LOCAL INDICES OF A VECTOR FIELD AT AN ISOLATED ZERO ON THE BOUNDARY

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ABSTRACT. We define two types of local indices of a vector field at an isolated zero on the boundary, and prove Poincaré-Hopf-type index theorems for certain vector fields on a compact smooth manifold which have only isolated zeros.

1. INTRODUCTION

The famous Poincaré-Hopf theorem states that the index $\mathrm{Ind}(V)$ of a continuous tangent vector field V on a compact smooth manifold X is equal to the Euler charactersitic $\chi(X)$ of X, if V has only isolated zeros away from the boundary and V points outward on the boundary of X. If you assume that the vectors on some of the boundary components point inward and point outward on the other components, then the formula will look like:

$$\mathrm{Ind}(V) = \chi(X) - \chi(\partial_- X) ,$$

where $\partial_- X$ denotes the union of the boundary components on which the vectors point inward. This can be observed by looking at the Morse function of the pair $(X, \partial_- X)$. In [4], M. Morse relaxed the requirement on the boundary behavior and obtained a formula

$$\mathrm{Ind}(V) + \mathrm{Ind}(\partial_- V) = \chi(X) .$$

Actually the requirement that the singularities are isolated are also relaxed. This formula has been rediscovered and extended by several authors [5] [1] [2]. In this paper we consider only vector fields whose zeros are isolated. But we allow zeros on the boundary.

Let X be an n-dimensional compact smooth manifold with boundary ∂X, and fix a Riemanian metric on X. We assume $n \geq 1$. For a continuous tangent vector field V on X and a point p of its boundary, we define the vector $\partial V(p)$ to be the orthogonal projection of $V(p)$ to the tangent space of ∂X at p. The tangent vector field ∂V on ∂X is called the *boundary* of V. $\partial^\perp V$ denotes the normal vector field on ∂X defined by $\partial^\perp V(p) = V(p) - \partial V(p)$. A zero p of ∂V is said to be of *type* $+$ if $V(p)$ is an outward vector. It is of *type* $-$ if $V(p)$ is an inward vector. It is of *type* 0 if it is also a zero of V.

Suppose p is an isolated zero of V. If p is in the interior of X, then the *local index* $\mathrm{Ind}(V, p)$ of V at p is defined as is well known; it is an integer. When p is on the boundary and is an isolated zero of ∂V, we will define the *normal local index* $\mathrm{Ind}_\nu(V, p)$ of V at p which is either an integer or a half-integer in the next section; when p is an isolated zero of $\partial^\perp V$, we will define the *tangential local index* $\mathrm{Ind}_\tau(V, p)$ of V at p. This may be a half-integer, too, when $n \leq 2$. These two local indices are not necessarily the same when they are both defined.

When the zeros of V and ∂V are all isolated, we define the *normal index* $\mathrm{Ind}_\nu(V)$ of V to be the sum of the local indices at the zeros in the interior and the normal

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2000 *Mathematics Subject Classification.* Primary 57R25; Secondary 55M25.

Key words and phrases. vector field, index, Euler characteristic.

The second author was partially supported by Grant-in-Aid for Scientific Research.

local indices at the zeros on the boundary. The sum of the local indices of ∂V at the zeros of type $+$ (*resp.* $-$, 0) is denoted $\mathrm{Ind}(\partial_+ V)$ (*resp.* $\mathrm{Ind}(\partial_- V)$, $\mathrm{Ind}(\partial_0 V)$).

Theorem 1. *Suppose X is an n-dimensional compact smooth manifold and V is a continuous tangent vector field on X. If V and ∂V have only isolated zeros, then the following equality holds:*

$$\mathrm{Ind}_\nu(V) + \frac{1}{2}\,\mathrm{Ind}(\partial_0 V) + \mathrm{Ind}(\partial_- V) = \chi(X) \ .$$

Remarks. (1) The local index of a zero of the zero vector field on a 0-dimensional manifold is always 1. So, when $n = 1$, $\mathrm{Ind}(\partial_0 V)$ is the number of the zeros on the boundary, and $\mathrm{Ind}(\partial_- V)$ is the number of boundary points at which the vector points inward.
(2) The special case where the vectors $V(p)$ are tangent to the boundary for all $p \in \partial X$ were discussed in [3]; see the review by J. M. Boardman in Mathematical Reviews.

