LOCAL INDICES OF A VECTOR FIELD AT AN ISOLATED ZERO ON THE BOUNDARY

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Abstract. We define two types of local indices of a vector field at an isolated zero on the boundary, and prove Poincaré-Hopf-type index theorems for certain vector fields on a compact smooth manifold which have only isolated zeros.

1. Introduction

The famous Poincaré-Hopf theorem states that the index \( \text{Ind}(V) \) of a continuous tangent vector field \( V \) on a compact smooth manifold \( X \) is equal to the Euler characteristic \( \chi(X) \) of \( X \), if \( V \) has only isolated zeros away from the boundary and \( V \) points outward on the boundary of \( X \). If you assume that the vectors on some of the boundary components point inward and point outward on the other components, then the formula will look like:

\[
\text{Ind}(V) = \chi(X) - \chi(\partial X),
\]

where \( \partial X \) denotes the union of the boundary components on which the vectors point inward. This can be observed by looking at the Morse function of the pair \((X, \partial X)\). In [4], M. Morse relaxed the requirement on the boundary behavior and obtained a formula

\[
\text{Ind}(V) + \text{Ind}(\partial_\nu V) = \chi(X).
\]

Actually the requirement that the singularities are isolated are also relaxed. This formula has been rediscovered and extended by several authors [5] [1] [2]. In this paper we consider only vector fields whose zeros are isolated. But we allow zeros on the boundary.

Let \( X \) be an \( n \)-dimensional compact smooth manifold with boundary \( \partial X \), and fix a Riemannian metric on \( X \). We assume \( n \geq 1 \). For a continuous tangent vector field \( V \) on \( X \) and a point \( p \) of its boundary, we define the vector \( \partial V(p) \) to be the orthogonal projection of \( V(p) \) to the tangent space of \( \partial X \) at \( p \). The tangent vector field \( \partial V \) on \( \partial X \) is called the boundary of \( V \). \( \partial_\nu V \) denotes the normal vector field on \( \partial X \) defined by \( \partial_\nu V(p) = V(p) - \partial V(p) \). A zero \( p \) of \( \partial V \) is said to be of type + if \( V(p) \) is an outward vector. It is of type − if \( V(p) \) is an inward vector. It is of type 0 if it is also a zero of \( V \).

Suppose \( p \) is an isolated zero of \( V \). If \( p \) is in the interior of \( X \), then the local index \( \text{Ind}(V, p) \) of \( V \) at \( p \) is defined as is well known; it is an integer. When \( p \) is on the boundary and is an isolated zero of \( \partial V \), we will define the normal local index \( \text{Ind}_\nu(V, p) \) of \( V \) at \( p \) which is either an integer or a half-integer in the next section; when \( p \) is an isolated zero of \( \partial_\nu V \), we will define the tangential local index \( \text{Ind}_\tau(V, p) \) of \( V \) at \( p \). This may be a half-integer, too, when \( n \leq 2 \). These two local indices are not necessarily the same when they are both defined.

When the zeros of \( V \) and \( \partial V \) are all isolated, we define the normal index \( \text{Ind}_\nu(V) \) of \( V \) to be the sum of the local indices at the zeros in the interior and the normal

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local indices at the zeros on the boundary. The sum of the local indices of \( \partial V \) at the zeros of type + (resp. -, 0) is denoted \( \text{Ind}(\partial_+ V) \) (resp. \( \text{Ind}(\partial_- V), \text{Ind}(\partial_0 V) \)).

**Theorem 1.** Suppose \( X \) is an \( n \)-dimensional compact smooth manifold and \( V \) is a continuous tangent vector field on \( X \). If \( V \) and \( \partial V \) have only isolated zeros, then the following equality holds:

\[
\text{Ind}_r(V) + \frac{1}{2} \text{Ind}(\partial_0 V) + \text{Ind}(\partial_- V) = \chi(X) .
\]

**Remarks.** (1) The local index of a zero of the zero vector field on a 0-dimensional manifold is always 1. So, when \( n = 1 \), \( \text{Ind}(\partial_0 V) \) is the number of the zeros on the boundary, and \( \text{Ind}(\partial_- V) \) is the number of boundary points at which the vector points inward.

(2) The special case where the vectors \( V(p) \) are tangent to the boundary for all \( p \in \partial X \) were discussed in [3]; see the review by J. M. Boardman in Mathematical Reviews.

