Arithmetic aspects of growth rates of hyperbolic Coxeter groups

Complex Hyperbolic Geometry and Related Topics

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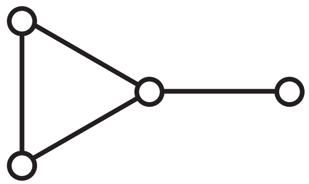
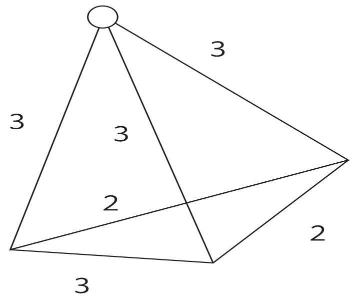
Contents

1. Growth rates of hyperbolic Coxeter groups

Hyperbolic Coxeter polyhedron $P = \cap_{i=1}^{k} H_i^- \subset \mathbb{H}^n$
all dihedral angles are π/n, $(n \in \mathbf{N} \cup \{\infty\})$
P is represented by its Coxeter diagram





ω Geometric Coxeter group (W, S)
$S = \{r_1, r_2, \cdots, r_k\}$, $W = \langle r_1, r_2, \cdots, r_k \rangle$

Growth function of (W, S)

$f_S(t) = \sum_{k \geq 0} a_k t^k = 1 + \sharp S t + \cdots$
where $a_k = \sharp\{g \in W \mid \ell_S(g) = k\}$

The growth rate of (W, S): $\tau := \limsup_{k \to \infty} \sqrt[k]{a_k}$
$= 1/R$ (R: the radius of convergence of $f_S(t)$)

$\tau > 1$ i.e. of exponential growth (de la Harpe 87?)

Theorem (Steinberg 68)

Let us denote by (W_T, T) the Coxeter subgroup of (W, S) generated by the subset $T \subseteq S$, and let its growth function be $f_T(t)$. Set $\mathcal{F} = \{T \subseteq S \ : \ W_T \text{ is finite }\}$. Then

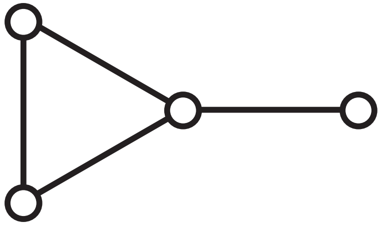
$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

Theorem (Solomon 66)

The growth function $f_S(t)$ of an irreducible finite Coxeter group (G, S) can be written as $f_S(t) = \prod_{i=1}^{k} [m_i + 1]$ where $[n] := 1 + t + \cdots + t^{n-1}$ and $\{m_1, m_2, \cdots, m_k\}$ is the set of exponents of (G, S).

$$\frac{1}{f_S(t^{-1})} = \tilde{Q}(t)/\tilde{P}(t) \Rightarrow f_S(t) = P(t)/Q(t)$$

where $P(t) = t^n \tilde{P}(t)$, $Q(t) = t^n \tilde{Q}(t)$.
Hence $R = 1/\tau$ is the smallest positive root of $Q(t)$.
Since $\tilde{Q}(t)$ is monic, $\tau > 1$ is an algebraic integer.



type of subgroup	growth function	number
A_3	$[2,3,4]$	2
$A_2 \times A_1$	$[2,2,3]$	1
A_2	$[2,3]$	4
$A_1 \times A_1$	$[2,2]$	2

$$\frac{1}{f_S(t^{-1})} = \frac{-2}{[2,3,4]} + \frac{-1}{[2,2,3]} + \frac{4}{[2,3]} + \frac{2}{[2,2]} + \frac{-4}{[2]} + 1.$$

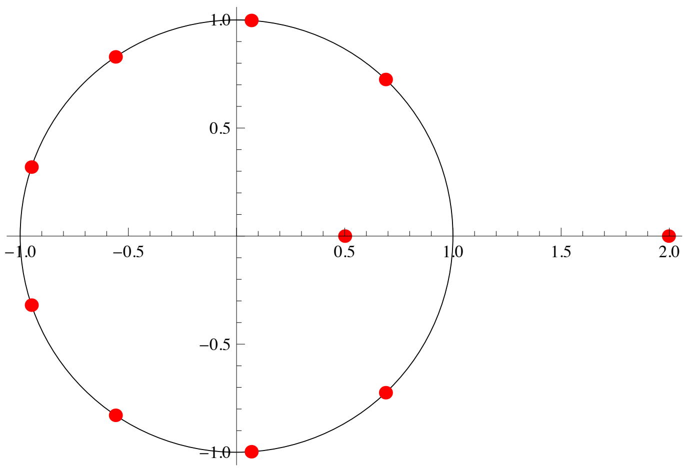
$$f_S(t) = \frac{(t+1)(t^2+1)(t^2+t+1)}{(t-1)(t^3+t-1)}.$$