When the zeros of V are isolated and the zeros of V on the boundary are the only zeros of $\partial^\perp V(p)$, we will define the *tangential index* $\mathrm{Ind}_\tau(V)$ of V to be the sum of the local indices of V at the zeros in the interior and the tangential local indices at the zeros on the boundary. If the dimension of X is bigger than 2, then the assumption on V forces the connected components of the boundary of X to be classified into the following two types:

(1) vectors point outward except at the isolated zeros,
(2) vectors point inward except at the isolated zeros.

The union of the components of the first type is denoted $\partial_+ X$, and the union of the components of the second type is denoted $\partial_- X$. If the dimension of X is 1, then the boundary components are single points; so the vector at the boundary either points outward, inward, or is zero, and accordingly the boundary ∂X is split into $\partial_+ X$, $\partial_- X$, and $\partial_0 X$.

Theorem 2. *Suppose X is an n-dimensional compact smooth manifold and V is a continuous tangent vector field on X. If the zeros of V are isolated and the zeros of V on the boundary are the only zeros of $\partial^\perp V(p)$, then the following equality holds:*

$$\mathrm{Ind}_\tau(V) = \begin{cases} \chi(X) & \text{if } n \text{ is even,} \\ \chi(X) - \chi(\partial_- X) & \text{if } n \geq 3, \\ \chi(X) - \frac{1}{2}\chi(\partial_0 X) - \chi(\partial_- X) & \text{if } n = 1. \end{cases}$$

In the last section, we will give an alternative formulation of these theorems.

2. Local Indices of an Isolated Zero on the Boundary

In this section, we describe the two local indices of a vector field V at an isolated zero on the boundary.

Let X be an n-dimensional compact smooth manifold with boundary ∂X. We fix an embedding of ∂X in a Euclidean space $\mathbb{R}^N$ of a sufficiently high dimension so that, under the identification $\mathbb{R}^N = 1 \times \mathbb{R}^N$, it extends to an an embedding of $(X, \partial X)$ in $([1,\infty) \times \mathbb{R}^N, 1 \times \mathbb{R}^N)$ such that $X \cap [1,2] \times \mathbb{R}^N = [1,2] \times \partial X$.

Now suppose p is an isolated zero sitting on the boundary ∂X. Let us take local cordinates $y_1, y_2, \ldots, y_n$ around p such that y_1 is equal to the first coordinate of $[1,\infty) \times \mathbb{R}^N$ and p corresponds to $a = (1, 0, \ldots, 0) \in \mathbb{R}^n$. V defines a vector field v on a neighborhood of a in the subset $y_1 \geq 1$. Choose a sufficiently small positive number ε so that the right half $D^n_+(a;\varepsilon)$ of the disk of radius ε with center at a is contained in this neighborhood, and a is the only zero of v in $D^n_+(a;\varepsilon)$. Let $H^{n-1}_+(a;\varepsilon)$ $(\subset \partial D^n_+(a;\varepsilon))$ denote the right hemisphere of radius ε with center at a. The vector field v induces a continuous map $\bar{v} : H_+^{n-1}(a;\varepsilon) \to S^{n-1}$ to the $(n-1)$-dimensional unit sphere by:

$$\bar{v}(x) = \frac{v(x)}{\|v(x)\|} .$$

Let $S^{n-2}(a;\varepsilon)$ denote the boundary sphere of $H_+^{n-1}(a;\varepsilon)$. When $n = 1$, we understand that it is an empty set. Assume that its image by $\bar{v}$ is not the whole sphere S^{n-1}. Pick up a "direction" $d \in S^{n-1} \setminus \bar{v}(S^{n-2}(a;\varepsilon))$, then $\bar{v}$ determines an integer, denoted $i(v, a; d)$, in $H_{n-1}(S^{n-1}, S^{n-1} \setminus \{d\}) = \mathbb{Z}$. Here we use the compatible orientations for $H_+^{n-1}(a;\varepsilon)$ and S^{n-1}. It is the algebraic intersection number of $\bar{v}$ with $\{d\} \subset S^{n-1}$, and is locally constant as a function of d. A pair of antipodal points $\{d, -d\}$ of S^{n-1} is said to be *admissible* if they are both in $S^{n-1} \setminus \bar{v}(S^{n-2}(a;\varepsilon))$. For such an admissible pair $\{\pm d\}$, we define a possibly-half-integer $i(v, a; \pm d)$ to be the average of the two integers $i(v, a; d)$ and $i(v, a; -d)$:

