\cdot}_X$ is a resolution of the constant sheaf R_X and $\mathcal{S}^{\cdot}_X(X)$ is the singular cochain complex of X. Acyclic models provides a system of maps $\{h^s : \mathcal{S}^{\cdot}_X(X) \otimes \mathcal{S}^{\cdot}_X(X) \to \mathcal{S}^{\cdot}_X(X)\}_{s\geq 0}$ extending the product on R_X (with higher homotopies). To construct such a system on $\mathcal{S}^{\cdot}_X$ it is necessary to obtain compatible systems for each open $U \subset X$. Rather than doing this one may proceed more simply using homological algebra to build a system

$$g^s : \mathcal{S}^{\cdot}_X \otimes \mathcal{S}^{\cdot}_X \to \mathcal{I}^{\cdot}_X, \quad s \geq o,$$

with $\mathcal{I}^{\cdot}_X$ an injective resolution of R_X, g^0 extending the product on R_X and g^s, $s \geq 1$, the usual higher homotopies. This is possible because $\mathcal{S}^{\cdot}_X \otimes \mathcal{S}^{\cdot}_X$ is a resolution of $R_X \otimes R_X$. Note that any two such systems are homotopic. Thus we may construct a symmetric signature map

$${}_{\triangle}H_n(X;R) \cong H_n(X;R) \longrightarrow Q^n(\mathcal{S}^{\cdot}_X).$$

Similar remarks apply to the sheaf of Alexander-Spanier cochains on $\mathcal{A}^{\cdot}_X$ on X. The sheaves $\mathcal{C}^{\cdot}_X$, $\mathcal{S}^{\cdot}_X$ and $\mathcal{A}^{\cdot}_X$ are naturally quasi-isomorphic and all symmetric signature maps may be identified.

To relate these sheaf theoretic symmetric signatures to the chain level symmetric signature [9;II,1.2] we note that $\mathrm{Hom}^{\cdot}(\Gamma_c\mathcal{S}^{\cdot}_X, R) = C^{\mathrm{lf}}(X)$ is the chain complex of locally finite singular simplices on X. Since $\Gamma_c\mathcal{S}^{\cdot}_X$ is a free R-module there is a quasi-isomorphism $\alpha : C^{\mathrm{lf}}(X) \to \mathrm{Hom}^{\cdot}(\Gamma_c\mathcal{S}^{\cdot}_X, I^{\cdot}) = C^{\mathrm{lf}}(X)'$. Let $\mathcal{S}^{\cdot}_X \to \mathcal{I}^{\cdot}_X$ be a quasi-isomorphism, where $\mathcal{I}^{\cdot}_X$ is some resolution of R_X by injective sheaves. Both sheaves are Γ_c-injective so that there is determined a quasi-isomorphism

$$\mathrm{Hom}^{\cdot}(\Gamma_c\mathcal{I}^{\cdot}_X, I^{\cdot}) \longrightarrow \mathrm{Hom}^{\cdot}(\Gamma_c\mathcal{S}^{\cdot}_X, I^{\cdot}).$$

Since $C^{\mathrm{lf}}(X)'$ is a complex of injective R-modules the chain map

$$\mathrm{Hom}^{\cdot}(\Gamma_c(g), I^{\cdot}) : W \otimes \mathrm{Hom}^{\cdot}(\Gamma_c\mathcal{I}^{\cdot}_X, I^{\cdot}) \longrightarrow C^{\mathrm{lf}}(X)' \otimes C^{\mathrm{lf}}(X)'$$

induced by the system g lifts up to homotopy to a chain map

$$\Psi'_X : W \otimes C^{\mathrm{lf}}(X)' \longrightarrow C^{\mathrm{lf}}(X)' \otimes C^{\mathrm{lf}}(X)'.$$

Let

$$\Psi_X : W \otimes C^{\mathrm{lf}}(X) \longrightarrow C^{\mathrm{lf}}(X) \otimes C^{\mathrm{lf}}(X)$$

be obtained by the application of acyclic models. Then, again by acyclic models, there is a homotopy

$$(\alpha \otimes \alpha) \cdot \Psi_X \simeq \Psi'_X \cdot (1 \otimes \alpha).$$

Thus we obtain a commutative diagram

$$\begin{array}{ccc}
{}_{\Delta}H_n(X;R) & \xrightarrow{\sigma^*} & Q^n(\mathcal{S}^{\cdot}_X) \\
\cong\big\downarrow & & \big\downarrow \\
H_n(X;R) & \xrightarrow{\sigma^*} & Q^n(C^{\mathrm{lf}}(X))
\end{array}$$

where the lower horizontal map is the chain level simply-connected symmetric signature of [9] taking values in the dual Q-groups. Similar considerations apply to the nonsimply-connected signature. ■

For a complex of sheaves $A^{\cdot}$ let $\mathcal{H}(A^{\cdot})$ denote the derived sheaf. Write $\mathcal{H}_i(A^{\cdot}) = \mathcal{H}^{-i}(A^{\cdot})$.

Example 3.2 An orientable R-homology manifold is a finite dimensional space X such that the derived sheaf $\mathcal{H}(D^{\cdot}_X)$ satisfies

$$\begin{aligned} \mathcal{H}_q(D^{\cdot}_X) &= 0, \quad q \neq n \\ \mathcal{H}_n(D^{\cdot}_X) &\cong R_X. \end{aligned}$$

It follows for each open subset $U \subset X$ the sheaf cohomology spectral sequence [1;1V.2]

$$E_2^{p,q} \cong H^p(U; \mathcal{H}^q D^{\cdot}_U) \implies H^*(U; D^{\cdot}_U),$$

collapses to an isomorphism

$$\omega_U : H^0(U;R) \cong H_n(U;R).$$

The isomorphisms are compatible with restriction so that for each $x \in X$ we have the commutative diagram

$$\begin{array}{ccc}
H^0(X;R) & \longrightarrow & (\mathcal{H}^0 R_X)_x \\
{\scriptstyle\gamma_X}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_x} \\
H_n(X;R) & \longrightarrow & (\mathcal{H}_n D^{\cdot}_X)_x
\end{array}$$

where the horizontal maps are restriction, the vertical maps are isomorphisms and

$$\gamma_x = \varinjlim_{U \ni x} \gamma_U : (\mathcal{H}^0 R_X)_x = \varinjlim_{U \ni x} H^0(U;R) \to (\mathcal{H}_n D^{\cdot}_X)_x = \varinjlim_{U \ni x} H_n(U;R).$$

Thus for each $x \in X$, $[X] = \gamma_X(1) \in H_n(X;R)$ restricts to a generator $[X]_x \in (\mathcal{H}_n D^{\cdot}_X)_x$, so that $[X]$ may be represented by a quasi-isomorphism $\alpha : R_X \to \Sigma^n D^{\cdot}_X$. It follows that the symmetric signature map takes $[X] \in H_n(X;R)$ to an n-dimensional symmetric Poincaré complex of sheaves $(\mathcal{C}^{\cdot}_X, \phi_X)$ on X. The symmetric signature

$$\sigma^*(X) \in L^n(X;R)$$

of an n-dimensional R-homology manifold is defined to be the cobordism class of the symmetric Poincaré complex $(\mathcal{C}^{\cdot}_X, \phi_X)$. ∎

The algebraic suspension chain map

$$\Sigma : \Sigma \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(A^{\cdot} \otimes A^{\cdot}, D^{\cdot}_X)) \longrightarrow \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(\Sigma A^{\cdot} \otimes \Sigma A^{\cdot}, D^{\cdot}_X))$$

of [9] determines a chain map

$$\Sigma\Phi_X : \Sigma \mathrm{Hom}^{\cdot}(\mathcal{C}^{\cdot}_X, D^{\cdot}_X) \longrightarrow \mathrm{Hom}^{\cdot}_{\mathbb{Z}[Z_2]}(W, \mathrm{Hom}^{\cdot}(\Sigma\mathcal{C}^{\cdot}_X \otimes \Sigma\mathcal{C}^{\cdot}_X, D^{\cdot}_X))$$

and hence the suspended symmetric signature

$$\Sigma\sigma^* : \Sigma H_n(X;R) \longrightarrow Q^{n+1}(\Sigma\mathcal{C}^{\cdot}_X).$$

This has a geometric interpretation. Let $p : X \times \mathbb{R} \to X$ be the projection map and ϕ the family of fibrewise compact supports on $X \times \mathbb{R}$. Let $p_\phi \mathcal{C}^{\cdot}_{X\times\mathbb{R}}$ be the direct image relative to ϕ, [1;IV,3]. Then there is a suspension quasi-isomorphism

$$\Sigma_X : \Sigma\mathcal{C}^{\cdot}_X \longrightarrow p_\phi \mathcal{C}^{\cdot}_{X\times\mathbb{R}}$$

of sheaves on X. (This is a special case of the Thom quasi-isomorphism for locally trivial bundles constructed below.) Consider the chain map

$$p_\phi(\Phi_{X\times\mathbb{R}}) : \mathrm{Hom}^{\cdot}(p_\phi\mathcal{C}^{\cdot}_{X\times\mathbb{R}}, D^{\cdot}_X) \longrightarrow \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(p_\phi\mathcal{C}^{\cdot}_{X\times\mathbb{R}} \otimes p_\phi\mathcal{C}^{\cdot}_{X\times\mathbb{R}}, D^{\cdot}_X)).$$

Then the diagram

$$\begin{array}{ccc}
\mathrm{Hom}^{\cdot}(p_\phi \mathcal{C}^{\cdot}_{X\times\mathbb{R}}, D^{\cdot}_X) & \xrightarrow{\Sigma^*_X} & \Sigma\mathrm{Hom}^{\cdot}(\mathcal{C}^{\cdot}_X, D^{\cdot}_X) \\
{\scriptstyle p_\phi \Phi_{X\times\mathbb{R}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \Sigma\Phi_X} \\
\mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(p_\phi \mathcal{C}^{\cdot}_{X\times\mathbb{R}} \otimes p_\phi \mathcal{C}^{\cdot}_{X\times\mathbb{R}}, D^{\cdot}_X)) & \xrightarrow{\Sigma^\%_X} & \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(\Sigma\mathcal{C}^{\cdot}_X \otimes \Sigma\mathcal{C}^{\cdot}_X, D^{\cdot}_X))
\end{array}$$

admits a (unique up to homotopy) chain homotopy

$$\Gamma_X : \Sigma\Phi_X \cdot \Sigma^*_X \simeq \Sigma^\%_X \cdot p_\phi \Phi_{X\times\mathbb{R}}.$$

This follows from the properties of resolutions and the injectivity of $D^{\cdot}_X$. In particular there is a commutative diagram

$$\begin{array}{ccc}
H^\phi_n(X\times\mathbb{R}; R) & \xrightarrow{p_\phi(\sigma^*)} & Q^{n+1}(p_\phi \mathcal{C}^{\cdot}_{X\times\mathbb{R}}) \\
{\scriptstyle \Sigma_*}\Big\downarrow & & \Big\downarrow{\scriptstyle \Sigma^\%} \\
\Sigma H_n(X; R) & \xrightarrow{\Sigma\sigma^*} & Q^{n+1}(\Sigma\mathcal{C}^{\cdot}_X)
\end{array}$$

relating algebraic to geometric suspension.

Let $Y \subset X$ with Y closed in X and $i : Y \to X$ the inclusion. A pair of sheaves $\beta : B^{\cdot} \to A^{\cdot}$ over (X, Y) consists of sheaves $A^{\cdot}$ over Y, $B^{\cdot}$ over X together with an i-cohomomorphism $\beta : B^{\cdot} \to A^{\cdot}$, [1;I.4]. Any i-cohomomorphism β may be factored

$$\beta : B^{\cdot} \overset{h(i)}{\to} i_* A^{\cdot} \to A^{\cdot}$$

where the last arrow is the natural i-cohomomorphism and $h(i)$ is a sheaf map. Define

$$\beta^\% : W^\% A^{\cdot} \to W^\% B^{\cdot} \,;\, \phi \mapsto h(i)^\%(i_*\phi)$$

and similarly for quadratic.

Definition 3.3. *An $(n+1)$-dimensional symmetric $\{$quadratic$\}$ Poincaré pair*

$$\begin{array}{c} (\beta : B^{\cdot} \to A^{\cdot}, (\delta\phi, \phi)) \\ \{ \quad (\beta : B^{\cdot} \to A^{\cdot}, (\delta\psi, \psi)) \quad \} \end{array}$$

on (X, Y) consists of a pair of sheaves $\beta : B^{\cdot} \to A^{\cdot}$ on (X, Y), an n-dimensional symmetric $\{$quadratic$\}$ Poincaré complex of sheaves $(A^{\cdot}, \psi)$ $\{(A^{\cdot}, \psi)\}$ on Y together with an $(n+1)$-dimensional symmetric $\{$quadratic$\}$ Poincaré pair

$$\begin{array}{c} (h : B^{\cdot} \to i_* A^{\cdot}, (\delta\phi, i_*\phi)) \\ \{ \quad (h : B^{\cdot} \to i_* A^{\cdot}, (\delta\psi, i_*\psi)) \quad \} \end{array}$$

on X. An $(n+1)$-dimensional symmetric pair $(\beta : B^{\cdot} \to A^{\cdot}, (\delta\phi, \phi))$ represents class in $Q^{n+1}(\beta) = H_{-n-1}(C(\beta^{\%}))$. Similarly for quadratic.

The complex of sheaves $\mathcal{C}^{\cdot}_{(X,Y)}$ on X is defined by the short exact sequence

$$0 \longrightarrow \mathcal{C}^{\cdot}_{(X,Y)} \longrightarrow \mathcal{C}^{\cdot}_X \xrightarrow{h(i)} i_*\mathcal{C}^{\cdot}_Y \longrightarrow 0.$$

Then $\mathcal{C}^{\cdot}_{(X,Y)}$ is a flabby sheaf and the relative cohomology groups of (X,Y) are given by

$$H^n(X,Y;R) = H^n(\Gamma(\mathcal{C}^{\cdot}_{(X,Y)})).$$

The complex of sheaves $D^{\cdot}_{(X,Y)}$ is defined by the short exact sequence

$$0 \longrightarrow i_*D^{\cdot}_Y \xrightarrow{Dh(i)} D^{\cdot}_X \longrightarrow D^{\cdot}_{(X,Y)} \longrightarrow 0.$$

Then $D^{\cdot}_{(X,Y)}$ is a flabby sheaf and the Borel-Moore relative homology groups of (X,Y) are given by

$$H_n(X,Y;R) = H^{-n}(X;D^{\cdot}_{(X,Y)}).$$

Let $\beta : \mathcal{C}^{\cdot}_X \longrightarrow \mathcal{C}^{\cdot}_Y$ be the canonical pair over (X,Y). See [1] for details.

Lemma 3.4. *There is a relative symmetric signature map*

$$\sigma^* : H_{n+1}(X,Y;R) \longrightarrow Q^n(\beta)$$

which is natural with respect to proper maps of pairs. An $(n+1)$-dimensional homology manifold $(X, \partial X)$ determines an $(n+1)$-dimensional symmetric Poincaréŕe pair

$$(\beta : \mathcal{C}^{\cdot}_X \to \mathcal{C}^{\cdot}_{\partial X}, (\delta\phi_X, \phi_{\partial X})).$$

The proof is similar to the absolute case and is left to the reader.