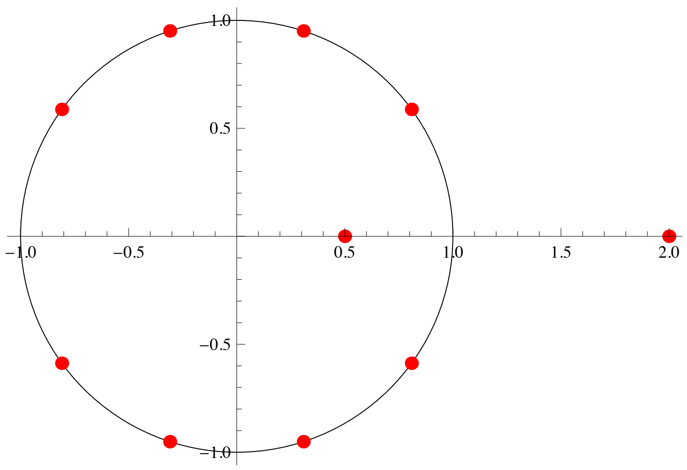
A real algebraic integer $\tau > 1$ is called:

(1) a *Salem number* if τ^{-1} is a conjugate of τ and all conjugates of τ other than τ and τ^{-1} lie on the unit circle. We assume also that there exists a conjugate on the unit circle.

(1') a *"Salem" number* if τ^{-1} is a conjugate of τ and all conjugates of τ other than τ and τ^{-1} lie on the unit circle (i.e. quadratic units are also "Salem" number).

QUIZ: Which is Salem or “Salem”?





A real algebraic integer $\tau > 1$ is called:

(2) a *Pisot number* if all algebraic conjugates of τ other than τ lie in the open unit disk.

(3) a *Perron number* if all of whose conjugates have strictly smaller absolute values.

Theorem (Cannon-Wagreich 92, Parry 93)

The growth rates of cocompact 2 and 3-dimensional hyperbolic Coxeter groups are "Salem" numbers.

Theorem (Floyd 92)

The growth rates of cofinite 2-dimensional hyperbolic Coxeter groups are Pisot numbers.

Theorem (K. and Umemoto 2012)

The growth rates of cofinite 3-dimensional hyperbolic Coxeter groups with 4 and 5 generators (i.e. simplexes, pyramids and prisms) are Perron numbers.

Remark

(1) Kellerhals and Perren (2011) observed numerically that many cocompact 4-dimensional hyperbolic Coxeter groups (including 5 and 6 generated groups) have Perron numbers as their growth rates.

(2) Kolpakov (2012) studied cofinite 3-dimensional hyperbolic Coxeter groups whose growth rated are Pisot numbers.

(3) Kellerhals (2011?) conjectured that every hyperbolic (W, S) has a Perron number as its growth rate. It seems to be a delicate problem heavily depending on the system of generators S:

An example of Machì:
$G = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$, $S = \{a, b^{\pm}\}$. Then
$f_S(t) = (1+t)(1+2t)/(1-t)(1-2t^2)$.

2. Cocompact 2 and 3-dimensional hyperbolic Coxeter groups

Theorem (Cannon-Wagreich 92, Parry 93)

The growth rates of cocompact 2 and 3-dimensional hyperbolic Coxeter groups are "Salem" numbers.

Proposition (Parry 93)

Let $c_2, \cdots, c_N \in \mathbf{N} \cup \{0\}$ be such that $\sum_{n=2}^{N} \frac{n-1}{n} c_n > 2$. Let $R(x)$ be the rational function

$$R(x) = \frac{x+1}{x-1} + \sum_{n=2}^{N} c_n \frac{x - x^n}{(x-1)(x^n-1)} = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are relatively prime $\mathbf{Z}$-polynomials. Then $P(x)$ is a product of distinct irreducible cyclotomic polynomials with exactly one "Salem" polynomial.