$$i(v, a; \pm d) = \frac{1}{2} i(v, a; d) + \frac{1}{2} i(v, a; -d) .$$

In the case of $n = 1$, there is only one admissible pair $\{\pm 1\} = S^0$, and

$$i(v, 1; \pm 1) = \begin{cases} \frac{1}{2} & \text{if } \bar{v}(1+\varepsilon) = 1, \\ -\frac{1}{2} & \text{if } \bar{v}(1+\varepsilon) = -1. \end{cases}$$

Definition. Suppose p is an isolated zero of ∂V. We may assume that ε is sufficiently small, and that the pair $\{\pm e_1\}$ with $e_1 = (1, 0, \ldots, 0) \in S^{n-1}$ is admissible. The *normal local index* $\mathrm{Ind}_\nu(V, p)$ of V at p is defined to be $i(v, a; \pm e_1)$.

Definition. Suppose p is an isolated zero of $\partial^\perp V$. We define the *tangential local index* $\mathrm{Ind}_\tau(V, p)$ of V at p as follows: If $n = 1$, then $\mathrm{Ind}_\tau(V, p) = i(v, 1; \pm 1)$. If $n \geq 2$, then set $S^{n-2} = \{e \in S^{n-1} | e \perp (1, 0, \ldots, 0)\}$. We may assume that ε is sufficiently small, and that, $S^{n-2} \subset S^{n-1} \setminus \bar{v}(S^{n-2}(a;\varepsilon))$. When $n = 2$, there is only one admissible pair in $S^{n-2} = S^0$. When $n \geq 3$, the value of $i(v, a; d)$ is independent of the choice of $d \in S^{n-2}$, and $i(v, a; \pm d) = i(v, a; d)$. So, for $n \geq 2$, we define $\mathrm{Ind}_\tau(V, p)$ to be $i(v, a; \pm d)$, where d is any point in S^{n-2}.

Remarks. (1) When $n = 1$, the two indices are the same.
(2) When $n \geq 3$, $\mathrm{Ind}_\tau(V, p)$ is an integer.

3. Proof of Theorem 1

We give a proof of Theorem 1. Assume that $(X, \partial X)$ is embedded in $([1, \infty) \times \mathbb{R}^N, 1 \times \mathbb{R}^N)$ as in the previous section. We consider the double DX of X:

$$DX = \partial([-1, 1] \times X) = \{\pm 1\} \times X \cup [-1, 1] \times \partial X .$$

DX can be embedded in $\mathbb{R} \times \mathbb{R}^N$ as the union of three subsets X_+, X_-, $[-1, 1] \times \partial X$, where X_+ is X itself, X_- is the image of the reflection $r : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N$ with respect to $0 \times \mathbb{R}^N$, and $\partial X \subset 1 \times \mathbb{R}^N$ is regarded as a subset of $\mathbb{R}^N$.

Let $V = V_+$ be the given tangent vector field on $X = X_+$. The reflection r induces a tangent vector field $r_*(V) = V_-$ on X_-. We can extend these to obtain a tangent vector field DV on DX by defining $DV(t, x)$ to be

$$\frac{t+1}{2} V_+(1, x) + \frac{1-t}{2} V_-(-1, x)$$

for $(t, x) \in [-1, 1] \times \partial X$. Note that, on $0 \times \partial X$, we obtain the boundary ∂V of V. There are four kinds of zeros of DV:

1. For each zero p of V in the interior of X, there are two zeros: the copy in the interior of X_+ and the copy in the interior of X_-. They have the same local index as the original one.