Theorem 3.5. *Let $f : X \to Y$ be a proper degree 1 map of n-dimensional homology manifolds. There is defined a symmetric signature*

$$\sigma^*(f) \in L^n(Y;R)$$

such that

$$f_*\sigma^*(X) = \sigma^*(Y) \oplus \sigma^*(f).$$

Furthermore

(1) *for $U \subset X$ open, $\sigma^*(f)_{|U} = \sigma^*(f_{|f^{-1}(U)})$,*
(2) *The assembly $As(\sigma^*(f)) \in L^n_{\mathrm{p}}(R[\pi_1 Y])$ is the chain level symmetric signature of [9],*
(3) *The signature is an invariant of the proper bordism class of f and is zero if f is a hereditary homotopy equivalence.*

Proof Since f is a degree 1 proper map $h(f) : (\mathcal{C}^{\cdot}_Y, \phi_Y) \to (f_*\mathcal{C}^{\cdot}_X, f_*\phi)$ is a map of n-dimensional symmetric Poincaré complexes of sheaves on Y. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\cdot}_Y & \xrightarrow{h(f)} & f_*\mathcal{C}^{\cdot}_X \\ \phi^0_Y \big\downarrow & & \big\downarrow f_*\phi^0_X \\ \Sigma^n D\mathcal{C}^{\cdot}_Y & \xleftarrow{Dh(f)} & \Sigma^n Df_*\mathcal{C}^{\cdot}_X \end{array}$$

and may form the ladder of cofibrations

$$\begin{array}{ccccccc} \Sigma^{-1}C(\omega) & \xrightarrow{e} & f_*\mathcal{C}^{\cdot}_X & \xrightarrow{\omega} & \Sigma^n D\mathcal{C}^{\cdot}_Y & \longrightarrow & C(\omega) \\ \| & & \big\uparrow (\phi^0_Y)' & & \big\uparrow \phi^0_Y & & \| \\ \Sigma^{-1}C(\omega) & \xrightarrow{e'} & W & \xrightarrow{\omega'} & \mathcal{C}^{\cdot}_Y & \longrightarrow & C(\omega) \end{array}$$

where $\omega = Dh(f) \cdot f_*\phi^0_X$. Define the n-dimensional symmetric Poincaré complex of sheaves on Y $(\mathcal{C}^{\cdot}_{f^!}, \phi_{f^!})$ by

$$\mathcal{C}^{\cdot}_{f^!} = \Sigma^{-1}C(\omega), \qquad \phi_{f^!} = e^{\%}(f_*\phi_X).$$

(Here $f^!$ denotes the Umkehr map rather than the right adjoint to $f_!$.) Then

$$(e, h(f)) : \mathcal{C}^{\cdot}_{f^!} \oplus \mathcal{C}^{\cdot}_Y \longrightarrow f_*\mathcal{C}^{\cdot}_X$$

is a quasi-isomorphism of sheaves on Y pulling $f_*\phi_X$ to $\phi_{f^!} \oplus \phi_Y$. To see this consider

$$(e, h(f))^* : Q^n(f_*\mathcal{C}^{\cdot}_X) \to Q^n(\mathcal{C}^{\cdot}_{f^!} \oplus \mathcal{C}^{\cdot}_Y) = Q^n(\mathcal{C}^{\cdot}_{f^!}) \oplus Q^n(\mathcal{C}^{\cdot}_Y) \oplus H_n(X; \mathcal{C}^{\cdot}_{f^!} \otimes \mathcal{C}^{\cdot}_Y).$$

Then the component of $(e, h(f))^*(f_*\phi_X)$ in $H_n(X; \mathcal{C}^{\cdot}_{f^!} \otimes \mathcal{C}^{\cdot}_Y)$ is represented by the map

$$\mathcal{C}^{\cdot}_{f^!} \otimes \mathcal{C}^{\cdot}_Y \xrightarrow{e \otimes h(f)} f_*\mathcal{C}^{\cdot}_X \otimes f_*\mathcal{C}^{\cdot}_X \xrightarrow{(f_*\phi)^0} \Sigma^n D^{\cdot}_Y$$

which is easily seen to be null-homotopic. ■

We now refine the above constructions to take account of normal data. In particular symmetric signatures (or more properly their Poincaré boundaries) are refined to quadratic signatures.

Let X be a finite dimensional compact ANR. Then X admits a closed embedding $i : X \hookrightarrow \mathbb{R}^N$, N large, for which there is a mapping cylinder neighbourhood $X \subset U \subset \mathbb{R}^N$, ie. U is a topological manifold neighbourhood of X and there is a proper map $p : \partial U \to X$ whose mapping cylinder is homeomorphic (rel. ∂U, X) to U, [8]. Let ν denote the pathspace fibration associated to p. Let ξ^{k-1} be a spherical fibration over X, $T\xi$ its Thom space and $\rho : S^N \to T\xi$ a given map. The triple (X, ξ, ρ) is a normal space (of formal dimension $N - k$) in the sense of Quinn [7]. We shall require additionally that there is a fibre map $b : \nu \to \xi$ over the identity such that, up to homotopy, ρ is the composition

$$S^N \xrightarrow{\text{coll}} T\nu \xrightarrow{Tb} T\xi.$$

where coll is the collapse map. It seems likely that the normal cobordism class of a general normal space (in the sense of Quinn) contains such a representative. In any case most normal spaces we consider will satisfy this additional condition (which is basically a transversality condition ensuring the existence of a compact ANR in the cobordism class).

Example 3.6 An n-dimensional orientable Poincaré space X is a compact finite dimensional ANR with a class $[X] \in H_n(X;\mathbb{Z})$ such that

$$[X]\cap = A(\phi^0_{\tilde{X}}) : H^*_c(\tilde{X}) \longrightarrow H^c_{n-*}(\tilde{X})$$

is an isomorphism of graded $\mathbb{Z}[\pi_1 X]$-modules. For such a space a standard argument shows ν is a spherical fibration. Thus taking $b = 1 : \nu \to \nu$ an oriented Poincaré space is naturally a normal space.

For any spherical fibration ξ over a locally contractible space X we may consider the Leray sheaf $\mathcal{H}^{\cdot}(\xi)$ on X given by

$$\mathcal{H}^{\cdot}(\xi) = \mathcal{H}^{\cdot}(q_* \mathcal{C}^{\cdot}_{(D_\xi, S_\xi)})$$

where $q : D_\xi \to X$ is the corresponding disk fibration.

Lemma 3.7. *If ξ has fibre homotopy equivalent to S^{k-1} then $\mathcal{H}^{\cdot}(\xi)$ is locally constant with stalks*

$$\mathcal{H}^{\cdot}(\xi)_x \cong H^*(D^k, S^{k-1}; \mathbb{Z}).$$

Proof For $x \in X$ let $U \ni x$ be a contractible open neighborhood. Choose a local trivialization $\phi : (D^k, S^{k-1}) \times U \longrightarrow q^{-1}(U)$ over U so that there is a quasi-isomorphism [6;2.7.7]

$$\phi_* : q_{|*} \mathcal{C}^{\cdot}_{q^{-1}(U)} \longrightarrow p_{2*} \mathcal{C}^{\cdot}_{(D^k, S^{k-1}) \times U}$$

of sheaves over X. Hence

$$\mathcal{H}(q)^{\cdot}_{|U} = \mathcal{H}(q_{|*} \mathcal{C}^{\cdot}_{q^{-1}(U)}) \cong \mathcal{H}(p_{2*} \mathcal{C}^{\cdot}_{(D^k, S^k) \times U}) \cong H^*(D^k, S^k)_U,$$

where the latter is the constant sheaf over U. ∎

The fibration ξ is said to be orientable if $\mathcal{H}^k(\xi)$ is isomorphic to the constant sheaf $\mathbb{Z}_X$ on X. If ξ is orientable then the sheaf cohomology spectral sequence of $q_* \mathcal{C}^{\cdot}_{(D_\xi, S_\xi)}$ collapses and, as in Example 3.2, there is a quasi-isomorphism

$$\gamma : \Sigma^k R_X \longrightarrow q_* \mathcal{C}_{(D_\xi, S_\xi)}$$

or in other words an orientation $\gamma \in \Gamma(q_* \mathcal{C}_{(D_\xi, S_\xi)})$. The slant product of γ with the pairing

$$\mathcal{C}^{\cdot}_X \otimes q_* \mathcal{C}^{\cdot}_{(D_\xi, S_\xi)} \to q_* \mathcal{C}^{\cdot}_{D_\xi} \otimes q_* \mathcal{C}^{\cdot}_{(D_\xi, S_\xi)} \to q_* \mathcal{C}^{\cdot}_{(D_\xi, S_\xi)}$$

determines a cup-product

$$\cup\gamma : \Sigma^k \mathcal{C}^{\cdot}_X \longrightarrow q_* \mathcal{C}^{\cdot}_{(D_\xi, S_\xi)}.$$

For ξ a trivial fibration this is easily seen to be a quasi-isomorphism. This is also the case for general ξ by local triviality. The induced map

$$\cup\gamma : H^*(X) \longrightarrow H^{*+k}(D_\xi, S_\xi)$$

is the Thom isomorphism. In the special case ξ an $\mathbb{R}^k$-bundle with projection $q : E_\xi \to X$ and associated disk bundle (D_ξ, S_ξ) there is a natural quasi-isomorphism

$$q_\phi \mathcal{C}^{\cdot}_{E_\xi} \to q_* \mathcal{C}^{\cdot}_{(D_\xi, S_\xi)}.$$

where ϕ is the family of fibrewise compact supports in E_ξ.

Suppose now (X, ξ, ρ) is an orientable normal space of formal dimension n, and that is ξ is orientable. Then $[X] = h(\rho) \cap \gamma \in H_n(X)$ is called the fundamental class of X. Let $(\mathcal{C}^{\cdot}_X, \phi_X)$ be the n-dimensional symmetric complex of sheaves on X obtained by evaluating the symmetric map on the fundamental class $[X]$. Compare the following theorem with [10;9.15]

Theorem 3.8. *The symmetric complex of sheaves $(\mathcal{C}^{\cdot}_X, \phi_X)$ admits a refinement to a normal complex of sheaves $(\mathcal{C}^{\cdot}_X, \theta_X)$ on X. In particular the symmetric boundary sheaf $(\partial\mathcal{C}^{\cdot}_X, \partial\phi_X)$ measuring the failure of local Poincaré duality in X admits a quadratic refinement $(\partial\mathcal{C}^{\cdot}_X, \psi_X)$.*

Recall [10;2] a normal refinement $\theta_X = (\phi_X, \gamma_X, \chi_X)$ of ϕ_X consists of a 0-cycle

$$\gamma \in \hat{W}^{\%} D\mathcal{C}^{\cdot}_X$$

where $\hat{W}$ is the standard complete free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$, together with a $(-n-1)$-chain

$$\chi_X \in \hat{W}^{\%} \mathcal{C}^{\cdot}_X$$

such that

$$J(\phi_X) - (\phi^0_X)^{\%}(\Sigma^n \gamma_X) = d\chi_X.$$

Such a refinement is equivalent to a quadratic refinement of the symmetric boundary $\partial\phi_X$. (See Weiss [13], Ranicki [10; 2.6]).

Proof of Theorem Let $p : D_\nu \to X$ denote the disk (or rather cone) fibration associated to ν so that there is a map

$$j : (U, \partial U) \subset (D_\nu, S_\nu) \xrightarrow{b} (D_\xi, S_\xi)$$

over X. This induces a map of sheaves on X [6;2.7.7]

$$j_* : q_* \mathcal{C}^{\cdot}_{(D_\xi, S_\xi)} \longrightarrow p_* \mathcal{C}^{\cdot}_{(U, \partial U)},$$

fitting into a commutative diagram

$$\begin{array}{ccc} q_*\mathcal{C}^{\cdot}_{(D_\xi,S_\xi)} & \xrightarrow{j_*} & p_*\mathcal{C}^{\cdot}_{(U,\partial U)} \\ \cup\gamma\big\uparrow & & \big\downarrow\omega \\ \Sigma^k\mathcal{C}^{\cdot}_X & \xrightarrow{\Sigma^k\phi^0_X} & \Sigma^{n+k}D\mathcal{C}^{\cdot}_X \end{array}$$

in which the vertical maps are quasi-isomorphisms. The only map not defined is ω which is constructed as follows. Since $(U,\partial U)$ is a topological manifold there is a quasi-isomorphism of flabby sheaves

$$\phi^0_U : \mathcal{C}^{\cdot}_{(U,\partial U)} \longrightarrow \Sigma^{n+k}D\mathcal{C}^{\cdot}_U.$$

But p is a proper map so

$$p_*(\phi^0_U) : p_*\mathcal{C}^{\cdot}_{(U,\partial U)} \longrightarrow \Sigma^{n+k}p_*D\mathcal{C}^{\cdot}_U \simeq \Sigma^{n+k}Dp_*\mathcal{C}^{\cdot}_U$$

is also a quasi-isomorphism. Also $p : U \to X$ has compact contractible fibres so the natural map $h(p) : \mathcal{C}^{\cdot}_X \to p_*\mathcal{C}^{\cdot}_U$ is a quasi-isomorphism [6;2.7.7] and we obtain the quasi-isomorphism

$$\omega : p_*\mathcal{C}^{\cdot}_{(U,\partial U)} \xrightarrow{p_*\phi^0_U} \Sigma^{n+k}Dp_*\mathcal{C}^{\cdot}_U \xrightarrow{Dh(p)} \Sigma^{n+k}D\mathcal{C}^{\cdot}_X.$$

Let $-\xi$ be a fibre homotopy inverse to ξ. Define $b' = b \oplus 1 : \nu \oplus -\xi \to \xi \oplus -\xi$. Since U is a space over X and $-\xi$ a fibration the Whitney sum $U' = U \oplus -\xi$ may be defined and there is a map $j' = j \oplus 1 : U \oplus -\xi \to \xi \oplus -\xi$ over X. Again there is a commutative diagram

$$\begin{array}{ccc} q'_*\mathcal{C}_{(D_{\xi\oplus-\xi},S_{\xi\oplus-\xi})} & \xrightarrow{j'_*} & p'_*\mathcal{C}^{\cdot}_{(U',\partial U')} \\ \cup\gamma'\big\uparrow & & \big\downarrow\omega' \\ \Sigma^{k+l}\mathcal{C}^{\cdot}_X & \xrightarrow{\Sigma^{k+l}\phi^0_X} & \Sigma^{n+k+l}D\mathcal{C}^{\cdot}_X \end{array}$$

with vertical maps quasi-isomorphisms.