Salem or "Salem"? (K. 2013)

dim=2: pentagon with angles $\pi/2, \pi/4, \pi/4, \pi/4, \pi/4$

$$
\begin{aligned}
1/f_S(x^{-1}) &= 1 + \frac{x - x^2}{(x+1)(x^2-1)} + \frac{4(x - x^4)}{(x+1)(x^4-1)} \\
&= \frac{(x^2 - 4x + 1)(x^2 + x + 1)}{(x+1)^2(x^2+1)}
\end{aligned}
$$

$$
f_S(x) = \frac{(x+1)^2(x^2+1)}{(x^2 - 4x + 1)(x^2 + x + 1)}
$$

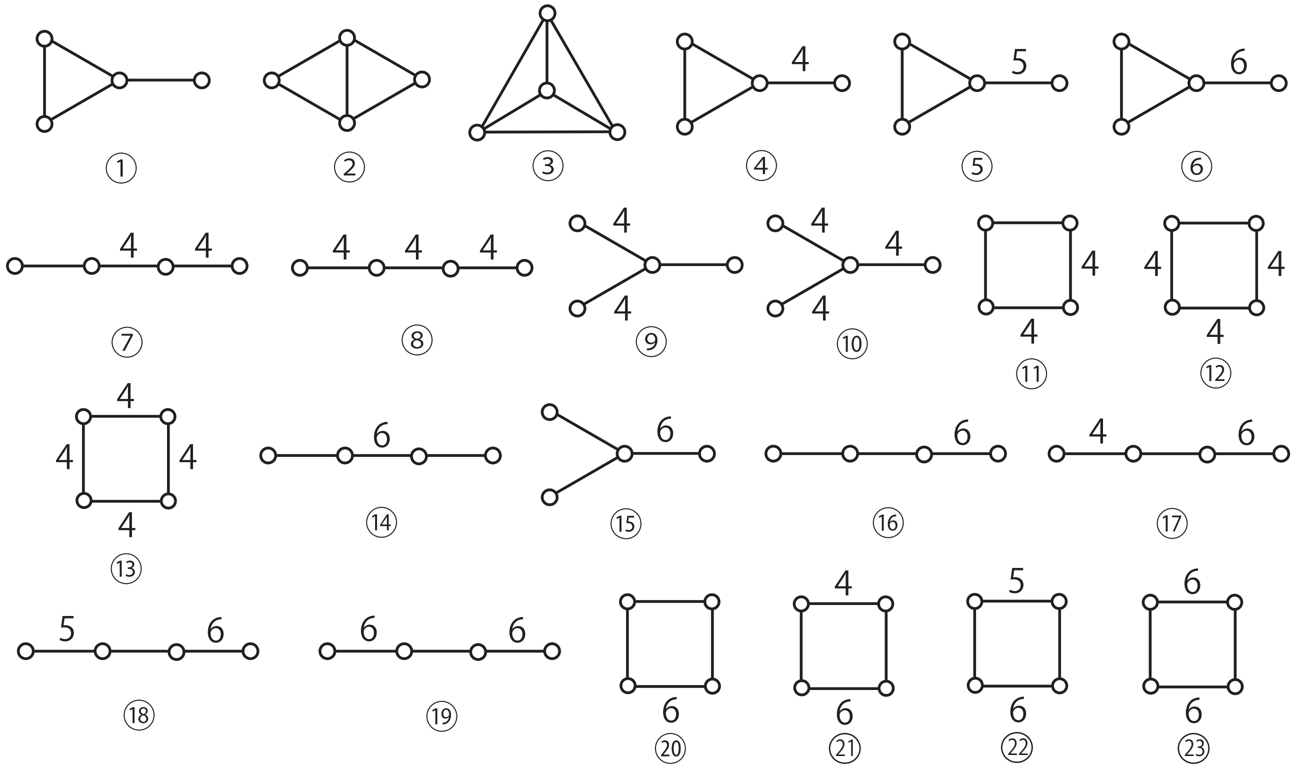
dim=3: Lambert cube

$$\begin{aligned} 1/f_S(x^{-1}) &= 1 - \frac{6}{x+1} + \frac{9}{(x+1)^2} + \frac{3}{(x+1)(x^3+x^2+x+1)} \\ &\quad - \frac{2}{(x+1)^3} - \frac{6}{(x+1)^2(x^3+x^2+x+1)} \\ &= \frac{(x-1)(x^2-3x+1)(x^2+x+1)}{(x+1)^3(x^2+1)} \end{aligned}$$

$$f_S(x) = \frac{(x+1)^3(x^2+1)}{(x-1)(x^2-3x+1)(x^2+x+1)}$$

3. Cofinite 3-dimensional hyperbolic Coxeter groups

Classification of cofinite 3-dim. Coxeter simplexes (Lannér 50)