When the zeros of \( V \) are isolated and the zeros of \( V \) on the boundary are the only zeros of \( \partial^+ V(p) \), we will define the **tangential index** \( \text{Ind}_r(V) \) of \( V \) to be the sum of the local indices of \( V \) at the zeros in the interior and the tangential local indices at the zeros on the boundary. If the dimension of \( X \) is bigger than 2, then the assumption on \( V \) forces the connected components of the boundary of \( X \) to be classified into the following two types:

1. vectors point outward except at the isolated zeros,
2. vectors point inward except at the isolated zeros.

The union of the components of the first type is denoted \( \partial_+ X \), and the union of the components of the second type is denoted \( \partial_- X \). If the dimension of \( X \) is 1, then the boundary components are single points; so the vector at the boundary either points outward, inward, or is zero, and accordingly the boundary \( \partial X \) is split into \( \partial_+ X, \partial_- X, \) and \( \partial_0 X \).

**Theorem 2.** Suppose \( X \) is an \( n \)-dimensional compact smooth manifold and \( V \) is a continuous tangent vector field on \( X \). If the zeros of \( V \) are isolated and the zeros of \( V \) on the boundary are the only zeros of \( \partial^+ V(p) \), then the following equality holds:

\[
\text{Ind}_r(V) = \begin{cases} 
\chi(X) & \text{if } n \text{ is even}, \\
\chi(X) - \chi(\partial_- X) & \text{if } n \geq 3, \\
\chi(X) - \frac{1}{2} \chi(\partial_0 X) - \chi(\partial_- X) & \text{if } n = 1.
\end{cases}
\]

In the last section, we will give an alternative formulation of these theorems.

**2. Local Indices of an Isolated Zero on the Boundary**

In this section, we describe the two local indices of a vector field \( V \) at an isolated zero on the boundary.

Let \( X \) be an \( n \)-dimensional compact smooth manifold with boundary \( \partial X \). We fix an embedding of \( \partial X \) in a Euclidean space \( \mathbb{R}^N \) of a sufficiently high dimension so that, under the identification \( \mathbb{R}^N = 1 \times \mathbb{R}^N \), it extends to an an embedding of \( (X, \partial X) \) in \( ([1, \infty) \times \mathbb{R}^N, 1 \times \mathbb{R}^N) \) such that \( X \cap [1, 2] \times \mathbb{R}^N = [1, 2] \times \partial X \).

Now suppose \( p \) is an isolated zero sitting on the boundary \( \partial X \). Let us take local coordinates \( y_1, y_2, \ldots, y_n \) around \( p \) such that \( y_1 \) is equal to the first coordinate of \([1, \infty) \times \mathbb{R}^N \) and \( p \) corresponds to \( a = (1, 0, \ldots, 0) \in \mathbb{R}^n \). \( V \) defines a vector field \( v \) on a neighborhood of \( a \) in the subset \( y_1 \geq 1 \). Choose a sufficiently small positive number \( \varepsilon \) so that the right half \( D_+^\varepsilon(a; \varepsilon) \) of the disk of radius \( \varepsilon \) with center at \( a \) is contained in this neighborhood, and \( a \) is the only zero of \( v \) in \( D_+^\varepsilon(a; \varepsilon) \).

Let \( H_+^{n-1}(a; \varepsilon) \) (\( \subset \partial D_+^\varepsilon(a; \varepsilon) \)) denote the right hemisphere of radius \( \varepsilon \) with center
at $a$. The vector field $v$ induces a continuous map $\bar{v} : H^{n-1}_+(a; \varepsilon) \to S^{n-1}$ to the $(n-1)$-dimensional unit sphere by:

$$\bar{v}(x) = \frac{v(x)}{|v(x)|}.$$  

Let $S^{n-2}(a; \varepsilon)$ denote the boundary sphere of $H^{n-1}_+(a; \varepsilon)$. When $n = 1$, we understand that it is an empty set. Assume that its image by $\bar{v}$ is not the whole sphere $S^{n-1}$. Pick up a “direction” $d \in S^{n-1}\setminus\bar{v}(S^{n-2}(a; \varepsilon))$, then $\bar{v}$ determines an integer, denoted $i(v, a; d)$, in $H_{n-1}(S^{n-1}, S^{n-1}\setminus\{d\}) = \mathbb{Z}$. Here we use the compatible orientations for $H^{n-1}_+(a; \varepsilon)$ and $S^{n-1}$. It is the algebraic intersection number of $\bar{v}$ with $\{d\} \subset S^{n-1}$, and is locally constant as a function of $d$. A pair of antipodal points $\{d, -d\}$ of $S^{n-1}$ is said to be admissible if they are both in $S^{n-1}\setminus\bar{v}(S^{n-2}(a; \varepsilon))$. For such an admissible pair $\{\pm d\}$, we define a possibly-half-integer $i(v, a; \pm d)$ to be the average of the two integers $i(v, a; d)$ and $i(v, a; -d)$:

$$i(v, a; \pm d) = \frac{1}{2} i(v, a; d) + \frac{1}{2} i(v, a; -d).$$

In the case of $n = 1$, there is only one admissible pair $\{\pm 1\} = S^0$, and

$$i(v, 1; \pm 1) = \begin{cases} 
\frac{1}{2} & \text{if } \bar{v}(1 + \varepsilon) = 1, \\
-\frac{1}{2} & \text{if } \bar{v}(1 + \varepsilon) = -1.
\end{cases}$$

**Definition.** Suppose $p$ is an isolated zero of $\partial V$. We may assume that $\varepsilon$ is sufficiently small, and that the pair $\{\pm e_1\}$ with $e_1 = (1, 0, \ldots, 0) \in S^{n-1}$ is admissible. The normal local index $\text{Ind}_e(V, p)$ of $V$ at $p$ is defined to be $i(v, a; \pm e_1)$.

**Definition.** Suppose $p$ is an isolated zero of $\partial^1 V$. We define the tangential local index $\text{Ind}_t(V, p)$ of $V$ at $p$ as follows: If $n = 1$, then $\text{Ind}_t(V, p) = i(v, 1; \pm 1)$. If $n \geq 2$, then set $S^{n-2} = \{e \in S^{n-1} | e \perp (1, 0, \ldots, 0)\}$. We may assume that $\varepsilon$ is sufficiently small, and that, $S^{n-2} \subset S^{n-1}\setminus\bar{v}(S^{n-2}(a; \varepsilon))$. When $n = 2$, there is only one admissible pair in $S^{n-2} = S^0$. When $n \geq 3$, the value of $i(v, a; d)$ is independent of the choice of $d \in S^{n-2}$, and $i(v, a; \pm d) = i(v, a; d)$. So, for $n \geq 2$, we define $\text{Ind}_t(V, p)$ to be $i(v, a; \pm d)$, where $d$ is any point in $S^{n-2}$.

**Remarks.** (1) When $n = 1$, the two indices are the same.
(2) When $n \geq 3$, $\text{Ind}_e(V, p)$ is an integer.

### 3. Proof of Theorem 1

We give a proof of Theorem 1. Assume that $(X, \partial X)$ is embedded in $([1, \infty) \times \mathbb{R}^N, 1 \times \mathbb{R}^N)$ as in the previous section. We consider the double $DX$ of $X$:

$$DX = \partial([-1, 1] \times X) = \{\pm 1\} \times X \cup [-1, 1] \times \partial X.$$  

$DX$ can be embedded in $\mathbb{R} \times \mathbb{R}^N$ as the union of three subsets $X_+, X_-, [-1, 1] \times \partial X$, where $X_+$ is $X$ itself itself, $X_-$ is the image of the reflection $r : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N$ with respect to $0 \times \mathbb{R}^N$, and $\partial X \subset 1 \times \mathbb{R}^N$ is regarded as a subset of $\mathbb{R}^N$.

Let $V = V_+$ be the given tangent vector field on $X = X_+$. The reflection $r$ induces a tangent vector field $r_*(V) = V_-$ on $X_-$. We can extend these to obtain a tangent vector field $DV$ on $DX$ by defining $DV(t, x)$ to be

$$\frac{t + 1}{2} V_+(1, x) + \frac{1 - t}{2} V_-(1, x)$$

for $(t, x) \in [-1, 1] \times \partial X$. Note that, on $0 \times \partial X$, we obtain the boundary $\partial V$ of $V$. There are four kinds of zeros of $DV$:

(1) For each zero $p$ of $V$ in the interior of $X$, there are two zeros: the copy in the interior of $X_+$ and the copy in the interior of $X_-$. They have the same local index as the original one.
(2) For each zero \( p = (1, x) \) of \( \partial V \) of type 0, the points \((t, x)\) are all zeros of \( DV \). Although these are not isolated, we can perturb the vector field in a very small neighborhood and make it into an isolated zero, whose local index is \( 2 \, \text{Ind}_0(V, p) \).