We may now construct the normal refinement $\theta_X = (\phi_X, \gamma_X, \chi_X)$. The Thom isomorphism associated to $-\xi$ determines a lift $[U'] \in H_{n+k+l}(U', \partial U')$ of the fundamental class $[U] \in H_{n+k}(U, \partial U)$. Similarly there is defined a lift $[D_{\xi\oplus-\xi}] \in H_{n+k+l}(D_{\xi\oplus-\xi}, S_{\xi\oplus-\xi})$. We then have

$$j_*([D_{\xi\oplus-\xi}]) = [U'].$$

Evaluating the symmetric signature map on $[U']$ determines an $(n+k+l)$-dimensional symmetric complex of sheaves on U'

$$(\mathcal{C}^{\cdot}_{U'} \to \mathcal{C}^{\cdot}_{\partial U'}, (\delta\phi_{U'}, \phi_{\partial U'}))$$

with algebraic Thom complex [10]

$$(\mathcal{C}^{\cdot}_{(U',\partial U')}, \delta\phi_{U'}/\phi_{\partial U'}).$$

Applying the direct image functor we obtain an $(n+k+l)$-dimensional symmetric complex of sheaves on X

$$(p'_*\mathcal{C}^{\cdot}_{(U',\partial U')}, p'_*(\delta\phi_{U'}/\phi_{\partial U'})).$$

Lift this across the quasi-isomorphism ω' to obtain (up to a boundary) a 0-cycle

$$\gamma_X \in \hat{W}^{\%} D\mathcal{C}^{\cdot}_X.$$

Now perform the same construction on the class $[D_{\xi\oplus -\xi}]$ to obtain an n-cycle

$$\gamma'_X \in \hat{W}^{\%}\mathcal{C}^{\cdot}_X.$$

By naturality of the symmetric signature we have up to a boundary

$$\gamma'_X = (\phi^0_X)^{\%}(\Sigma^n \gamma_X).$$

Finally $\xi \oplus -\xi$ is fibre homotopy equivalent to the trivial fibration so up to a boundary, which is determined by the chain homotopy Γ_X evaluated on $[X]$, we may identify

$$J(\gamma'_X) = J(\phi_X).$$

This shows there is a cochain

$$\chi_X \in \hat{W}^{\%}\mathcal{C}^{\cdot}_X$$

such that

$$J(\phi_X) - (\phi^0_X)^{\%}(S^n\gamma_X) = d\chi_X.$$

Thus setting $\theta_X = (\phi_X, \gamma_X, \chi_X)$ we obtain an n-dimensional normal complex of sheaves $(\mathcal{C}^{\cdot}_X, \theta_X)$ on X refining the symmetric signature $(\mathcal{C}^{\cdot}, \phi)$. It can be shown that the equivalence class of this normal complex (see Weiss [13]) is independent of the choices made.

■

Finally we have

Theorem 3.9. *Let $(f,b): M^n \to X^n$ be a normal map with M an n-dimensional topological manifold and X an n-dimensional Poincaré space. Then there is defined a quadratic signature*

$$\sigma_*(f,b) \in L_n(X;R)$$

such that

$$(1+T)\sigma_*(f,b) = \sigma^*(f) \in L^n(X;R)$$

and with assembly the Wall-Ranicki quadratic signature of (f,b).

Proof Take $R = \mathbb{Z}$. We give the construction of a stable Umkehr map for f which preserves symmetric signatures. The construction of a quadratic refinement

$\sigma(f, b) \in L_n(X)$ of the symmetric signature $\sigma(f) \in L^n(X)$ follows in the standard way (cf. [9;2.3]). The normal map determines a mapping cylinder neighbourhood W of X, with associated fibration a fibre homotopy trivial spherical fibration, together with an embedding of M in W with trivial normal bundle. Thus there is an inclusion as an open set $j : M \times \mathbb{R}^{k+l+1} \hookrightarrow W$ into the interior of W. This determines a sheaf map

$$F : j_* \mathcal{C}^{\cdot}_{(M \times D^{k+l+1}, M \times S^{k+l})} \to \mathcal{C}^{\cdot}_{(W, \partial W)}$$

of sheaves over W. This is a stable Umkehr map for f under the identifications $\Sigma^{k+l+1} \mathcal{C}^{\cdot}_M \simeq \Gamma_M \mathcal{C}^{\cdot}_{M \times \mathbb{R}^{k+l+1}} \simeq \mathcal{C}^{\cdot}_{(M \times D^{k+l+1}, M \times S^{k+l})}$ and $\Sigma^{k+l+1} \mathcal{C}^{\cdot}_X \simeq \Gamma_X \mathcal{C}^{\cdot}_{(W, \partial W)} \simeq \Gamma_X \mathcal{C}^{\cdot}_{(X \times D^{k+l+1}, X \times S^{k+l})}$, the latter by virtue of the fibre homotopy equivalence g. Finally let $\mathcal{C}^{\cdot}_{(M \times D^{k+l+1}, M \times S^{k+l})}$ and $\mathcal{C}^{\cdot}_{(W, \partial W)}$ be given the 'Thom' symmetric structure obtained by pulling back the natural symmetric structures on $\mathcal{C}^{\cdot}_{(M \times D^{k+l+1})}$ and $\mathcal{C}^{\cdot}_W$. Then these symmetric structures correspond to the suspension of ϕ_M and ϕ_X under the Umkehr identification. Furthermore the sheaf map F preserves these structures.

∎

4. Decomposing sheaves on $X \times \mathbb{R}$.

In this section we consider the various possibilities for decomposing an n-dimensional symmetric Poincaré sheaf $(A^{\cdot}, \phi)$ on a space of the form $Y = X \times \mathbb{R}$. Analogous results for quadratic sheaves may be obtained. A good example to keep in mind is of a proper Morse function $f : M^n \to \mathbb{R}$ with M a manifold (without boundary) and the associated n-dimensional symmetric Poincaré sheaf $(f_* \mathcal{C}^{\cdot}_M, f_* \phi)$ on $\mathbb{R}$. In this case the algebraic decompositions of sheaves described in this section can be constructed directly from corresponding geometric decompositions of M. Write $Y_0 = X \times \{0\}$, $Y_- = X \times (-\infty, 0]$ and $Y_+ = X \times [0, +\infty)$. The first sheaf to be considered is the restriction $A^{\cdot}_0 = i^* A^{\cdot}$ of $A^{\cdot}$ to the subspace $X = Y_0 \subset Y$. This will be given the structure of an $(n-1)$-dimensional symmetric sheaf $(A^{\cdot}_0, \phi_0)$ on X. In general it will not be Poincaré. In our example the symmetric sheaf $((f_* \mathcal{C}^{\cdot}_M)_0, (f_* \phi)_0)$ may be obtained (up to quasi-isomorphism) from the subspace $M_0 = f^{-1}(0) \subset M$ together with a class $[M_0] \in H_{n-1}(M_0)$ using Proposition 3.1, and M_0 will be a manifold if 0 is a regular value of f. Similarly we may define an n-dimensional symmetric pair $(A^{\cdot}_- \to A^{\cdot}_0, (\phi_-, \phi_0))$ over (Y_-, Y_0), where $A^{\cdot}_- = i^*_- A^{\cdot}$ is the restriction of $A^{\cdot}$ to Y_-. Again it will not be Poincaré in general. The Morse function example suggests a further sheaf over X: an n-dimensional symmetric Poincaré sheaf $(A^{\cdot}_0 \to \partial_X A^{\cdot}, (\delta\phi, \partial_X \phi)$ which is the 'germ Poincaré cobordism' of $A^{\cdot}$ at X and corresponds to the manifold with boundary $(f^{-1}[-\epsilon, \epsilon], f^{-1}(-\epsilon) \cup f^{-1}(\epsilon))$ where $\epsilon > 0$ is small. This simple geometric picture fails if f is not Morse. Nevertheless the $(n-1)$-dimensional symmetric Poincaré boundary $\partial_X A^{\cdot}$ may still be constructed for arbitrary $A^{\cdot}$. It decomposes as a Poincaré symmetric sheaf into a direct sum $\partial_- A^{\cdot} \oplus \partial_+ A^{\cdot}$ corresponding to the subspaces Y_- and Y_+. Lastly we have the symmetric Poincaré pair $(A^{\cdot}_{<0} \to \partial_- A^{\cdot}, (\phi_{<0}, \partial_- \phi))$ on (Y_-, Y_0), where $A^{\cdot}_{<0}$ is the direct image on Y_- of the restriction of $A^{\cdot}$ to $Y_{<0} = X \times (-\infty, 0)$. The original Poincaré sheaf $A^{\cdot}$ may be reconstructed by gluing the sheaves $A^{\cdot}_{<0}$, $A^{\cdot}_0$ and $A^{\cdot}_{>0}$

together along their boundaries, (or rather gluing their direct image on Y). Here are the details.

Let

$$\mathcal{C}^{\cdot} = \mathcal{C}^{\cdot}_{Y}, \quad \mathcal{C}^{\cdot}_{\pm} = \mathcal{C}^{\cdot}_{Y_\pm}, \quad \mathcal{C}^{\cdot}_{0} = \mathcal{C}^{\cdot}_{Y_0} = \mathcal{C}^{\cdot}_{X}$$

be the canonical sheaves on Y, $Y_\pm$ and Y_0 respectively and

$$i_0 : Y_0 \to Y, \quad i_0^{\pm} : Y_0 \to Y_\pm, \quad i_\pm : Y_\pm \to Y$$

the inclusion maps. The short exact sequence of sheaves on Y

$$0 \to \mathcal{C}^{\cdot} \to i_-\mathcal{C}^{\cdot}_- \oplus i_+\mathcal{C}^{\cdot}_+ \to i_0\mathcal{C}^{\cdot}_0 \to 0 \tag{4.1}$$

determines the Mayer-Vietoris long exact cohomology sequence for the triad of spaces $(Y; Y_\pm, Y_0)$. Let $V \subset Y_0$ be an open set and $V_0 = V \cap Y$, etc.. Restricting (4.1) to V and applying the functor Γ_c determines short exact sequences ((4.1) consists of flabby sheaves) of R-module chain complexes

$$0 \to \Gamma_c(\mathcal{C}^{\cdot}_{|V}) \to \Gamma_c(\mathcal{C}^{\cdot}_{-|V_-}) \oplus \Gamma_c(\mathcal{C}^{\cdot}_{+|V_+}) \to \Gamma_c(\mathcal{C}^{\cdot}_{0|V_0}) \to 0, \quad V \subset Y \text{ open} \tag{4.2}$$

Apply the functor $\mathrm{Hom}^{\cdot}(-, I^{\cdot})$ to obtain the short exact sequence of sheaves on Y

$$0 \to i_0 D^{\cdot}_0 \xrightarrow{\alpha} i_- D^{\cdot}_- \oplus i_+ D^{\cdot}_+ \xrightarrow{\beta} D^{\cdot} \to 0 \tag{4.3}$$

where

$$D^{\cdot} = D^{\cdot}_{Y}, \quad D^{\cdot}_{\pm} = D^{\cdot}_{Y_\pm}, \quad D^{\cdot}_{0} = D^{\cdot}_{Y_0} = D^{\cdot}_{X}.$$

We shall construct a splitting $\gamma_- : D^{\cdot} \to i_- D^{\cdot}_- \oplus i_+ D^{\cdot}_+$ of (4.3). The map γ_- is not a chain map. It is constructed from a map $\tau_- : i_0^- \mathcal{C}^{\cdot}_0 \to \mathcal{C}^{\cdot}_-$ such that $(i_-\tau_-, 0) : i_0\mathcal{C}^{\cdot}_0 \to i_-\mathcal{C}^{\cdot}_- \oplus i_+\mathcal{C}^{\cdot}_+$ is a splitting of (4.1). Again τ_- is not a chain map. Let $p_- : Y_- \to Y_0$ be the projection map and $\beta_- : R_{Y_0} \to p_-^* R_{Y_0}$ the canonical (p_-)-cohomomorphism. This extends naturally to a (p_-)-cohomomorphism $\mathcal{C}^{\cdot}(\beta_-)$: $\mathcal{C}^{\cdot}_0 \to \mathcal{C}^{\cdot}$ which factors as a composition

$$\mathcal{C}^{\cdot}_0 \to p_-^* \mathcal{C}^{\cdot}_0 \xrightarrow{h_-} \mathcal{C}^{\cdot},$$

where h_- is a sheaf map. Since p_-^* is exact it follows that h_- determines maps $h_- : p_-^*\mathcal{L}^{\cdot}_0 \to \mathcal{L}^{\cdot}_-$ (recall $\mathcal{L}^{\cdot}$ is the subsheaf of cocycles of $\mathcal{C}^{\cdot}$). Define $\tau_-^q : i_0^- \mathcal{C}^q_0 \to \mathcal{C}^q_-$ to be the composition

$$i_0^- \mathcal{C}^q_0 = i_0^- \mathcal{C}^0(\mathcal{L}^{q-1}_0) \xrightarrow{g_-} \mathcal{C}^0(p_-^*\mathcal{L}^{q-1}_0) \xrightarrow{\mathcal{C}^0(h_-)} \mathcal{C}^0(\mathcal{L}^{q-1}_-) = \mathcal{C}^q_-$$

where g_- is 'extension by zero', so that on sections over an open set $U \subset Y_-$ it is the inclusion

$$\prod_{x \in Y_0 \cap U} (\mathcal{L}^{q-1}_0)_x \subset \prod_{y \in U} (\mathcal{L}^{q-1}_0)_{p_-(y)}.$$

The map τ_- restricts to a map of sections with compact support and hence determines a splitting of (4.2). Applying $\mathrm{Hom}^{\cdot}(-, I^{\cdot})$ we obtain a splitting $\gamma_- : D^{\cdot} \to i_- D^{\cdot}_- \oplus i_+ D^{\cdot}_+$ of (4.3) (in fact we get a splitting on the left, but we construct the right splitting γ_- from this in the usual way).

We shall require the following elementary lemma. Let

$$0 \to A^{\cdot} \xrightarrow{\alpha} B^{\cdot} \xrightarrow{\beta} C^{\cdot} \to 0$$

be a short exact sequence of cochain complexes in an abelian category. Let p : $C(\alpha) \to C^{\cdot}$ be the natural map, where $C(\alpha)$ is the chain mapping cone of α.

Lemma 4.1. *Suppose for each $q \in \mathbb{Z}$ there is given a map $\gamma^q : C^q \to B^q$ splitting $0 \to A^q \to B^q \to C^q \to 0$. Then there is determined a chain map $\kappa : C^{\cdot} \to C(\alpha)$ such that $\kappa \cdot p \simeq 1$ and $p \cdot \kappa = 1$.*

Proof. Let $\bar{\gamma}^q = (-)^q \gamma^q$ so that $\bar{\gamma} : C^{\cdot} \to B^{\cdot}$ satisfies $\beta(d\bar{\gamma} + \bar{\gamma}d) = 0$. Let $\omega : C^{\cdot} \to \Sigma^{-1}A^{\cdot}$ be the chain map satisfying $\alpha \cdot \omega = d\bar{\gamma} + \bar{\gamma}d : C^{\cdot} \to \Sigma^{-1}B^{\cdot}$. Define the chain map

$$\kappa : C^{\cdot} \to C(\alpha); c \mapsto (\gamma(c), \omega(c)), \quad c \in C.$$

Then $p \cdot \kappa = 1$ and $\kappa \cdot p \simeq 1$ via the homotopy

$$H : C(\alpha) \to C(\alpha); (b, a) \mapsto (0, (-)^{|a|}\bar{a})$$

where $\alpha(\bar{a}) = b - \gamma\beta(b)$. ■

With the same notation as in the proof of the lemma, define maps of sheaves on Y

$$S = \omega_- : D^{\cdot} \to \Sigma^{-1} i_0 D_0^{\cdot}$$

where $\alpha \cdot \omega_- = d \cdot \bar{\gamma}_- + \bar{\gamma}_- \cdot d$, and

$$S_- = \pi_1 \cdot \bar{\gamma}_- : D^{\cdot} \to i_- D_-^{\cdot}.$$

If $\alpha_- = \pi_1 \cdot \alpha : i_0 D_0^{\cdot} \to i_- D_-^{\cdot}$ then the relation

$$\alpha_- \cdot S = d \cdot S_- + S_- \cdot d \tag{4.4}$$

holds. Note that S is a quasi-isomorphism when restricted to X since $D_{\pm}^{\cdot}$ are acyclic here.