(2) For each zero $p = (1, x)$ of ∂V of type 0, the points (t, x) are all zeros of DV. Although these are not isolated, we can perturb the vector field in a very small neighborhood and make it into an isolated zero, whose local index is $2\,\mathrm{Ind}_\nu(V, p)$.
(3) For each zero $p = (1, x) \in \partial X$ of ∂V of type $-$, the point $(0, x)$ is an isolated zero of DV whose local index is equal to $\mathrm{Ind}(\partial V, p)$.
(4) For each zero $p = (1, x) \in \partial X$ of ∂V of type $+$, the point $(0, x)$ is an isolated zero of DV whose local index is equal to $-\mathrm{Ind}(\partial V, p)$.

One can verify the computation of the local indices in cases (2), (3), and (4) above as follows: First define the local coordinates $y_1, \ldots, y_n$ around $(0, x)$ extending the y_i's around $p = (1, x)$ described in §2 by

$$\begin{cases} y_1(t, *) = t & \text{for all } t \le 1 \\ y_i(t, x') = y_i(1, x') & \text{if } i = 2, \ldots, n \text{ and } -1 \le t \le 1, \\ y_i(t, x'') = y_i(-t, x'') & \text{if } i = 2, \ldots, n \text{ and } t \le -1\ . \end{cases}$$

Let $r : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}$ be the reflection $r(t, x') = (-t, x')$ and consider the map

$$D\bar{v} : r(H_+^{n-1}(a; \varepsilon)) \cup [-1, 1] \times S^{n-2}(a; \varepsilon) \cup H_+^{n-1}(a; \varepsilon) \to S^{n-1}$$

induced from DV, and compute the algebraic intersection number with $e_1 = (1, 0, \ldots, 0)$ in case (2) and with $e_2 = (0, 1, 0, \ldots, 0)$ in cases (3) and (4). Note that (3) and (4) do not occur when $n = 1$. Let $\bar{v} : H_+^{n-1}(a; \varepsilon) \to S^{n-1}$ be the map induced by V as in §2. Note that $\bar{v}$ can be defined not only for an isolated zero of ∂V of type 0 but also for a zero of type ± 1. $D\bar{v}$ is the double of $\bar{v}$ in the sense that it is $\bar{v}$ on the subset $H_+^{n-1}(a; \varepsilon)$ and that it is the composite $r \circ \bar{v} \circ r$ on the subset $r(H_+^{n-1}(a; \varepsilon))$; therefore, for $q \in r(H_+^{n-1}(a; \varepsilon))$, $D\bar{v}(q) = e_1$ if and only if $\bar{v}(r(q)) = -e_1$. In case (2), the vectors on the subset $[-1, 1] \times S^{n-2}(a; \varepsilon)$ and e_1 are never parallel; so the algebraic intersection of $D\bar{v}$ with e_1 is $i(v, a; e_1) + i(v, a; -e_1) = 2\,\mathrm{Ind}_\nu(V, p)$. In case (3) (*resp.* (4)), we may assume that all the vectors $D\bar{v}((t, x'))$ $(t \neq 0)$ point away from (*resp.* toward) the hyperplane $y_1 = 0$; therefore, the local index is equal to $\mathrm{Ind}(\partial V, p)$ (*resp.* $-\mathrm{Ind}(\partial V, p)$), since the y_1 direction is preserved (*resp.* reversed) in case (3) (*resp.* (4)).

Apply the Poincaré-Hopf index theorem to DV and ∂V; we obtain the following equalities:

$$2\,\mathrm{Ind}_\nu(V) + \mathrm{Ind}(\partial_- V) - \mathrm{Ind}(\partial_+ V) = 2\chi(X) - \chi(\partial X)\ ,$$
$$\mathrm{Ind}(\partial_0 V) + \mathrm{Ind}(\partial_- V) + \mathrm{Ind}(\partial_+ V) = \chi(\partial V)\ .$$

The desired formula follows immediately from these.

4. Proof of Theorem 2

When $n = 1$, the normal local index and the tangential local index are the same; therefore, the $n = 1$ case follows from Theorem 1. So we assume that $n \geq 2$.