(3) For each zero \( p = (1, x) \in \partial X \) of \( DV \) of type -, the point \((0, x)\) is an isolated zero of \( DV \) whose local index is equal to \( \text{Ind}(\partial V, p) \).

(4) For each zero \( p = (1, x) \in \partial X \) of \( DV \) of type +, the point \((0, x)\) is an isolated zero of \( DV \) whose local index is equal to \(- \, \text{Ind}(\partial V, p)\).

One can verify the computation of the local indices in cases (2), (3), and (4) above as follows: First define the local coordinates \( y_1, \ldots, y_n \) around \((0, x)\) extending the \( y_i \)'s around \( p = (1, x) \) described in §2 by

\[
\begin{align*}
y_1(t, \ast) &= t \\
y_i(t, x') &= y_i(1, x') \\
y_i(t, x'') &= y_i(-t, x'')
\end{align*}
\]

for all \( t \leq 1 \) if \( i = 2, \ldots, n \) and \(-1 \leq t \leq 1\),

\[
y_i(t, x'') = y_i(-t, x'') \quad \text{if } i = 2, \ldots, n \text{ and } t \leq -1.
\]

Let \( r : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1} \) be the reflection \( r(t, x') = (-t, x') \) and consider the map

\[
DV : r(H^{n-1}_n(a; \varepsilon)) \cup [-1, 1] \times S^{n-2}(a; \varepsilon) \cup H^{n-1}_n(-a; \varepsilon) \to S^{n-1}
\]

induced from \( DV \), and compute the algebraic intersection number with \( e_1 = (1, 0, \ldots, 0) \) in case (2) and with \( e_2 = (0, 1, 0, \ldots, 0) \) in cases (3) and (4). Note that (3) and (4) do not occur when \( n = 1 \). Let \( \bar{v} : H^{n-1}_n(a; \varepsilon) \to S^{n-1} \) be the map induced by \( V \) as in §2. Note that \( \bar{v} \) can be defined not only for an isolated zero of \( DV \) of type 0 but also for a zero of type \( \pm 1 \). \( \bar{v} \) is the double of \( v \) in the sense that it is \( \bar{v} \) on the subset \( H^{n-1}_n(a; \varepsilon) \) and that it is the composite \( r \circ \bar{v} \circ r \) on the subset \( r(H^{n-1}_n(a; \varepsilon)) \); therefore, for \( q \in r(H^{n-1}_n(a; \varepsilon)) \), \( \bar{v}(q) = e_1 \) if and only if \( v(r(q)) = -e_1 \). In case (2), the vectors on the subset \([-1, 1] \times S^{n-2}(a; \varepsilon) \) and \( e_1 \) are never parallel; so the algebraic intersection of \( \bar{v} \) with \( e_1 \) is \( i(v, a; e_1) + i(v, a; -e_1) = 2 \, \text{Ind}_0(V, p) \). In case (3) (resp. (4)), we may assume that all the vectors \( \bar{v}(t, x') \) \((t \neq 0)\) point away from (resp. toward) the hyperplane \( y_1 = 0 \); therefore, the local index is equal to \( \text{Ind}(\partial V, p) \) (resp. \(- \, \text{Ind}(\partial V, p)\)), since the \( y_1 \) direction is preserved (resp. reversed) in case (3) (resp. (4)).

Apply the Poincaré-Hopf index theorem to \( DV \) and \( \partial V \); we obtain the following equalities:

\[
2 \, \text{Ind}_0(V) + \text{Ind}(\partial_- V) - \text{Ind}(\partial_+ V) = 2 \chi(X) - \chi(\partial X),
\]

\[
\text{Ind}(\partial_0 V) + \text{Ind}(\partial_- V) + \text{Ind}(\partial_+ V) = \chi(\partial V).
\]

The desired formula follows immediately from these.

4. Proof of Theorem 2

When \( n = 1 \), the normal local index and the tangential local index are the same; therefore, the \( n = 1 \) case follows from Theorem 1. So we assume that \( n \geq 2 \).