To illustrate these constructions consider the special case X a point, so that $Y = \mathbb{R}$. Then $\tau_- : i_0^- \mathcal{C}_0^{\cdot} = i_0^- R \to \mathcal{C}_{\mathbb{R}_-}^{\cdot}$ takes $1 \in R$, regarded as a section of $i_0^- R$, to the section $\tau_-(1) \in \mathcal{C}_{\mathbb{R}_-}^0(\mathbb{R}_-)$ given by

$$\tau_-(1)(t) = \begin{cases} 1 \in R, & t = 0 \\ 0 \in R, & t < 0. \end{cases}$$

It follows that $d \cdot (i_- \bar{\tau}_-) + (i_- \bar{\tau}_-) \cdot d : i_0 R \to \Gamma_c \mathcal{C}_{\mathbb{R}}^1$ takes $1 \in R$, regarded as a section of $i_0 R$, to the image in $\mathcal{L}_{\mathbb{R}}^1(\mathbb{R})$ of the step serration $s_0 \in \mathcal{C}_{\mathbb{R}}^0(\mathbb{R})$ given by

$$s_0(t) = \{\, 1, \quad t \geq 0, 0, \quad t < 0.$$

This is a generator for $H_c^1(\mathbb{R}; R)$. Similarly the map $S : D_{\mathbb{R}}^{\cdot} \to \Sigma^{-1} i_0 D_0^{\cdot} = \Sigma^{-1} I^{\cdot}$ is, over $0 \in \mathbb{R}$, evaluation on this generator.

Now suppose $A^{\cdot}$ is a complex of sheaves on Y. Define sheaves $A_-^{\cdot}$ on Y_- and $A_0^{\cdot}$ on Y_0 by restriction, so that there is a natural sheaf map $h_- : A_-^{\cdot} \to i_0^- A_0^{\cdot}$ of sheaves on Y_- (or equivalently an (i_0^-)-cohomomorphism $\theta_- : A_-^{\cdot} \to A_0^{\cdot}$). For a

$(-n)$-chain $\phi \in W^{\%}A^{\cdot}$ (not necessarily a cocycle) define a $(-n)$-chain $\phi_- \in W^{\%}A^{\cdot}_-$ and a $(-n+1)$-chain $\phi_0 \in W^{\%}A^{\cdot}_0$ by

$$(\phi_-)^s : A^{\cdot}_- \otimes A^{\cdot}_- \overset{i^*_-\phi^s}{\to} \Sigma^n i^*_- D^{\cdot}_Y \overset{i^*_- S_-}{\to} \sigma^n D^{\cdot}_-, \quad s \geq 0,$$

and

$$(\phi_0)^s : A^{\cdot}_0 \otimes A^{\cdot}_0 \overset{i^*_0\phi^s}{\to} \Sigma^n i^*_0 D^{\cdot}_Y \overset{i^*_0 S}{\to} \Sigma^{n-1} D^{\cdot}_0, \quad s \geq 0.$$

The relation

$$d(\phi_-)^s = h^{\%}_-(i^-_0(\phi_0)^s) - i^*_-(S_- \cdot (d\phi)^s), \quad s \geq 0 \tag{4.5}$$

follows from (4.4). Thus, for example, if $(A^{\cdot}, \phi)$ is an n-dimensional symmetric sheaf on Y then

$$d(\phi_-) = h^{\%}_-(i^-_0(\phi_0))$$

so that $(\theta_- : A^{\cdot}_- \to A^{\cdot}_0, (\phi_-, \phi_0))$ is an n-dimensional symmetric pair on (Y_-, Y_0). More generally, if $(g : E^{\cdot} \to C^{\cdot}, (\delta\phi, \phi))$ is an n-dimensional symmetric pair on Y we have from (4.5)

$$\begin{aligned} d(\delta\phi_-)^s &= h^{\%}_-(i^-_0(\delta\phi_0)^s) - i^*_-(S_- \cdot (d\delta\phi)^s) \\ &= h^{\%}_-(i^-_0(\delta\phi_0)^s) - g^{\%}_-(\phi_-)^s, \quad s \geq 0, \end{aligned}$$

so that there is defined an n-dimensional n-dimensional symmetric pair

$$((h_-, g_-) : D^{\cdot}_- \to i^-_0 D^{\cdot}_0 \oplus_{i^-_0 C^{\cdot}_0} C^{\cdot}_-, (\delta\phi_-, (i^-_0\delta\phi_0, \phi_-)))$$

of sheaves on Y_-. In particular we may split symmetric cobordisms.

In a similar fashion we may obtain from $(A^{\cdot}, \phi)$ an n-dimensional symmetric pair $(\theta_+ : A^{\cdot}_+ \to A^{\cdot}_0, (\phi_+, \phi_0))$ on Y_+. The short exact sequence

$$0 \to A^{\cdot} \to i_- A^{\cdot} \oplus i_+ A^{\cdot} \to i_0 A^{\cdot} \to 0$$

recovers $A^{\cdot}, \phi)$ from the pairs $A^{\cdot}_\pm \to A^{\cdot}_0, (\phi_\pm, \phi_0))$. We have shown

Lemma 4.2. *Any symmetric sheaf on Y may be expressed as the union along Y_0 of symmetric pairs on (Y_-, Y_0) and (Y_+, Y_0).*

Next we discuss the construction of the 'germ cobordism' on X of an n-dimensional symmetric Poincaré sheaf $(A^{\cdot}, \phi)$ on Y. This will be an n-dimensional symmetric Poincaré pair $(A^{\cdot}_0 \to \partial_X A^{\cdot}, (\delta\phi, \partial_X A^{\cdot}))$ on X. In what follows we shall assume $A^{\cdot}$ is the canonical resolution of some complex of sheaves. Most examples we encounter are of this form and since for any symmetric sheaf $(B^{\cdot}, \phi)$ the symmetric structure ϕ can always be lifted to the canonical resolution $\mathcal{C}^{\cdot}(B^{\cdot})$ it is not a restriction.) Assuming $A^{\cdot}$ is of this form and $V \subset Y$ is open, then any $s \in A^{\cdot}(V)$ may be regarded as a serration of some sheaf on Y. For $K \subset V$ form the truncated serration which is identical to s on K and zero elsewhere. This defines a new section $s_K \in A^{\cdot}(V)$. Of course $-_K : A^{\cdot}(V) \to A^{\cdot}(V)$ is not a chain map.

Recall $A_0^{\cdot} = i^*A^{\cdot}$ is the restriction of $A^{\cdot}$ to X. Let $\Gamma_X A^{\cdot}$ denote the subsheaf of $A^{\cdot}$ consisting of germs of sections of $A^{\cdot}$ with support in X. The short exact sequence of flabby sheaves on Y

$$0 \to \Gamma_X A^{\cdot} \to A^{\cdot} \to \delta_X A^{\cdot} \to 0 \tag{4.6}$$

defines a sheaf $\delta_X A^{\cdot}$ on Y. In fact $\delta_X A^{\cdot} = k_* k^* A^{\cdot}$ where $k : Y - X \hookrightarrow Y$. Let $i^! = i^* \Gamma_X A^{\cdot}$ (so that $i^!$ is a right adjoint to the direct images with compact support functor $i_!$; this is Poincaré-Verdier duality, [6;3.1.12]) The short exact sequence of flabby sheaves on X

$$0 \to i^! A^{\cdot} \xrightarrow{j} i^* A^{\cdot} \to \partial_X A^{\cdot} \xrightarrow{p} 0 \tag{4.7}$$

defining $\partial_X A^{\cdot}$ is the pullback of (4.6) to X. Then $\partial_X A^{\cdot}$ is a direct sum of sheaves $\partial_- A^{\cdot} \oplus \partial_+ A^{\cdot}$ corresponding to the subspaces Y_- and Y_+ of Y.

Construct a commutative diagram

$$\begin{array}{ccc} i^! DA^{\cdot} & \xrightarrow{\sigma_1} & Di^* A^{\cdot} \\ {\scriptstyle j_{DA^{\cdot}}}\downarrow & & \downarrow{\scriptstyle Dj} \\ i^* DA^{\cdot} & \xrightarrow{\sigma_2} & Di^! A^{\cdot} \end{array}$$

as follows. Firstly define $\sigma_1 : i^! DA^{\cdot} \to Di^* A^{\cdot}$. Let $U \subset X$ be open. Then

$$(i^! DA^{\cdot})(U) = \varinjlim_{V \supset i(U)} (\Gamma_X DA^{\cdot})(V)$$

where V ranges over all open neighbourhoods of $i(U)$ in Y. Fix such a V and define $\sigma_1(V) : (\Gamma_X DA^{\cdot})(V) \to (Di^* A^{\cdot})(U)$, or equivalently

$$\sigma_1(V) : \mathrm{Hom}^{\cdot}_X(\Gamma_c A^{\cdot}_{|V}, I^{\cdot}) \to \mathrm{Hom}^{\cdot}(\Gamma_c(i^* A^{\cdot}_{|U}), I^{\cdot})$$

by

$$\sigma_1(V)(f)(s) = f(s_{X'}),$$

where $f \in \mathrm{Hom}_X(\Gamma_c A^{\cdot}_{|V}, I^{\cdot})$ and $s \in \Gamma_c(i^* A^{\cdot}_{|U})$. Here X' is a thin closed neighbourhood of X in Y (say $X \times [-\epsilon, \epsilon]$, ϵ small) and for $t \in A^{\cdot}(V)$ a representative of s, $s_{X'} = t_{X'} \in \Gamma A^{\cdot}_{|V}$. We leave the verification that $\sigma_1(V)$ is a chain map to the reader. Taking the limit over all $V \supset i(U)$ we obtain the sheaf map σ_1.

Again let $U \subset X$ be open and $V \supset i(U)$ an open neighbourhood of $i(U)$ in Y. Define

$$\sigma_2(V) : \mathrm{Hom}^{\cdot}(\Gamma_c A^{\cdot}_{|V}, I^{\cdot}) \to \mathrm{Hom}^{\cdot}(\Gamma_c((i^* \Gamma_X A^{\cdot})_{|U}), I^{\cdot})$$

to be the dual of the inclusion $\Gamma_c((i^* \Gamma_x A^{\cdot})_{|U}) \subset \Gamma_c A^{\cdot}_{|V}$, that is, the inclusion of the subcomplex of $\Gamma_c A^{\cdot}_{|V}$ of sections with compact support contained in X. Again taking limits over V determines the sheaf map σ_2.

Lemma 4.3. *The sheaf maps σ_1 and σ_2 are quasi-isomorphisms.*

Proof. That σ_1 is a quasi-isomorphism follows from [6;3.1.13]. To see that σ_2 is a quasi-isomorphism note that σ_2 is in fact the dual of the composition of quasi-isomorphisms

$$i^! A^{\cdot} \stackrel{i^! e}{\to} i^! D^2 A^{\cdot} \stackrel{\sigma_1 DA^{\cdot}}{\to} Di^* DA^{\cdot}.$$

∎

Now to the construction of the symmetric structure $(\delta\phi, \partial_X \phi)$ on $A_0^{\cdot} \to \partial_X A^{\cdot}$. Let $V \subset Y$ be open. The short exact sequence

$$0 \to \Gamma_X DA^{\cdot}(V) \stackrel{j'}{\to} DA^{\cdot}(V) \stackrel{p'}{\to} \delta_X DA^{\cdot}(V) \to 0 \tag{4.8}$$

may be written

$$0 \to \mathrm{Hom}_X^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot}) \stackrel{j'}{\to} \mathrm{Hom}^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot}) \stackrel{p'}{\to} \delta_X \mathrm{Hom}^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot}) \to 0 \tag{4.9}$$

By construction $\delta_X \mathrm{Hom}^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot})$ is the quotient R-module chain complex consisting of equivalence classes $[f]$ of maps $f \in \mathrm{Hom}^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot})$, where $[f_1] = [f_2]$ iff for all $s \in \Gamma_c A_{|V}^{\cdot}$ such that $|s| \cap X = \emptyset$ we have $f_1(s) = f_2(s)$. The short exact sequence (4.8) naturally splits by the map

$$\gamma_X' : \mathrm{Hom}^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot}) \to \mathrm{Hom}_X^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot})$$

given by

$$\gamma_X'(f)(s) = \begin{cases} f(s), & |s| \cap X \neq \emptyset, \\ 0, & |s| \cap X = \emptyset. \end{cases}$$

The map γ_X' is not a chain map. As usual this induces a splitting on the right

$$\rho_X' : \delta_X \mathrm{Hom}^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot}) \to \mathrm{Hom}^{\cdot}(\Gamma_c A^{\cdot}, I^{\cdot})$$

given by $\rho_X'([f]) = f - (j' \cdot \gamma_X')(f)$. The map ρ_X' is not a chain map. Applying lemma 4.1 we obtain a chain map

$$\omega'(V) : \Sigma \delta_X \mathrm{Hom}^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot}) \to \mathrm{Hom}_X^{\cdot}(\Gamma_c A_{|V}^{\cdot}, I^{\cdot})$$

satisfying $j' \cdot \omega' = d \cdot \bar{\rho}_X' + \bar{\rho}_X' \cdot d$, and inducing on homology the connecting homomorphism for (4.8). Let $\omega' : \delta_X DA^{\cdot} \to \Sigma^{-1} \Gamma_X DA^{\cdot}$ be the induced sheaf map. Write $\omega' : \partial_X DA^{\cdot} \to \Sigma^{-1} DA^{\cdot}$ also for its restriction to X.

The following lemma follows easily from the definitions.

Lemma 4.4. *The composition*

$$\partial_X DA^{\cdot} \xrightarrow{\omega'} \Sigma^{-1} i^! DA^{\cdot} \xrightarrow{\sigma_1} \Sigma^{-1} Di^* A^{\cdot} \xrightarrow{Dj} \Sigma^{-1} Di^! A^{\cdot}$$

is the zero map. In particular $\sigma_1 \cdot \omega'$ lifts to a map $\sigma_3 : \partial_X DA^{\cdot} \to \Sigma^{-1} D\partial_X A^{\cdot}$.