- $(t-1)(t^4+t^3+t^2+t-1)$

- $(t-1)(3t^2+t-1)$

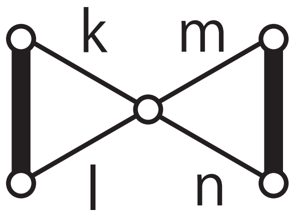
- $(t-1)(t^7+t^6+2t^5+2t^4+t^3+t^2-1)$

- $(t-1)(t^9+t^7+t^6+t^4+t^2+t-1)$

- $(t-1)(2t^5+t^4+t^2+t-1)$

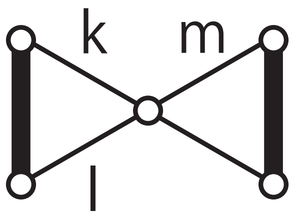
- $(t-1)(t^7+t^6+t^5+t^4+t^3-1)$

Classification of cofinite 3-dim.Coxeter pyramids (Tumarkin 2004)



$k = 2, 3, 4;$ $m = 2, 3, 4;$

$l = 3, 4;$ $n = 3, 4.$



$k = 5, 6;$ $m = 2, 3;$

$l = 2, 3, 4, 5, 6.$

- $(k,l,m,n) = (2,3,2,3):\ (t-1)(t^5+2t^4+2t^3+t^2-1)$

- $(2,3,2,4):\ (t-1)(t^7+t^6+2t^5+t^4+2t^3+t-1)$

- $(2,3,3,3):\ (t-1)(t^4+2t^3+t^2+t-1)$

- $(2,3,3,4):\ (t-1)(t^7+2t^6+2t^5+2t^4+2t^3+t^2+t-1)$

- $(2,3,4,4):\ (t-1)(t^5+t^4+t^3+2t-1)$

- $(2,4,2,4):\ (t-1)(t^4+2t^3+t^2+t-1)$

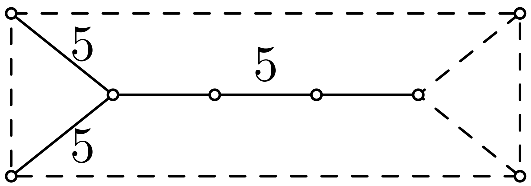
Proposition (K. and Umemoto 2012)

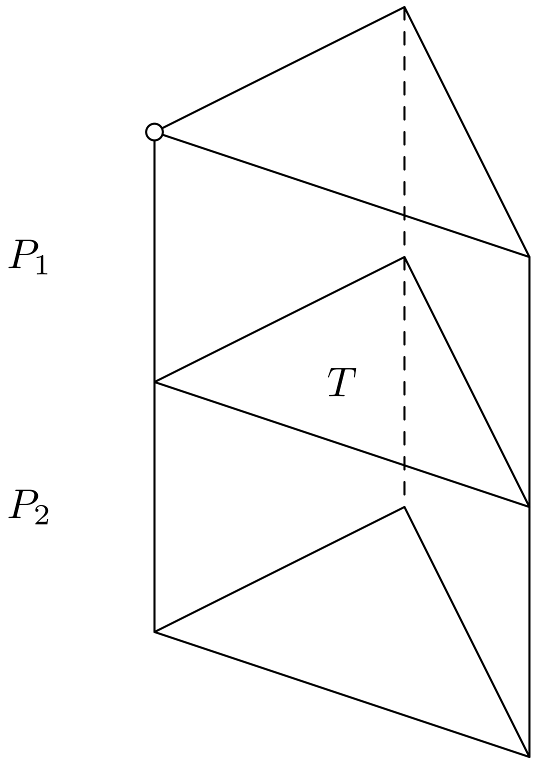
Consider the $\mathbf{Z}$-polynomial of degree $n \geq 2$

$$g(t) = \sum_{k=1}^{n} a_k t^k - 1,$$

where a_k is a non-negative integer. We also assume that the greatest common divisor of $\{k \in \mathbb{N} \mid a_k \neq 0\}$ is 1. Then there is a real number r_0, $0 < r_0 < 1$ which is the unique zero of $g(t)$ having the smallest absolute value of all zeros of $g(t)$.

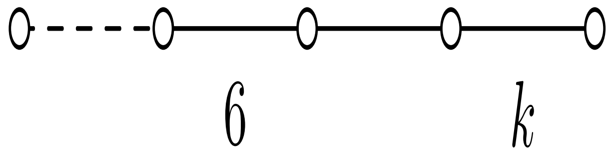
Classification of cofinite 3-dim. Coxeter prisms (Kaplinskaya 74)





P: a non-compact hyp. Coxeter prism with $k \geq 7$.