Let \( DX \) be the double of \( X \) and let us use the same notation as in the first paragraph of the previous section. We will define the twisted double \( DV \) of the vector field \( V \) on \( X \) as follows: \( \tilde{V}_+ = V \) is a vector field on \( X = X_+ \). Consider \( -V \); the reflection \( r \) induces a vector field \( \bar{V}_- = v_-(V) \) on \( X_- \). Extend these to obtain a tangent vector field \( \bar{V}_D \) on \( DX \) by defining \( \bar{V}_D(t, x) \) to be

\[
\frac{t+1}{2} \bar{V}_+(1, x) + \frac{1-t}{2} \bar{V}_-(1, x)
\]

for \((t, x) \in [-1, 1] \times \partial X \). In general, if \( V(p) \) is tangent to \( \partial X \) at \( p = (1, x) \in \partial X \), then the twisted double \( \bar{V}_D \) has a corresponding zero \((0, x)\). We are assuming that
this happens only when $p$ is a zero of $V$. Thus there are only two types of zeros of $\tilde{DV}$:

1. For each zero $p$ of $V$ in the interior of $X$, there are two zeros: the copy in the interior of $X_+$ which has the same local index as $\text{Ind}(V, p)$ and the copy in the interior of $X_-$ whose local index is equal to $(-1)^n \text{Ind}(V, p)$.

2. For each zero $p = (1, x)$ of $V$ on the boundary of $X$, the points $(t, x)$ are all zeros of $\tilde{DV}$. Although these are not isolated, we can perturb the vector field in a very small neighborhood and make it into an isolated zero, whose local index is equal to $2 \text{Ind}_+(V, p)$ if $n$ is even and is equal to 0 if $n$ is odd.

The computation of the local index in case (2) can be done in the following way. Let us use the notation in the previous section. In this case we consider

$$\tilde{Dv} : r(H_+^{n-1}(a; \varepsilon)) \cup [-1, 1] \times S^{n-2}(a; \varepsilon) \cup H_{+n}^{n-1}(a; \varepsilon) \to S^{n-1}$$

induced from $\tilde{DV}$, and compute the algebraic intersection number with $e_2 = (0, 1, 0, \ldots, 0)$. $\tilde{Dv}$ is the twisted double of $v$ in the sense that it is $v$ on the subset $H_+^{n-1}(a; \varepsilon)$ and that it is the composite $r \circ A \circ \bar{v} \circ r$ on the subset $r(H_{+n}^{n-1}(a; \varepsilon))$, where $A : S^{n-1} \to S^{n-1}$ is the antipodal map; therefore, for $q \in r(H_{+n}^{n-1}(a; \varepsilon))$,

$$\tilde{Dv}(q) = e_2 \text{ if and only if } \bar{v}(r(q)) = -e_2.$$ 

The vectors on the subset $[-1, 1] \times S^{n-2}(a; \varepsilon)$ and $e_2$ are never parallel; so the algebraic intersection of $\tilde{Dv}$ with $e_2$ is $i(v, a; e_1) + (-1)^n i(v, a; -e_1)$ which is equal to $2 \text{Ind}_+(V, p)$ if $n$ is even and is equal to 0 if $n$ is odd.

So, if $n$ is even, the Poincaré-Hopf formula for $\tilde{DV}$ reduces to the desired formula $\text{Ind}_+ V = \chi(X)$.

Next we consider the case where $n \geq 3$. As we mentioned in the first section, the components of $\partial X$ are classified into two types:

1. Vectors point outward except at the isolated zeros,
2. Vectors point inward except at the isolated zeros.

Suppose that $p$ is an isolated zero of $V$ on a connected component $C$ of $\partial X$ and that $C$ is of the first type. Consider a small neighborhood of $p$ and coordinates $(y_1, \ldots, y_n)$ as in §2. The vector field $v$ along $y_1 = 1$ can be thought of as a map $\varphi(y_2, \ldots, y_n) = (z_1, 2, \ldots, z_n)$ from an open set $U \subset \mathbb{R}^{n-1}$ to $\mathbb{R}^n$ satisfying $z_1 \geq 0$. The equality holds if and only if $(y_2, \ldots, y_n) = (0, 0, \ldots, 0)$. Choose a very small number $\varepsilon > 0$. Using a homotopy

$$\max \{\varepsilon - (y_2^2 + y_3^2 + \cdots + y_n^2), 0\}(-t, 0, \ldots, 0) + \varphi(y_2, \ldots, y_n),$$

one can add a collar along $C$ and extend the vector field $V$ over the added collar. Repeat this process if there are more zeros on $C$ until the vector points outward along the new boundary component. The zeros on the boundary component $C$ now lies in the interior, and the local indices are equal to the corresponding tangential local indices. We can do a similar modification in the case of the second type component, and move all the zeros on the boundary into the interior. Now apply the Poincaré-Hopf theorem to get:

$$\text{Ind}_+ V = \chi(X) - \chi(\partial_- X).$$

This completes the proof.