We can now define the symmetric structure $(\delta\phi, \partial_X\phi)$ on the pair $A^{\cdot}_0 \to \partial_X A^{\cdot}$. Define the system of maps $\delta\phi$ by

$$(\delta\phi)^s : A^{\cdot}_0 \otimes A^{\cdot}_0 \xrightarrow{i^*\phi^s} \Sigma^n i^* D^{\cdot}_Y \xrightarrow{\gamma'_X} \Sigma^n i^! D^{\cdot}_Y \xrightarrow{\sigma_1} \Sigma^n D^{\cdot}_X, \quad s \geq 0,$$

and the system $\partial_X\phi$ by

$$(\partial_X\phi)^s : \partial_X\phi \otimes \partial_X\phi \xrightarrow{\partial_X\phi^s} \Sigma^n \partial_X D^{\cdot}_Y \xrightarrow{\sigma_3} \Sigma^{n-1} D\partial_X R_Y \xrightarrow{Dp} \Sigma^{n-1} D^{\cdot}_X, \quad s \geq 0.$$

Alternatively we may describe the symmetric structure in adjoint form

$$(\delta\phi)^s : i^* A^{\cdot} \xrightarrow{i^*\phi^s} \Sigma^n \partial_X DA^{\cdot} \xrightarrow{\gamma'_X} \Sigma^n i^! DA^{\cdot} \xrightarrow{\sigma_1} \Sigma^n Di^* A^{\cdot}$$

and

$$(\partial_X\phi)^s : \partial_X A^{\cdot} \xrightarrow{\partial_X\phi^s} \Sigma^n \partial_X DA^{\cdot} \xrightarrow{\sigma_3} \Sigma^{n-1} D\partial_X A^{\cdot}$$

This will be useful when verifying Poincaré duality.

Lemma 4.5. *The pair $(A^{\cdot}_0 \to \partial_X A^{\cdot}, (\delta\phi, \partial_X\phi))$ is an n-dimensional symmetric pair on X.*

Proof. We must show $\partial_X\phi$ is a cocycle as well as the relation $d(\delta\phi) = p^{\#}(\partial_X\phi)$. But p is onto so that $p^{\#}$ is into. Thus we need only demonstrate the latter relation. Think of $(\delta\phi)^s$ as a map with domain the tensor product of $A^{\cdot}_0$ with itself. Similarly for $\partial_X\phi$. Now $p^{\#}(\partial_X\phi)^s = Dp \cdot \sigma_3 \cdot \partial_X\phi^s \cdot (p \otimes p) = Dp \cdot \sigma_3 \cdot \cdot i^*\phi^s$. Applying the symmetric relations of ϕ (which is a cocycle) shows that $d(\delta\phi)^s = \sigma_1 \cdot (d \cdot \bar{\gamma}'_X + \bar{\gamma}'_X \cdot d) \cdot i^*\phi^s$. The result follows.

The following is also true.

Lemma 4.6. *The pair $(A^{\cdot}_0 \to \partial_X A^{\cdot}, (\delta\phi, \partial_X\phi))$ is an n-dimensional symmetric Poincaré pair on X.*

Proof. Since $\partial_X\phi^0$ is a quasi-isomorphism we need only show σ_3 is a quasi-isomorphism. Consider the ladder of derived sheaves on X

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \Sigma\mathcal{H}(i^! DA^{\cdot}) & \xrightarrow{i'_*} & \Sigma\mathcal{H}(i^* DA^{\cdot}) & \xrightarrow{p'_*} & \Sigma\mathcal{H}(\partial_X DA^{\cdot}) & \xrightarrow{\omega'} & \mathcal{H}(I^! DA^{\cdot}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \sigma_{1*}} & & \downarrow{\scriptstyle \sigma_{2*}} & & \downarrow{\scriptstyle \sigma_{3*}} & & \downarrow{\scriptstyle \sigma_{1*}} & \\
\cdots \longrightarrow & \Sigma\mathcal{H}(Di^* A^{\cdot}) & \xrightarrow{(Dj)_*} & \Sigma\mathcal{H}(Di^! A^{\cdot}) & \xrightarrow{\delta} & \mathcal{H}(D\partial_X A^{\cdot}) & \xrightarrow{(Dp)_*} & \mathcal{H}(di^* A^{\cdot}) & \longrightarrow \cdots
\end{array}$$

where δ is the connecting homomorphism for the short exact sequence 4.8. The rows are exact and the outer squares commute. Further σ_1 and σ_2 are quasi-isomorphisms. Thus we need only show the middle square commutes and the result will follow from the five-lemma. The derived sheaf of a sheaf $A^\cdot$ is the sheaf generated by the presheaf $U \mapsto H^*(A^\cdot(U))$, so that it is sufficient to prove for each $U \subset X$ open the diagram

$$\begin{tikzcd}
\Sigma i^* DA^\cdot(U) \arrow[r, "p'"] \arrow[d, "\sigma_2"'] & \Sigma \partial_X DA^\cdot(U) \arrow[d, "\sigma_3"] \\
\Sigma Di^! A^\cdot(U) \arrow[r, "\delta"] & D\partial_X A^\cdot(U)
\end{tikzcd}$$

commutes on taking homology. This follows if for all $V \supset i(U)$, open in Y,

$$\begin{tikzcd}
\Sigma \mathrm{Hom}^\cdot(\Gamma_c A^\cdot_{|V}, I^\cdot) \arrow[r, "p'"] \arrow[d, "\sigma_2"'] & \Sigma \delta_X \mathrm{Hom}^\cdot(\Gamma_c A^\cdot_{|V}, I^\cdot) \arrow[d, "\sigma_3"] \\
\Sigma \mathrm{Hom}^\cdot(\Gamma_c((i^*\Gamma_X A^\cdot)_{|U}), I^\cdot) \arrow[r, "\delta"] & \mathrm{Hom}^\cdot(\Gamma_c((i^*\delta_X A^\cdot)_{|U}), I^\cdot)
\end{tikzcd}$$

commutes on homology. Let $f \in \Sigma \mathrm{Hom}^\cdot(\Gamma_c A^\cdot_{|V}, I^\cdot)$ be a cocycle. Then by definition of σ_3

$$(Dp \cdot \sigma_3 \cdot p')(f) = d(\sigma_1 \cdot \tau'_X)(f).$$

The connecting homomorphism δ takes $\sigma_2(f)$ to an element $dg \in \mathrm{Hom}^\cdot(\Gamma_c((i^*\delta A^\cdot)_{|U}), I^\cdot)$ where $g \in \Sigma\mathrm{Hom}^\cdot(\Gamma_c((i^*A^\cdot)_{|U}), I^\cdot)$ satisfies $Dj(g) = \sigma_2(f)$. The result follows if we can identify $Dj \cdot \sigma_1 \cdot \tau'_X = \sigma_2$. Let $s \in \Gamma_c((i^*\Gamma_X A^\cdot)_{|U})$. Then $\sigma_2(f)(s) = f(k(s))$ where $k : \Gamma_c((i^*\Gamma_X A^\cdot)_{|U}) \subset \Gamma_c A^\cdot_{|V}$. Also $(Di \cdot \sigma_1 \cdot \tau'_X)(f)(s) = (\sigma_1 \cdot \tau'_X)(f)(i(s)) = \tau'_X(f)(t_{X'})$ where $t \in A^\cdot(V)$ represents $i(s) \in \Gamma_c((i^*A^\cdot)_{|U})$. But we may choose $t = k(s)$ so $\tau'_X(t_{X'}) = \tau'_X(f)((k(s))_{X'}) = \tau'_X(f)(k(s)) = f(k(s))$. This completes the proof that $(\partial_X A^\cdot, \partial_X \phi)$ is an $(n-1)$-dimensional symmetric Poincaré sheaf on X. Finally it is cl

$$A^\cdot_0 \overset{i^*\phi^0}{\to} \Sigma^n i^* DA^\cdot \overset{\sigma_2}{\to} \Sigma^n Di^! A^\cdot.$$

■

Note that the algebraic Thom complex of the symmetric pair $(A^\cdot_0 \to \partial_X A^\cdot, (\delta\phi, \partial_X\phi))$ may be identified with the n-dimensional symmetric sheaf $(i^! A^\cdot, i^!\phi)$ on X, where $i^!\phi$ is the system of maps

$$(i^!\phi)^s : i^! A^\cdot \otimes i^! A^\cdot \overset{i^!\phi^s}{\to} \Sigma^n i^! D^\cdot_Y \overset{\sigma_1}{\to} \sigma^n D^\cdot_X.$$

(The algebraic Thom complex corresponds geometrically to replacing the germ cobordism over X by the quotient space obtained by collapsing the boundary to a point. In other words, the Thom complex of the normal bundle of M_0 in M.)

The $(n-1)$-dimensional symmetric Poincaré sheaf $(\partial_X A^\cdot, \partial_X\phi)$ is the direct sum of Poincaré sheaves $(\partial_- A^\cdot, \partial_-\phi)$ and $(\partial_+ A^\cdot, \partial_+\phi)$ corresponding to the subspaces Y_- and Y_+.

There is one final Poincaré pair over Y that we will require. We shall just give the definition and omit the proof of its properties. Build a sheaf $A^\cdot_{<0}$ on Y by first restricting $A^\cdot$ to $Y_{<0} = X \times (-\infty, 0)$ and then taking the direct image on Y. In fact $A^\cdot_{<0} = \delta_- A^\cdot$ where the latter is the subsheaf of $\delta_X A^\cdot$ with support in Y_-. Then there is a natural map $\delta_- A^\cdot \to i_* \partial_- A^\cdot$. Define a system of maps

$$(\delta_-)^s : \delta_- A^\cdot \otimes \delta_- A^\cdot \stackrel{\delta_- \phi^s}{\to} \Sigma^n \delta_- D^\cdot_Y \stackrel{\rho'_-}{\to} \Sigma^n D^\cdot_Y$$

where ρ'_- is the restriction of γ'_X to $\delta_- D^\cdot_Y$. We state without proof

Lemma 4.7. *The pair $(\delta_- A^\cdot \to i_* \partial_- A^\cdot, (\delta_- \phi, i_* \partial_- \phi))$ is an n-dimensional symmetric Poincaré pair on Y.*

The n-dimensional symmetric Poincaré sheaf $(A^\cdot, \phi)$ on Y may be reconstructed as the union of the three n-dimensional symmetric Poincaré pairs $\delta_- A^\cdot \to i_* \partial_- A^\cdot$, $i_* A^\cdot_0 \to i_* \partial_X A^\cdot$ and $\delta_+ A^\cdot \to i_* \partial_+ A^\cdot$ along their boundaries. To check that ϕ is recovered in this way from $\delta_- \cup i_* \phi_0 \cup \delta_+ \phi$ one must verify that the composition

$$D^\cdot_Y \xrightarrow{(p'_-, q, p'_+)} \delta_- D^\cdot_Y \cup i^* i_* D^\cdot_Y \cup \delta_+ D^\cdot_Y \xrightarrow{(\rho'_-, PV \cdot i_* \sigma_1 \cdot i_* \tau'_X, \rho'_+)^T} D^\cdot_Y$$

is the identity map. Here $q : D^\cdot_Y \to i_* i^* D^\cdot_Y$ is the natural map and PV is the Poincaré-Verdier duality map. This reduces to verifying the identity $PV \cdot i_* \sigma_1 \cdot i_* \tau'_X + \gamma'_X \cdot p' = 1$. However $PV \cdot i_* \sigma_1 = i'$ so we must show $i' \cdot \tau'_X + \gamma'_X \cdot p' = 1$ which is true since τ'_X and ρ'_X are complimentary splittings.

The functors ∂_- and ∂_+ defined above determine homomorphisms

$$\partial_- = \partial_+ : L^n(X \times \mathbb{R}; R) \to L^{n-1}(X; R), \quad n > 0.$$

These maps are isomorphisms. Inverses are given by the cylinder construction

$$c : L^{n-1}(X; R) \to L^n(X \times \mathbb{R}; R), \quad n > 0,$$

which we now describe.

Let X and Y be spaces. Then there are natural maps

$$D^\cdot_X \hat{\otimes} D^\cdot_Y \underset{\nu}{\overset{\mu}{\rightleftarrows}} D^\cdot_{X \times Y},$$

of sheaves on $X \times Y$. Let $(A^\cdot_1, \phi_1)$ and $(A^\cdot_2, \phi_2)$ be symmetric Poincaré sheaves on X and Y of dimension m and n respectively. Define their product symmetric Poincaré sheaf $(A^\cdot_1 \hat{\otimes} A^\cdot_2, \phi_1 \hat{\otimes} \phi_2)$ on $X \times Y$ by setting $\phi_1 \hat{\otimes} \phi_2$ to be the composition

$$\begin{aligned} W \stackrel{\Delta}{\to} W \otimes W \stackrel{\phi_1 \otimes \phi_2}{\to} & \mathrm{Hom}^\cdot(A^\cdot_1 \otimes A^\cdot_1, D^\cdot_X) \otimes \mathrm{Hom}^\cdot(A^\cdot_2 \otimes A^\cdot_2, D^\cdot_Y) \\ \stackrel{\hat{}}{\to} & \mathrm{Hom}^\cdot((A^\cdot_1 \otimes A^\cdot_1) \hat{\otimes} (A^\cdot_2 \otimes A^\cdot_2), D^\cdot_X \hat{\otimes} D^\cdot_Y) \\ \stackrel{\mu_*}{\to} & \mathrm{Hom}^\cdot((A^\cdot_1 \hat{\otimes} A^\cdot_2) \otimes (A^\cdot_1 \otimes A^\cdot_2), D^\cdot_{X \times Y}). \end{aligned}$$

Let $(B^{\cdot},\phi)$ be an $(n-1)$-dimensional symmetric Poincaré sheaf on X. Let $(R_{\mathbb{R}},\phi_{\mathbb{R}})$ be the 1-dimensional symmetric Poincaré pair on $\mathbb{R}$ given by

$$\phi^0_{\mathbb{R}} : R_{\mathbb{R}} \otimes R_{\mathbb{R}} \xrightarrow{\cdot} R_{\mathbb{R}} \xrightarrow{[\mathbb{R}]} \Sigma D^{\cdot}_{\mathbb{R}}$$

and $\phi^s_{\mathbb{R}} = 0$, $s > 0$. Define the cylinder on $(B^{\cdot},\phi)$ to be the n-dimensional symmetric Poincaré sheaf $(cB^{\cdot}, c\phi) = (B^{\cdot}\hat{\otimes}R_{\mathbb{R}}, \phi\hat{\otimes}\phi_{\mathbb{R}})$ on $X\times\mathbb{R}$. Then by construction $c\phi$ is given by

$$(c\phi)^s : (B^{\cdot}\hat{\otimes}R_{\mathbb{R}}) \otimes (B^{\cdot}\hat{\otimes}R_{\mathbb{R}}) \to (B^{\cdot}\otimes B^{\cdot})\hat{\otimes}R_{\mathbb{R}} \xrightarrow{\phi^s\hat{\otimes}[\mathbb{R}]} \Sigma^{n-1}D^{\cdot}_X\hat{\otimes}\Sigma D^{\cdot}_{\mathbb{R}} \xrightarrow{\mu} \Sigma^n D^{\cdot}_{X\times\mathbb{R}}.$$

Suppose now $(A^{\cdot},\phi)$ is an n-dimensional symmetric Poincaré sheaf on $X\times\mathbb{R}$. Then pushing the pairs $A^{\cdot}_{<0} \to i_*\partial_- A^{\cdot}$ and $A^{\cdot}_{>0} \to i_*\partial_+ A^{\cdot}$ to $-\infty$ and $+\infty$ gives a cobordism of $(A^{\cdot},\phi)$ to

$$(c\partial_- A^{\cdot} \cup A^{\cdot}_0 \cup c\partial_+ A^{\cdot}, c\partial_-\phi \cup \delta\phi \cup c\partial_+\phi)$$

obtained by gluing along the boundaries $\partial_- A^{\cdot}$ and $\partial_+ A^{\cdot}$. At this point we can either push $A^{\cdot}_0$ to $-\infty$ or $+\infty$. Thus we have

$$(c\partial_- A^{\cdot}, c\partial_-\phi) \simeq (A^{\cdot},\phi) \simeq (c\partial_+ A^{\cdot}, c\partial_+\phi).$$

In particular $c\cdot\partial_- = c\cdot\partial_+$ determines the identity map on $L^n(X\times\mathbb{R};R)$. It is easily shown $\partial_-\cdot c = \partial_+\cdot c$ is the identity on $L^{n-1}(X;\mathbb{R})$. We have shown

Corollary 4.8. *Let $(A^{\cdot},\phi)$ be an n-dimensional symmetric Poincaré sheaf on $Y = X\times\mathbb{R}$. Then $(A^{\cdot},\phi)$ is Poincaré cobordant to the union of Poincaré pairs on (Y_-,Y_0) and (Y_+,Y_0). Furthermore the cobordism may be taken to be stationary in an arbitrarily small neighbourhood of $Y_0 = X$.*

Proof. We may cobord $(A^{\cdot},\phi)$ so that over a neighbourhood of X it is the cylinder on a $(n-1)$-dimensional symmetric Poincaré sheaf. Proposition 4.1 applied to such a cylinder leads to Poincaré pairs.