Let DX be the double of X and let us use the same notation as in the first paragraph of the previous section. We will define the twisted double $\tilde{D}V$ of the vector field V on X as follows: $\tilde{V}_+ = V$ is a vector field on $X = X_+$. Consider $-V$; the reflection r induces a vector field $\tilde{V}_- = v_*(-V)$ on X_-. Extend these to obtain a tangent vector field $\tilde{D}V$ on DX by defining $\tilde{D}V(t, x)$ to be

$$\frac{t+1}{2}\tilde{V}_+(1, x) + \frac{1-t}{2}\tilde{V}_-(-1, x)$$

for $(t, x) \in [-1, 1] \times \partial X$. In general, if $V(p)$ is tangent to ∂X at $p = (1, x) \in \partial X$, then the twisted double $\tilde{D}V$ has a corresponding zero $(0, x)$. We are assuming that this happens only when p is a zero of V. Thus there are only two types of zeros of $\tilde{D}V$:

(1) For each zero p of V in the interior of X, there are two zeros: the copy in the interior of X_+ which has the same local index as $\operatorname{Ind}(V,p)$ and the copy in the interior of X_- whose local index is equal to $(-1)^n \operatorname{Ind}(V,p)$.
(2) For each zero $p=(1,x)$ of V on the boundary of X, the points (t,x) are all zeros of $\tilde{D}V$. Although these are not isolated, we can perturb the vector field in a very small neighborhood and make it into an isolated zero, whose local index is equal to $2\operatorname{Ind}_\tau(V,p)$ if n is even and is equal to 0 if n is odd.

The computation of the local index in case (2) can be done in the following way. Let us use the notation in the previous section. In this case we consider

$$\tilde{D}\bar{v} : r(H_+^{n-1}(a;\varepsilon)) \cup [-1,1]\times S^{n-2}(a;\varepsilon) \cup H_+^{n-1}(a;\varepsilon) \to S^{n-1}$$

induced from $\tilde{D}V$, and compute the algebraic intersection number with $e_2 = (0,1,0,\dots,0)$. $\tilde{D}\bar{v}$ is the twisted double of $\bar{v}$ in the sense that it is $\bar{v}$ on the subset $H_+^{n-1}(a;\varepsilon)$ and that it is the composite $r\circ A\circ\bar{v}\circ r$ on the subset $r(H_+^{n-1}(a;\varepsilon))$, where $A : S^{n-1}\to S^{n-1}$ is the antipodal map; therefore, for $q\in r(H_+^{n-1}(a;\varepsilon))$, $\tilde{D}\bar{v}(q) = e_2$ if and only if $\bar{v}(r(q)) = -e_2$. The vectors on the subset $[-1,1]\times S^{n-2}(a;\varepsilon)$ and e_2 are never parallel; so the algebraic intersection of $\tilde{D}\bar{v}$ with e_2 is $i(v,a;e_1)+(-1)^n i(v,a;-e_1)$ which is equal to $2\operatorname{Ind}_\tau(V,p)$ if n is even and is equal to 0 if n is odd.

So, if n is even, the Poincaré-Hopf formula for $\tilde{D}V$ reduces to the desired formula $\operatorname{Ind}_\tau V = \chi(X)$.

Next we consider the case where $n\geq 3$. As we mentioned in the first section, the components of ∂X are classified into two types:

(1) vectors point outward except at the isolated zeros,
(2) vectors point inward except at the isolated zeros.

Suppose that p is an isolated zero of V on a connected component C of ∂X and that C is of the first type. Consider a small neighborhood of p and coordinates $\{y_1,\dots,y_n\}$ as in §2. The vector field v along $y_1=1$ can be thought of as a map $\varphi(y_2,\dots,y_n) = (z_1,z_2,\dots,z_n)$ from an open set $U\subset\mathbb{R}^{n-1}$ to $\mathbb{R}^n$ satisfying $z_1\geq 0$. The equality holds if and only if $(y_2,\dots,y_n) = (0,\dots,0)$. Choose a very small number $\varepsilon>0$. Using a homotopy

$$\max\{\varepsilon - (y_2^2+y_3^2+\cdots+y_n^2),0\}(-t,0,\dots,0)+\varphi(y_2,\dots,y_n)\ ,$$

one can add a collar along C and extend the vector field V over the added collar. Repeat this process if there are more zeros on C until the vector points outward along the new boundary component. The zeros on the boundary component C now lies in the interior, and the local indices are eaual to the corresponding tangential local indices. We can do a similar modification in the case of the second type component, and move all the zeros on the boundary into the interior. Now apply the Poincaré-Hopf theorem to get:

$$\operatorname{Ind}_\tau V = \chi(X) - \chi(\partial_- X)\ .$$

This completes the proof.