5. An Alternative Formulation

Let $V$ be a continuous vector field on an $n$-dimensional compact smooth manifold $X$ whose zeros are isolated. In the previous sections, we considered the zeros of $V$ as the only singular points, and defined the normal/tangential index as the sum of local indices only at the zeros. In this section, the zeros of $\partial V$ (in the normal index case) and the zeros of $\partial^2 V$ (in the tangential index case) are also regarded
as singular points of $V$. Note that the definition of the normal (resp. tangential) local index at an isolated zero on the boundary given in §2 is valid for an isolated zero of $\partial V$ (resp. $\partial^1 V$).

**Definition.** When the zeros of $V$ and $\partial V$ are all isolated, the *expanded normal index* $\text{Ind}_n^\tau(V)$ of $V$ is defined to be the sum of the local indices of $V$ at the interior zeros of $V$ and the normal local indices of $V$ at the zeros of $\partial V$. When the zeros of $V$ and $\partial^1 V$ are all isolated, the *expanded tangential index* $\text{Ind}_n^\tau(V)$ of $V$ is defined to be the sum of the local indices of $V$ at the interior zeros of $V$ and the tangential local indices of $V$ at the zeros of $\partial^1 V$.

**Remark.** Note that the tangential local index $\text{Ind}_r(V, p)$ at an isolated zero $p$ of $\partial^1 V$ is equal to zero if $n \geq 3$ and $p$ is not a zero of $V$; this can be observed by choosing $d \in S^{n-2}$ to be not equal to $\pm v(p)$. Also note that, if $n = 1$, the zeros of $\partial^1 V$ are automatically the zeros of $V$. Therefore, $\text{Ind}_r^\tau(V) = \text{Ind}_n^\tau(V)$ if $n \neq 2$.

**Theorem 3.** Suppose $X$ is an $n$-dimensional compact smooth manifold and $V$ is a continuous tangent vector field on $X$. If $V$ and $\partial V$ have only isolated zeros, then the following equality holds:

$$\text{Ind}_n^\tau(V) = \left\{ \begin{array}{ll} \chi(X) & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{array} \right.$$  

**Proof.** Immediate from the proof of Theorem 2. □

**Theorem 4.** Suppose $X$ is an $n$-dimensional compact smooth manifold and $V$ is a continuous tangent vector field on $X$. If $V$ and $\partial^1 V$ have only isolated zeros, then the following equality holds:

$$\text{Ind}_n^\tau(V) = \left\{ \begin{array}{ll} \chi(X) & \text{if } n \text{ is even}, \\ \chi(X) - \chi(\partial X) & \text{if } n \geq 3, \\ \chi(X) - \frac{1}{2} \chi(\partial^1 X) - \chi(\partial X) & \text{if } n = 1. \end{array} \right.$$  

**Proof.** The only difference between Theorem 2 and Theorem 4 is the existence of the isolated zeros of $\partial^1 V$ that are not the zeros of $V$. Since there is nothing to prove when $n = 1$, we assume that $n > 1$.

Suppose $n$ is even. There are three types of zeros of $\hat{D} V$, not two; the third type is an isolated zero $(0, x)$ corresponding to $p = (1, x)$ such that $V(p)$ is a non-zero tangent vector of $\partial X$ as mentioned above. The local index of $\hat{D} V$ is $2 \text{Ind}_n^\tau(V, p)$. Therefore the Poincaré-Hopf formula for $\hat{D} V$ gives $2 \text{Ind}_n^\tau(V) = 2 \chi(X)$.

Next suppose $n \geq 3$. Follow the proof of Theorem 2, treating the zeros of $\partial^1 V$ like the zeros of $V$ on the boundary, and apply the Poincaré-Hopf theorem. □

**References**

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