Corollary 4.9. *The rational Pontrjagin classes and $\mathcal{L}$-genus of a PL-manifold are topological invariants.*

Proof. Let $h : M'^m \to M^m$ be a homeomorphism of PL-manifolds. Let $N^{4k} \subset M^m\times\mathbb{R}^j$, ($j$ large), be a simply-connected PL-submanifold with trivial normal bundle. Let $g_0 : W' \to N\times\mathbb{R}^i$ be the restriction of $h\times 1 : M'\times\mathbb{R}^j \to M\times\mathbb{R}^j$ to the codimension 0 submanifold $N\times\mathbb{R}^i$, $i = m+j-4k$, so that g_0 is a homeomorphism of open PL-manifolds. Choose a proper homotopy from g_0 to a PL-map g_1 such that g_1 is PL-transverse to $N \subset N\times\mathbb{R}^i$. Let $g_{1|} : N' \to N$ be the induced degree one map of manifolds. Since the rational Pontrjagin classes and $\mathcal{L}$-genus are detected by the signatures of submanifolds such as N it is sufficient to prove that $\sigma(N') = \sigma(N) \in L^{4k}(\mathbb{Z})$.

Consider the n-dimensional symmetric Poincaré sheaf $\mathcal{C}^{\cdot}_{g_0}$ on $N \times \mathbb{R}^i$. Since g_0 is a homeomorphism $[\mathcal{C}^{\cdot}_{g_0}] = 0 \in L^{4k+i}(N \times \mathbb{R}^i)$. Now g_1 is proper homotopic to g_0 so that $[\mathcal{C}^{\cdot}_{g_0}] = [\mathcal{C}^{\cdot}_{g_1}] = 0$. Let $\partial_- : L^{4k+i}(N \times \mathbb{R}^i) \to L^{4k}(N)$ be the splitting isomorphism. Since g_1 is transverse to N it follows that $\partial_-[\mathcal{C}^{\cdot}_{g_1}] = [\mathcal{C}^{\cdot}_{g_{1|}}] \in L^{4k}(N)$, where $g_{1|} : N' \to N$ is the restriction of g_1 to N. Thus $[\mathcal{C}^{\cdot}_{g_{1|}}] = 0 \in L^{4k}(N)$ and so its assembly in $L^{4k}(\mathbb{Z})$ is also zero. But by theorem 3.5 the assembly is just $\sigma(N') - \sigma(N) \in L^{4k}(\mathbb{Z})$. Hence $\sigma(N') = \sigma(N) \in L^{4k}(\mathbb{Z})$.

The original proof of Novikov of the topological invariance of the rational Pontrjagin classes used the Bass-Heller-Swan calculation $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}^i]) = 0$, $i \geq 0$, to show that a homeomorphism $M \times \mathbb{R}^i \to W$ of PL-manifolds determines a PL-homeomorphism $M' \times \mathbb{R}^i \to W$, in the case $\pi_1(M) = 0$. Instead we use the weaker transversality properties of sheaves to prove invariance, thus avoiding the need for Bass-Heller-Swan.

5. The L-homology groups.

The symmetric {quadratic} L-group $L^n(X;R)$ $\{L_n(X;R)\}$ of a space X is by definition the group of cobordism classes of n-dimensional symmetric {quadratic} Poincaré complexes of sheaves of R-modules on X. A proper map $f : X \to Y$ determines a homomorphism

$$f_* : L^n(X;R) \to L^n(Y;R)$$
$$\{ \quad f_* : L_n(X;R) \to L_n(Y;R) \quad \}$$

depending only on the proper homotopy class of f. We shall omit further mention of the ring R in the notation.

Let (X, Y) be a pair with Y closed in X. Define the relative symmetric {quadratic} L-groups $L^{n+1}(X,Y)$ $\{L_{n+1}(X,Y)\}$ of the pair (X,Y) to be the group of cobordism classes of $(n+1)$-dimensional symmetric {quadratic} Poincaré pairs over (X,Y) designed to fit into the long exact sequence of L-groups

$$\cdots \to L^{n+1}(X,Y) \xrightarrow{\partial} L^n(Y) \xrightarrow{j_*} L^n(X) \xrightarrow{i_*} L^n(X,Y) \to \ldots$$

$$\{ \quad \cdots \to L_{n+1}(X,Y) \xrightarrow{\partial} L_n(Y) \xrightarrow{j_*} L_n(X) \xrightarrow{i_*} L_n(X,Y) \to \ldots \quad \}$$

Thus a null-cobordism of an $(n+1)$-dimensional symmetric Poincaré pair

$$(\beta : D^{\cdot} \to C^{\cdot}, (\delta\phi, \phi))$$

over (X,Y) consists of an $(n+1)$-dimensional symmetric Poincaré pair

$$(B^{\cdot} \to C^{\cdot}, (\omega, \phi))$$

over Y and an $(n+2)$-dimensional symmetric Poincaré pair

$$(Q^{\cdot} \to i_* B^{\cdot} \oplus_{i_* C^{\cdot}} D^{\cdot}, (\Omega, i_*\omega \oplus -\delta\phi))$$

over (X,Y). Similarly for a quadratic pair.

Let $(X;X_1,X_2)$ be a triad with $X_0 = X_1 \cap X_2 \neq \emptyset$. Let $i_j : X_j \subset X$ and $i_0 : X_0 \subset X$ be the inclusion maps. Suppose we are given n-dimensional symmetric pairs

$$(\beta_i : D_i^\cdot \to C^\cdot, (\delta\phi_i, \phi_0)) \quad \text{over } (X_i, X_0), \quad i = 1,2.$$

We may assume the β_i are onto or else replace with mapping cylinders. Then consider the diagram $\mathcal{F}$ of sheaves over X

$$\mathcal{F} = \{i_1 D_1^\cdot \to i_0 C^\cdot \leftarrow i_2 D_2^\cdot\}$$

writing $i = i_*$ for the direct image functor. Interchanging the functor $W^\%$ with limits gives a chain map

$$\Phi_{\mathcal{F}} : \varinjlim W^\% \mathcal{F} \to W^\% \varprojlim \mathcal{F}$$

where

$$W^\% \mathcal{F} = \{W^\% i_1 D_1^\cdot \longleftarrow W^\% i_0 C^\cdot \longrightarrow W^\% i_2 D_2^\cdot\}$$

The symmetric pairs $(\beta_i : D_i^\cdot \to C^\cdot, (\delta\phi_i, \phi_0))$ determine a cycle

$$(\delta\phi_1, \delta\phi_2) \in (\varinjlim W^\% \mathcal{F})_{-n-1}.$$

Definition 5.1. The union of symmetric complexes

$$(D_1^\cdot \cup_{C^\cdot} D_2^\cdot, \delta\phi_1 \cup_{\phi_0} \delta\phi_2)$$

is defined to be the $(n+1)$-dimensional symmetric complex

$$(W^\% \varinjlim \mathcal{F}, \Phi_{\mathcal{F}}(\delta\phi_1, \delta\phi_2))$$

over X.

A Poincaré splitting of an n-dimensional symmetric complex of sheaves $(A^\cdot, \phi)$ on X consists of Poincaré pairs of sheaves

$$(\beta_i : D_i^\cdot \to C^\cdot, (\delta\phi_i, \phi_0)) \quad \text{over } (X_i, X_0), \quad i = 1,2.$$

together with a quasi-isomorphism of symmetric complexes of sheaves

$$(A^\cdot, \phi) \longrightarrow (D_1^\cdot \cup_{C^\cdot} D_2^\cdot, \delta\phi_1 \cup_{\phi_0} \delta\phi_2).$$

As shown in section 4 any Poincaré sheaf over $Y = X \times \mathbb{R}$ is cobordant to a sheaf admitting such a splitting over (Y_-, Y_0) and (Y_+, Y_0).

For a pair (X,Y) with Y closed in X we say a set $U \subset Y$, open in X, is strongly contained in Y if there exists a triad decomposition $(X; X_+, X_-)$ of X with $U \subset X_- \subset \text{int}Y$ and X_0 bicollared in $Y - U$. Write $Y_\pm = X_\pm \cap Y, Y_0 = X_0 \cap Y = X_0$.

Theorem 5.2. *(Excision) Let the open set U in X be strongly contained in Y. Then the inclusion map $(X-U, Y-U) \subset (X, Y)$ induces an isomorphism*

$$\Phi : L^n(X-U, Y-U) \to L^n(X, Y).$$

Similarly for quadratic.

Proof. We verify Φ is onto. Let $(\beta : D^{\cdot} \to C^{\cdot}, (\delta\phi, \phi))$ be an n-dimensional symmetric Poincaré pair over (X, Y). Construct complexes $(D^{\cdot}_{\pm}, \delta\phi_{\pm})$ over $X_{\pm}$, $(C^{\cdot}_{\pm}, \phi_{\pm})$ over $Y_{\pm}$ and $(D^{\cdot}_0, \delta\phi_0), (C^{\cdot}_0, \phi_0)$ over X_0 by splitting β along X_0. This is achieved by first deforming β over a neighbourhood of X_0 so that it is the cylinder on a Poincaré pair using the techniques of section 4. Then the pair

$$((\beta_+, \theta_+) : D^{\cdot}_+ \to C^{\cdot}_+ \cup_{C^{\cdot}_0} i^+_0 D^{\cdot}_0, (\delta\phi_+, \phi_+ \cup_{\phi_0} i^+_0 \delta\phi_0)$$

is an n-dimensional symmetric Poincaré pair over (X_+, Y_+) and hence $(X-U, Y-U)$, extending by zero. The image of this pair under the map Φ is obtained again by extending by zero over (X, Y). The $(n-1)$-dimensional symmetric Poincaré complex

$$(C^{\cdot}_+ \cup_{C^{\cdot}_0} i_* D^{\cdot}_0,\ \phi_+ \cup_{\phi_0} i_* \delta\phi_0)$$

over Y is cobordant (over Y) to $(C^{\cdot}, \phi)$ via the cobordism

$$((0, \theta_-, 0, \beta_-) : D^{\cdot}_- \to (C^{\cdot}_+ \cup_{C_0} i_* D^{\cdot}_0) \oplus (C^{\cdot}_+ \cup_{C_0} C^-), (\delta\phi_-, (\phi_+ \cup_{\phi_0} \delta\phi_0) \oplus (\phi_+ \cup_{\phi_0} \phi_-))$$

which is equivalent to

$$(D^{\cdot}_- \to (C^{\cdot}_+ \cup_{C_0} i_* D^{\cdot}_0) \oplus C^{\cdot}, (\delta\phi_-, (\phi_+ \cup_{\phi_0} \delta\phi_0) \oplus \phi)).$$

Finally the n-dimensional symmetric complex

$$(D^{\cdot}_+ \cup_{(C^{\cdot}_+ \cup_{C^{\cdot}_0}) i_* D^{\cdot}_0} D^{\cdot}_- \cup_{C^{\cdot}} D^{\cdot}, (\delta\phi_+ \cup \delta\phi_- \cup \delta\phi))$$

is the boundary of the cylinder on $(\beta : D^{\cdot} \to C^{\cdot}, (\delta\phi, \phi))$. This shows Φ is onto. The verification that Φ is into is similar. ∎

We conclude that the functors $L^*(-; R)$ (and $L_*(-; R)$) are a homology theory on the category of locally compact topological spaces and proper maps.

6. Comparison with Simplicial Stacks.

Let K be a locally finite simplicial complex. In this section we identify the simplicial $\mathbb{L}$-homology groups of K [10] with the sheaf theoretic L-homology groups of the geometric realisation $|K|$ constructed above. In particular there is a functor of categories with weak chain duality

$$\left\{ \begin{array}{c} \text{stacks on } K \\ \mathit{Hom}^{\cdot}(-, \Delta^{\cdot}_X) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{sheaves on } |K| \\ \mathit{Hom}^{\cdot}(-, D^{\cdot}_X) \end{array} \right\}$$

inducing an isomorphism on L-groups.

The category with weak chain duality of stacks on a simplicial complex was studied in [10] and is the basic ingredient for the $\mathbb{L}$-homology groups of a simplicial complex. (Actually the stacks considered in [10] have free stalks. In comparing with sheaves we work with perfect stalks.) For a locally finite simplicial complex K let $\mathcal{K}$ be the category with objects simplices of K and maps $\sigma \to \tau$ face relations $\sigma \leq \tau$. The category of stacks $\mathrm{Sh}(K)$ is the functor category of covariant functors $M : \mathcal{K} \to R$-mod. A cochain complex of stacks $M^\cdot$ is perfect if $M^\cdot(\sigma)$ is perfect for each $\sigma \in K$. The category $\mathrm{Sh}(K)^\cdot$ is the category of (bounded) perfect cochain complexes of stacks on K. For stacks $M^\cdot$, $N^\cdot$ define the stack $\mathit{Hom}^\cdot(M^\cdot, N^\cdot)$ by

$$\mathit{Hom}^\cdot(M^\cdot, N^\cdot)(\sigma) = \mathrm{Hom}^\cdot(M^\cdot_{|\mathrm{st}(\sigma)}, N^\cdot_{|\mathrm{st}(\sigma)}), \quad \sigma \in K,$$

where $M^\cdot_{|\mathrm{st}(\sigma)}$ is the restriction of $M^\cdot$ to the full subcategory of $\mathcal{K}$ with objects in the open star $\mathrm{st}(\sigma)$.