5. An Alternative Formulation

Let V be a continuous vector field on an n-dimensional compact smooth manifold X whose zeros are isolated. In the previous sections, we considered the zeros of V as the only singular points, and defined the normal/tangential index as the sum of local indices only at the zeros. In this section, the zeros of ∂V (in the normal index case) and the zeros of $\partial^\perp V$ (in the tangential index case) are also regarded as singular points of V. Note that the definition of the normal (*resp.* tangential) local index at an isolated zero on the boundary given in §2 is valid for an isolated zero of ∂V (*resp.* $\partial^{\perp} V$).

Definition. When the zeros of V and ∂V are all isolated, the *expanded normal index* $\operatorname{Ind}_{\nu}^{*}(V)$ of V is defined to be the sum of the local indices of V at the interior zeros of V and the normal local indices of V at the zeros of ∂V. When the zeros of V and $\partial^{\perp} V$ are all isolated, the *expanded tangential index* $\operatorname{Ind}_{\tau}^{*}(V)$ of V is defined to be the sum of the local indices of V at the interior zeros of V and the tangential local indices of V at the zeros of $\partial^{\perp} V$.

Remark. Note that the tangential local index $\operatorname{Ind}_{\tau}(V, p)$ at an isolated zero p of $\partial^{\perp} V$ is equal to zero if $n \geq 3$ and p is not a zero of V; this can be observed by choosing $d \in S^{n-2}$ to be not equal to $\pm \bar{v}(p)$. Also note that, if $n = 1$, the zeros of $\partial^{\perp} V$ are automatically the zeros of V. Therefore, $\operatorname{Ind}_{\tau}^{*}(V) = \operatorname{Ind}_{\tau}(V)$ if $n \neq 2$.

Theorem 3. *Suppose X is an n-dimensional compact smooth manifold and V is a continuous tangent vector field on X. If V and ∂V have only isolated zeros, then the following equality holds:*

$$\operatorname{Ind}_{\nu}^{*}(V) = \begin{cases} \chi(X) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Immediate from the proof of Theorem 2. □

Theorem 4. *Suppose X is an n-dimensional compact smooth manifold and V is a continuous tangent vector field on X. If V and $\partial^{\perp} V$ have only isolated zeros, then the following equality holds:*

$$\operatorname{Ind}_{\tau}^{*}(V) = \begin{cases} \chi(X) & \text{if } n \text{ is even,} \\ \chi(X) - \chi(\partial_{-} X) & \text{if } n \geq 3, \\ \chi(X) - \frac{1}{2}\chi(\partial_0 X) - \chi(\partial_{-} X) & \text{if } n = 1. \end{cases}$$

Proof. The only difference between Theorem 2 and Theorem 4 is the existence of the isolated zeros of $\partial^{\perp} V$ that are not the zeros of V. Since there is nothing to prove when $n = 1$, we assume that $n > 1$.

Suppose n is even. There are three types of zeros of $\tilde{D}V$, not two; the third type is an isolated zero $(0, x)$ corresponding to $p = (1, x)$ such that $V(p)$ is a non-zero tangent vector of ∂X as mentioned above. The local index of $\tilde{D}V$ is $2 \operatorname{Ind}_{\tau}^{*}(V, p)$. Therefore the Poincaré-Hopf formula for $\tilde{D}V$ gives $2 \operatorname{Ind}_{\tau}^{*}(V) = 2\chi(X)$.

Next suppose $n \geq 3$. Follow the proof of Theorem 2, treating the zeros of $\partial^{\perp} V$ like the zeros of V on the boundary, and apply the Poincaré-Hopf theorem. □

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