Example 6.1. The stack of local simplicial chains $\Delta^\cdot_K \in \mathrm{Sh}(K)^\cdot$ is given by

$$\Delta^\cdot_X(\sigma) = \Delta(K, K - \mathrm{st}(\sigma))$$

where we index so that $\Delta^\cdot_X$ is a cochain complex concentrated in non-positive degrees.

In order to deal with perfect stacks (rather than just stacks with free stalks) it is necessary to replace $\Delta^\cdot_X$ by the stack $\Delta^\cdot_X \otimes_R I^\cdot$ where $I^\cdot$ is a fixed injective resolution of R of finite length. We shall again denote the latter stack $\Delta^\cdot_X$ (and only make reference in future to this stack).

For $M^\cdot \in \mathrm{Sh}(K)^\cdot$ define the dual stack $DM^\cdot \in \mathrm{Sh}(K)^\cdot$ by

$$DM^\cdot = \mathit{Hom}^\cdot(M^\cdot, \Delta^\cdot_X).$$

As usual there is a natural transformation $e : 1 \to D^2$.

Theorem 6.2. *The triple $(Sh(K)^\cdot, D, e)$ is a category with weak chain duality.*

Proof We must prove that D is exact and for any $M^\cdot \in \mathrm{Sh}(K)^\cdot$ the natural map $e(M^\cdot)$ is a quasi-isomorphism (that is, a quasi-isomorphism on each stalk.)

To show that D is exact consider the following (equivalent) construction of the dual of a stack. For $M \in \mathrm{Sh}(K)$ define $D'M$ by

$$(D'M)^r(\sigma) = \Sigma_{\tau \geq \sigma} \mathrm{Hom}^\cdot(M[\tau], I^\cdot)^{r+|\tau|}$$

with boundary map $d : (D'M)^r(\sigma) \to (D'M)^{r+1}(\sigma)$ given by

$$d(f) = d_{I^\cdot} \cdot f + (-)^r \sum_{\tau \geq \partial_i \tau \geq \sigma} (-)^i f \cdot \phi_{\partial_i \tau, \tau} \in \sum_{\tau' \geq \sigma} \mathrm{Hom}^\cdot(M[\tau'], I^\cdot)^{r+1+|\tau'|}$$

for $f \in \mathrm{Hom}^{\cdot}(M[\tau], I^{r+|\tau|})$, $\tau \geq \sigma$. (Here $\partial_i\tau$ is the i-th face of τ and $\phi_{\partial_i\tau,\tau}$: $M(\partial_i\tau) \to M(\tau)$ the structure map.) For $\sigma' \leq \sigma$ the structure map

$$(D'M)^r(\sigma') \to (D'M)^r(\sigma)$$

is the projection

$$\sum_{\tau' \geq \sigma'} \mathrm{Hom}^{\cdot}(M[\tau'], I^{\cdot})^{r+|\tau'|} \to \sum_{\tau \geq \sigma} \mathrm{Hom}^{\cdot}(M[\tau], I^{\cdot})^{r+|\tau'|}$$

This defines D' on $\mathrm{Sh}(K)$ and there is a canonical extension to $\mathrm{Sh}(K)^{\cdot}$. Clearly D' is an exact functor. But for any $M \in \mathrm{Sh}(K)$ there is a natural isomorphism

$$D'M \longrightarrow Hom^{\cdot}(M, \Delta_X^{\cdot}).$$

To see this define a natural transformation

$$\Phi_M : D'M \to Hom^{\cdot}(M, \Delta_K^{\cdot})$$

on $\sigma \in K$ by

$$\Phi_M^r(\sigma) : \sum_{\sigma' \geq \sigma} \mathrm{Hom}^{\cdot}(M(\sigma'), I^{\cdot})^{r+|\sigma'|} \to \mathrm{Hom}^r(M_{|\mathrm{st}(\sigma)}, (\Delta_K^{\cdot})_{|\mathrm{st}(\sigma)})$$

by

$$\Phi_M^r(\sigma)(\sum_{\sigma' \geq \sigma} f_{\sigma'})(\tau) = (f_{\tau'} \cdot \phi_{\tau,\tau'})_{\tau' \geq \tau} : M[\tau] \to \sum_{\tau' \geq \tau} I^{r+|\tau'|}$$

for $\tau \geq \sigma$.

Define an inverse to Φ_M

$$\Psi_M : Hom^{\cdot}(M^{\cdot}, \Delta_K^{\cdot}) \to D'M$$

as follows. On $\sigma \in K$

$$\Psi_M^r(\sigma) : \mathrm{Hom}^r(M_{|\mathrm{st}(\sigma)}, (\Delta_K^{\cdot})_{|\mathrm{st}(\sigma)}) \to \sum_{\sigma' \geq \sigma} \mathrm{Hom}^{\cdot}(M(\sigma'), I^{\cdot})^{r+|\sigma'|}$$

is given by

$$\Psi_M^r(\sigma)(\{g_{\sigma'}\}_{\sigma' \geq \sigma}) = \sum_{\sigma' \geq \sigma} (g_{\sigma'})_{\sigma'}$$

where the maps

$$g_{\sigma'} : M(\sigma') \to \sum_{\tau' \geq \sigma'} I^{r+|\tau'|}, \quad \sigma' \geq \sigma$$

should be thought of as a natural transformation

$$\{g_{\sigma'}\}_{\sigma' \geq \sigma} : M_{|\mathrm{st}(\sigma)} \to (\Delta_K^r)_{|st(\sigma)}$$

and $(g_{\sigma'})_{\sigma'}$ is the composition of $g_{\sigma'}$ with the projection to the σ' component. It follows that $\Delta^{\cdot}_X$ is a complex of injective objects in $\mathrm{Sh}(K)^{\cdot}$.

Finally we show $e(M^{\cdot})$ is a quasi-isomorphism. It will be sufficient to demonstrate this for $M \in \mathrm{Sh}(K)$. For $\sigma \in K$ and $\tau \in \mathrm{st}(\sigma)$ let $C(\tau,\sigma) = \Delta(\tau, \mathrm{link}_\tau(\sigma))$. Then for $\tau > \sigma$, the complex $C(\tau,\sigma)$ is acyclic and $C(\sigma,\sigma) = R$. Applying D twice we obtain from the definitions

$$(D^2M)(\sigma) = \sum_{\tau\geq\sigma} C(\tau,\sigma) \otimes_R \ \mathrm{Hom}^{\cdot}(\ \mathrm{Hom}^{\cdot}(M(\tau), I^{\cdot}), I^{\cdot}).$$

Define the transformation e with $e(M)(\sigma) : M(\sigma) \to (D^2M)(\sigma)$ given by the composition

$$M(\sigma) \xrightarrow{i_\sigma} \sum_{\tau\geq\sigma} C(\tau,\sigma) \otimes_R M[\tau] \xrightarrow{1\otimes e} \sum_{\tau\geq\sigma} C(\tau,\sigma) \otimes_R \ \mathrm{Hom}^{\cdot}(\ \mathrm{Hom}^{\cdot}[M(\tau), I^{\cdot})I^{\cdot})$$

where e is the natural transformation of R-modules. It follows that e is a quasi-isomorphism for perfect stacks. ■

Let $|K|$ denote the geometric realisation of K. Let $\mathrm{Sh}_K(|K|)$ denote the full subcategory of $\mathrm{Sh}(|K|)$ consisting of sheaves on $|K|$ which are constant on the interior of each simplex. Define

$$Q : \mathrm{Sh}(K) \longrightarrow \mathrm{Sh}_K(|K|)$$

taking a stack M to the sheaf QM on $|K|$ where QM is the constant sheaf $M(\sigma)$ on the interior of the simplex $\sigma \in K$. Sections are 0-cocycles in the cochain complex $C(K;M)^{\cdot}$ of K with coefficients in M. Define

$$P : \mathrm{Sh}(|K|) \to \mathrm{Sh}(K)$$

taking a sheaf A on $|K|$ to the stack PA with $PA(\sigma) = A(\mathrm{st}(\sigma))$, $\sigma \in K$. Structure maps are induced by restriction of sections. It is straightforward to show there is an equivalence of categories

$$\mathrm{Sh}(K) \underset{P}{\overset{Q}{\rightleftarrows}} \mathrm{Sh}_K(|K|).$$

Both functors Q and P commute with *Hom* in the obvious way and preserve perfect complexes. Thus we have

Theorem 6.3. *There is an equivalence of categories with weak chain duality*

$$(Sh(K)^{\cdot}, Hom^{\cdot}(-, \Delta^{\cdot}_X), e) \underset{P}{\overset{Q}{\rightleftarrows}} (Sh_K(|K|)^{\cdot}, Hom^{\cdot}(-, Q\Delta^{\cdot}_X), e).$$

Note also that $Q\Delta^{\cdot}_X$ is a complex of injective objects in the category $\mathrm{Sh}_K(|K|)^{\cdot}$.

Let $F : \mathrm{Sh}_K(|K|)^{\cdot} \to \mathrm{Sh}(|K|)$ be the inclusion functor. Then F admits a right adjoint G, [6].

Lemma 6.4. *There is a natural quasi-isomorphism*

$$FQ\Delta^{\cdot}_K \longrightarrow D^{\cdot}_{|K|}.$$

Proof Let $X = |K|$. Let $\mathfrak{S}^{\cdot}_X$ be the cosheaf on X obtained from the precosheaf $U \mapsto C(U)^{\cdot}$ of singular chains in U by taking the direct limit along subdivisions, [1;V,1]. Then $\mathfrak{S}^{\cdot}_X$ is a flabby coresolution of the constant cosheaf so that the sheaf $Hom^{\cdot}(\mathfrak{S}^{\cdot}_X, I^{\cdot})$ is a flabby resolution of R_X. It follows there is a quasi-isomorphism of injective sheaves

$$\mathcal{S}^{\cdot}_X = Hom^{\cdot}(\Gamma_c Hom^{\cdot}(\mathfrak{S}^{\cdot}_X, I^{\cdot}), I^{\cdot}) \to D^{\cdot}_X = Hom^{\cdot}(\Gamma_c \mathcal{C}^{\cdot}_X, I^{\cdot}).$$

Let $C^{\cdot}_X$ be the sheaf of local singular chains. Then there is a quasi-isomorphism [1;V,2]

$$C^{\cdot}_X \longrightarrow \mathcal{S}^{\cdot}_X.$$

But we also have the natural quasi-isomorphism

$$FQ\Delta^{\cdot}_K \longrightarrow C^{\cdot}_X$$

obtained from the natural map from simplicial chains to singular chains. The lemma follows. ■

By the universal properties of the functor G this quasi-isomorphism factors as

$$FQ\Delta^{\cdot}_K \longrightarrow GD^{\cdot}_X \xrightarrow{s} D^{\cdot}_X$$

where $s : GD^{\cdot}_X \to D^{\cdot}_X$ is the natural map.

Lemma 6.5. *The map $s : GD^{\cdot}_X \to D^{\cdot}_X$ is a quasi-isomorphism.*

Proof By [6;8.1.9] it is sufficient to show that the derived sheaf of $D^{\cdot}_X$ is constant on the interior of simplices, ie. $D^{\cdot}_X$ is weakly K-constructible. Fix $\sigma \in K$. For $i \geq |\sigma|$ let $U^i \subset st(\sigma)$ be the union of the interiors of the i-simplices of $st(\sigma)$. Let $n = n(\sigma)$ be the smallest integer such that $U^r = \emptyset,\ r > n$. For $|\sigma \ \leq k \leq n$ set

$$st(\sigma)_k = \bigcup_{k \leq i \leq n} U^i$$

so that

$$st(\sigma)_n \subset st(\sigma)_{n-1} \subset \cdots \subset st(\sigma)_{|\sigma|} = st(\sigma),$$

and $st(\sigma)_k$ is open in $st(\sigma)_{k-1},\ |\sigma| < k \leq n$. The derived sheaf of $D^{\cdot}_{|K|}$ is constant on $st(\sigma)_n$ since $st(\sigma)_n \subset st(\sigma)$ is open and consists of a disjoint union of the interiors of n-simplices. Assume by induction that $D^{\cdot}_{|k|}$ has constant derived sheaf when restricted to the interiors of simplices in $st(\sigma)_k$, for some $|\sigma| < k \leq n$. The inclusions

$$st(\sigma)_k \xrightarrow{i_k} st(\sigma)_{k-1} \xleftarrow{j_k} st(\sigma)_{k-1} \setminus st(\sigma)_k$$

determine a distinguished triangle

$$j_{k!}j_k^! D^{\cdot}_{st(\sigma)_{k-1}} \to D^{\cdot}_{st(\sigma)_{k-1}} to i_{k_*} D^{\cdot}_{st(\sigma)_k} \to \dots .$$

But $j_k^! D^{\cdot}_{st(\sigma)_{k-1}} = D^{\cdot}_{st(\sigma)_{k-1}\setminus st(\sigma)_k}$ which is a disjoint union of interiors $(k-1)$-implies and so has constant derived sheaf. Hence $j_{k!}j_k^! D^{\cdot}_{st(\sigma)_{k-1}}$ also has constant derived sheaf on interiors. By induction this is also the case for $i_{k_*} D^{\cdot}_{st(\sigma)_k}$. It follows that $D^{\cdot}_{st(\sigma)_{k-1}}$ has constant derived sheaf on the interior of simplices. By induction $D^{\cdot}_{|K|}$ is weakly K-constructible on $st(\sigma)$ for all σ, and hence on $|K|$. ∎

We conclude there is a natural quasi-isomorphism in $\mathrm{Sh}_K(|K|)$

$$t : Q\Delta^{\cdot}_K \longrightarrow GD^{\cdot}_{|K|}.$$

It follows from the adjointness properties of G that $GD^{\cdot}_{|K|}$ is a complex of injective objects in $\mathrm{Sh}_K(|K|)$ so that t is a chain equivalence. Suppose then $A^{\cdot} \in \mathrm{Sh}_K(|K|)$. Then there is a chain equivalence

$$Hom^{\cdot}(A^{\cdot}, Q\Delta^{\cdot}_K) \xrightarrow{t_*} Hom^{\cdot}(A^{\cdot}, GD^{\cdot}_{|K|})$$

and applying F we obtain the quasi-isomorphism

$$F Hom^{\cdot}(A^{\cdot}, Q\Delta^{\cdot}_K) \longrightarrow F Hom^{\cdot}(A^{\cdot}, GD^{\cdot}_{|K|}) \cong Hom^{\cdot}(FA^{\cdot}, D^{\cdot}_{|K|}),$$

the latter map from the adjoint relation. In conclusion we have

Lemma. *The inclusion functor determines a functor of categories with weak chain duality*

$$(Sh_K(|K|)^{\cdot}, Hom^{\cdot}(-, Q\Delta^{\cdot}_K), e) \longrightarrow (Sh(|K|)^{\cdot}, Hom^{\cdot}(-, D^{\cdot}_{|K|}), e).$$

This functor induces a natural transformation on L-groups which is an isomorphism for K a point. Thus

Corollary 6.6. *There are natural isomorphisms*

$$\begin{array}{c} H_*(K; \mathbb{L}^{\cdot}) \longrightarrow L^*(|K|) \\ \{ \quad H_*(K; \mathbb{L}_{\cdot}) \longrightarrow L_*(|K|) \quad \} \end{array}$$

where $\mathbb{L}^{\cdot}$ $\{\mathbb{L}_{\cdot}\}$ *is the projective symmetric* $\{$*quadratic*$\}$ $\mathbb{L}$*-spectrum.*

7. Appendix: Weak Chain Duality.

We recall the definition of a category with chain duality [8]. Such an enriched category is the natural setting for abstract algebraic Poincaré complexes. We give the definition in terms of cochain complexes in accordance with standard practise in sheaf theory.

Let $\mathbb{A}$ be an additive category and $\mathbb{B}(\mathbb{A})$ the category of bounded cochain complexes $A^\cdot$ of objects in $\mathbb{A}$ (so that $A^i = 0$ for all but a finite number of i) with morphisms chain maps. Let $D : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ be a given additive contravariant functor. Then there is determined an additive contravariant extension $D : \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A})$ taking a complex $A^\cdot$ of $\mathbb{B}(\mathbb{A})$ to the total complex of the bicomplex with $(i,j)^{th}$ component $(DA^{-i})^j$. We have (cf. [8])

Definition 7.1 An additive category $\mathbb{A}$ with chain duality (D,e) consists of an additive contravariant functor $D : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ together with a natural transformation $e : 1 \to D^2$ such that for all A in $\mathbb{A}$

1. $D(e(A)) \cdot e(DA) = 1_{DA} : DA \to D^3A \to DA$
2. $e(A) : A \to D^2A$ is a chain homotopy equivalence.

The main example is the category $\mathbb{M}_p(R)$ of finitely generated projective modules over a commutative ring R with chain duality

$$D : \mathbb{M}_p(R) \to \mathbb{M}_p(R) \subset \mathbb{B}(\mathbb{M}_p(R)); P \longmapsto P^* = \mathrm{Hom}_R(P, R),$$

and

$$e(P) : P \to P^{**}$$

the usual isomorphism.

More generally let K be a locally finite simplicial complex regarded as a category with one object for each simplex $\sigma \in K$ and one morphism for each face relation $\sigma \leq \tau$. Then the (stack) category $\mathbb{M}_p(R)[K]$ of covariant functors $P : K \to \mathbb{M}_p(R)$ and natural transformations may be given the structure of a category with chain duality, (see [8] for details.)

Given a category with chain duality Ranicki defines algebraic Poincaré complexes of objects in the category and cobordism groups of such objects. For the first example $\mathbb{M}_p(R)$ the cobordism groups are the (projective) algebraic L-groups of the ring R [8] while in the second the cobordism groups are the homology groups of the simplicial complex K with algebraic $\mathbb{L}$-spectra coefficients [8].

When we consider the category of sheaves over a topological space the following definition of a category with weak chain duality is more useful.

Definition 7.2 An abelian category $\mathbb{A}$ with weak chain duality (D, e) consists of an exact contravariant functor $D : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ together with a natural transformation $e : 1 \to D^2$ such that for all A in $\mathbb{A}$

1. $D(e(A)) \cdot e(DA) = 1_{DA} : DA \to D^3A \to DA$
2. $e(A) : A \to D^2A$ is a quasi-isomorphism (=homology isomorphism).

The main example is the category $\mathbb{M}(R)$ of finitely generated modules over a ring R of finite cohomological dimension with chain duality

$$D : \mathbb{M}(R) \to \mathbb{B}(\mathbb{M}(R)); M \longmapsto DM = \operatorname{Hom}^{\cdot}(M, I^{\cdot}),$$

where $I^{\cdot}$ is an injective resolution of R of finite length and $\operatorname{Hom}^{\cdot}$ is the mapping chain complex. Again $e(M) : M \to D^2 M$ is the usual map which in this case is a quasi-isomorphism. We shall see that the inclusion $R \to I^{\cdot}$ induces an isomorphism of L-groups.

As in [8] we may define the symmetric (quadratic) Q-groups of a complex in $\mathbb{B}(\mathbb{A})$ for $\mathbb{A}$ a category with weak chain duality. Because we consider cochain complexes here instead of chain complexes in [8] we give details.

For a complex $A^{\cdot}$ in $\mathbb{B}(\mathbb{A})$ and $n \in \mathbb{Z}$ let $\Sigma^n A^{\cdot}$ be the n^{th} suspension complex with

$$(\Sigma^n A^{\cdot})^i = A^{i-n},$$

$$d_{\Sigma^n A^{\cdot}} = (-)^n d_{A^{\cdot}} : (\Sigma^n A^{\cdot})^i = A^{i-n} \to (\Sigma^n A^{\cdot})^{i+1} = A^{i+1-n}.$$

For complexes $A^{\cdot}, B^{\cdot}$ in $\mathbb{B}(\mathbb{A})$ let $\operatorname{Hom}^{\cdot}(A^{\cdot}, B^{\cdot})$ be the mapping complex of abelian groups with elements $f \in \operatorname{Hom}^i(A^{\cdot}, B^{\cdot})$ degree i maps $f : A^{\cdot} \to B^{\cdot}$ of graded objects and differential given by $df = d_{B^{\cdot}} \cdot f + (-)^{i+1} f \cdot d_{A^{\cdot}}$.

Let $\mathbb{Z}[\mathbb{Z}_2]$ be the group ring with $\mathbb{Z}_2$ the multiplicative group of order 2 generated by $T \in \mathbb{Z}_2$. Let $f \in \operatorname{Hom}^i(A^{\cdot}, DB^{\cdot})$. Define the element $Tf \in \operatorname{Hom}^i(B^{\cdot}, DA^{\cdot})$ as follows. The element f is by definition a collection of degree i maps $\{f_j : A^j \to (DB^{\cdot})^{j+i}\}_{j \in \mathbb{Z}}$. Applying the functor D gives a collection of maps

$$\{Df_j : \sum_{q-p=j+1} D((DA^p)^q) \longrightarrow DA^j\}_{j \in \mathbb{Z}}.$$

Form the new collection

$$\left\{ \sum_{a-b=k+i} (-)^{kb} Df_b : \sum_{a-b=k+i} \; \sum_{q-p=b+i} D((DA^p)^q)^a \longrightarrow \sum_{a-b=k+i} (DA^b)^a \right\}_{k \in \mathbb{Z}}$$

and write this as a collection of maps

$$\left\{(D^2 B^{\cdot})^k \longrightarrow (DA^{\cdot})^{k+i}\right\}_{k \in \mathbb{Z}}.$$

Precomposing with the natural transformation $e(B^{\cdot}) : B^{\cdot} \to D^2 B^{\cdot}$ gives the collection of degree i maps

$$\left\{B^k \longrightarrow (DA^{\cdot})^{k+i}\right\}_{k \in \mathbb{Z}}$$

which by definition is the element $Tf \in \operatorname{Hom}^i(B^{\cdot}, DA^{\cdot})$. The mapping complex $\operatorname{Hom}^i(B^{\cdot}, DA^{\cdot})$ then becomes a complex of $\mathbb{Z}[\mathbb{Z}_2]$-modules.

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$- module $\mathbb{Z}$ regarded as a cochain complex. Thus

$$W^i = \begin{cases} \mathbb{Z}[\mathbb{Z}_2] & i \leq 0 \\ 0 & i > 0 \end{cases}$$

and
$$d_W = 1 + (-)^i T : W^i = \mathbb{Z}[\mathbb{Z}_2] \to W^{i+1} = \mathbb{Z}[\mathbb{Z}_2],\ i < 0.$$
For a complex $A^{\cdot}$ in $\mathbb{B}(\mathbb{A})$ define complexes of abelian groups

$$\begin{aligned} W^{\%}A^{\cdot} &= \mathrm{Hom}^{\cdot}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}^{\cdot}(A^{\cdot}, DA^{\cdot})) \\ W_{\%}A^{\cdot} &= W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}^{\cdot}(A^{\cdot}, DA^{\cdot}). \end{aligned}$$

A cocycle $\phi \in (W^{\%}A^{\cdot})^n$ ($\psi \in (W_{\%}A^{\cdot})^n$) consists of a collection of maps

$$\begin{gathered} \left\{\phi^s : A^{\cdot} \longrightarrow DA^{\cdot}\right\}_{s\geq 0} \text{ of degree } n-s \\ \left(\left\{\psi^s : A^{\cdot} \longrightarrow DA^{\cdot}\right\}_{s\geq 0} \text{ of degree } n+s\right) \end{gathered}$$

satisfying the relations

$$\begin{aligned} d\phi^s + (-)^{n+s+1}\phi^s d + (-)^{n+1}\phi^{s-1} + (-)^{n+s}T\phi^{s-1} = 0, \\ (d\psi^s + (-)^{n+1}\psi^s d + (-)^s\psi^{s+1} - T\psi^{s+1} = 0). \end{aligned}$$

A map of complexes $f : A^{\cdot} \to B^{\cdot}$ determines maps

$$\begin{aligned} & f^{\%} : W^{\%}B^{\cdot} \to W^{\%}A^{\cdot}; f^{\%}(\phi)^s = Df \cdot \phi^s \cdot f : A^{\cdot} \to DA^{\cdot} \\ & f_{\%} : W_{\%}B^{\cdot} \to W^{\%}A^{\cdot}; f_{\%}(\psi)^s = Df \cdot \psi^s \cdot f : A^{\cdot} \to DA^{\cdot}. \end{aligned}$$

Define the symmetric (quadratic) $Q-$groups of a complex $A^{\cdot}$, denoted $Q^n(A^{\cdot})$ ($Q_n(A^{\cdot})$) to be the abelian groups

$$\begin{aligned} Q^n(A^{\cdot}) &= H^n(W^{\%}A^{\cdot}) \equiv H_{-n}(W^{\%}A^{\cdot}) \\ (Q_n(A^{\cdot}) &= H^n(W^{\%}A^{\cdot}) \equiv H_{-n}(W_{\%}A^{\cdot})). \end{aligned}$$

An object A in $\mathbb{A}$ is said to be flat (with respect to the chain duality D) if DA is a complex of injective objects in the category $\mathbb{A}$. For example, projective objects in A are flat.

Let $f : A^{\cdot} \to B^{\cdot}$ be a quasi-isomorphism of complexes of flat objects.

A complex $A^{\cdot}$ of objects in $\mathbb{A}$ is said to be finite dimensional if it admits a resolution (=quasi-isomorphism) $P^{\cdot} \to A^{\cdot}$ where $P^{\cdot}$ is a bounded complex of projective objects in $\mathbb{A}$. Define the algebraic bordism category [8]

$$\Lambda(\mathbb{A}) = (\mathbb{A}, \mathbb{B}, \mathbb{C})$$

where $\mathbb{B} \subset \mathbb{B}(\mathbb{A})$ is the full-subcategory of finite dimensional complexes and $\mathbb{C} \subset \mathbb{B}$ is the full-subcategory of acyclic complexes. Thus by definition an $n-$dimensional symmetric (quadratic) Poincaré complex $(A^{\cdot}, \phi)$ ($(A^{\cdot}, \psi)$) in $\Lambda(\mathbb{A})$ consists of a complex $A^{\cdot}$ in $\mathbb{B}$ such that the chain map $\phi_0 : A^{\cdot} \to \Sigma^n DA^{\cdot}$ ($\psi_0 + T\psi_0 : A^{\cdot} \to \Sigma^n DA^{\cdot}$) has chain mapping cone in $\mathbb{C}$, ie. is a quasi-isomorphism. Define also the algebraic bordism category
$$\Lambda'(\mathbb{A}) = (\mathbb{A}, \mathbb{B}', \mathbb{C}')$$

where $\mathbb{B}' \subset \mathbb{B}(\mathbb{A})$ is the full subcategory of bounded complexes of projective objects and $\mathbb{C}' \subset \mathbb{B}'$ is the full-subcategory of contractible complexes.

For any algebraic bordism category $\Lambda(\mathbb{A}) = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ the $n-$dimensional symmetric (quadratic) $L-$group $L^n(\Lambda)$ $(L_n(\Lambda))$ is the abelian group of cobordism classes of $n-$dimensional symmetric (quadratic) $\mathbb{C}-$Poincaréé complexes.

Let $\mathbb{M}_p(R)$ be the category with chain duality $D'A^{\cdot} = \mathrm{Hom}^{\cdot}_R(A^{\cdot}, R)$ and $\mathbb{M}(R)$ the category with weak chain duality $DA^{\cdot} = \mathrm{Hom}^{\cdot}_R(A^{\cdot}, I^{\cdot})$. Then inclusion $\mathbb{M}_p(R) \subset \mathbb{M}(R)$ determines a functor of algebraic bordism categories

$$F : \Lambda'(\mathbb{M}_p(R)) \longrightarrow \Lambda(\mathbb{M}(R)).$$

Lemma 7.3. *The induced morphisms*

$$F : L^*(\Lambda') \longrightarrow L^*(\Lambda)$$
$$F : L_*(\Lambda') \longrightarrow L_*(\Lambda)$$

are isomorphisms.

In particular the L-groups $L^*_p(R)$ $(L^p_*(R))$ of [8] coincide with the L-groups $L^*(\Lambda)$ $(L_*(\Lambda))$ defined here. We shall write $L^*(R)$ $(L_*(R))$ for $L^*(\Lambda)$ $(L_*(\Lambda))$.

Proof. We will show $F : L^n(\Lambda') \longrightarrow L^n(\Lambda)$ is an isomorphism. The argument for the quadratic $L-$groups is entirely similar. Construct an inverse $G : L^n(\Lambda) \to L^n(\Lambda')$ by taking projective resolutions as follows. Let $(A^{\cdot}, \phi)$ be an $n-$dimensional symmetric Poincaréé complex in $\Lambda(\mathbb{M}(R))$. Let $h : P^{\cdot} \to A^{\cdot}$ be a projective resolution of finite length, and let $(P^{\cdot}, h^{\%}(\phi))$ be the induced $n-$dimensional symmetric Poincaréé complex. The inclusion $R \to I^{\cdot}$ determines a quasi-isomorphism

$$\mathrm{Hom}^{\cdot}(P^{\cdot}, \mathrm{Hom}^{\cdot}(P^{\cdot}, R)) \to \mathrm{Hom}^{\cdot}(P^{\cdot}, \mathrm{Hom}^{\cdot}(P^{\cdot}, I^{\cdot}))$$

and hence an isomorphism $Q'^n(P^{\cdot}) \longrightarrow Q^n(P^{\cdot})$ where Q' denotes the $Q-$group with respect to the duality D'. Thus we may lift the symmetric class $h^*(\phi)$ to a class $\phi' \in Q'^n(P^{\cdot})$. This class is easily seen to be $\mathbb{C}'-$Poincaré. Define $G(A^{\cdot}, \phi) = (P^{\cdot}, \phi')$. Then G is an inverse to F.

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