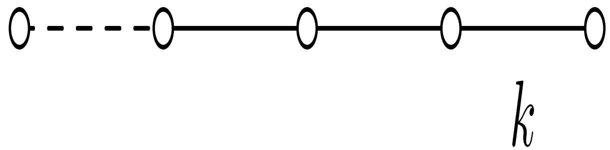
Then the growth function $f_{P_1}(t)$ of P_1 of the non-compact straight hyperbolic Coxeter prism P_1 with Coxeter diagram



can be calculated as

$$\frac{(t+1)^3(t^2-t+1)(t^2+t+1)(t^{k-1}+\cdots+t+1)}{(t-1)Q_1(t)}$$

where $Q_1(t) = 2t^{k+4} + 3t^{k+3} + 4t^{k+2} + 5t^{k+1} + 6t^k + \cdots + 6t^6 + 5t^5 + 3t^4 + 2t^3 + t^2 - 1$, while the growth function $f_{P_2}(t)$ of the compact straight hyperbolic Coxeter prism P_2 with Coxeter diagram



is equal to

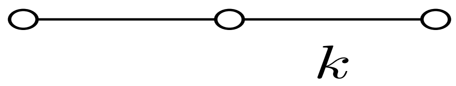
$$\frac{(t+1)^3(t^2+1)(t^2+t+1)(t^{k-1}+\cdots+t+1)}{(t-1)Q_2(t)}.$$

where $Q_2(t) = -t^{k+5} - t^{k+4} + 2t^{k+2} + 4t^{k+1} + 5t^k + \cdots + 5t^5 + 4t^4 + 2t^3 - t - 1.$

Now P is the "amalgam" of P_1 and P_2 along T, the growth function $f_P(t)$ of P satisfies

$$\frac{1}{f_P(t)} = \frac{1}{f_{P_1}(t)} + \frac{1}{f_{P_2}(t)} - \left(\frac{1-t}{1+t}\right)\frac{1}{f_T(t)}$$

where $f_T(t)$ is the growth function of the hyperbolic triangle T with Coxeter diagram



$$\frac{(t+1)^2(t^2+t+1)(t^{k-1}+\cdots+t+1)}{t^{k+3}+t^{k+2}-t^k-\cdots-t^3+t+1}.$$

As a conclusion, $f_P(t)$ of the prism P can be written as

$$\frac{(t+1)^3(t+1)^2(t^2-t+1)(t^2+t+1)(t^{k-1}+\cdots+t+1)}{(t-1)Q(t)}$$

where

$$\begin{aligned} Q(t) &= 2t^{k+6}+4t^{k+5}+7t^{k+4}+10t^{k+3}+12t^{k+2}+14t^{k+1} \\ &+ 15t^k+\cdots+14t^7+12t^6+9t^5+6t^4+3t^3+t^2-1. \end{aligned}$$

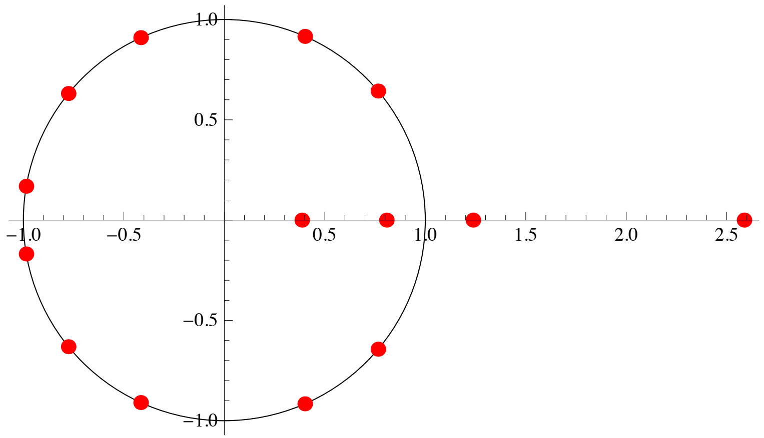
Theorem (K. and Umemoto 2012)

The growth rates of cofinite 3-dimensional hyperbolic Coxeter groups with 4 and 5 generators (i.e. simplexes, pyramids and prisms) are Perron numbers.

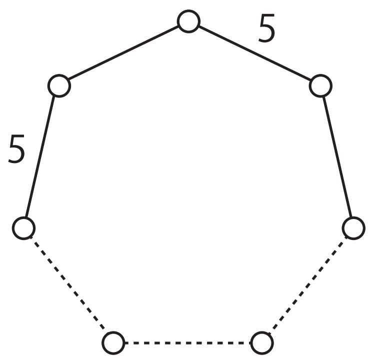
4. 2-Salem numbers as growth rates of 4-dim. Coxeter groups

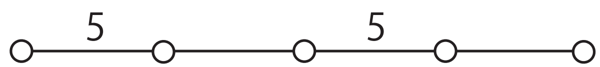
Definition (Samet 52, Kerada 95)

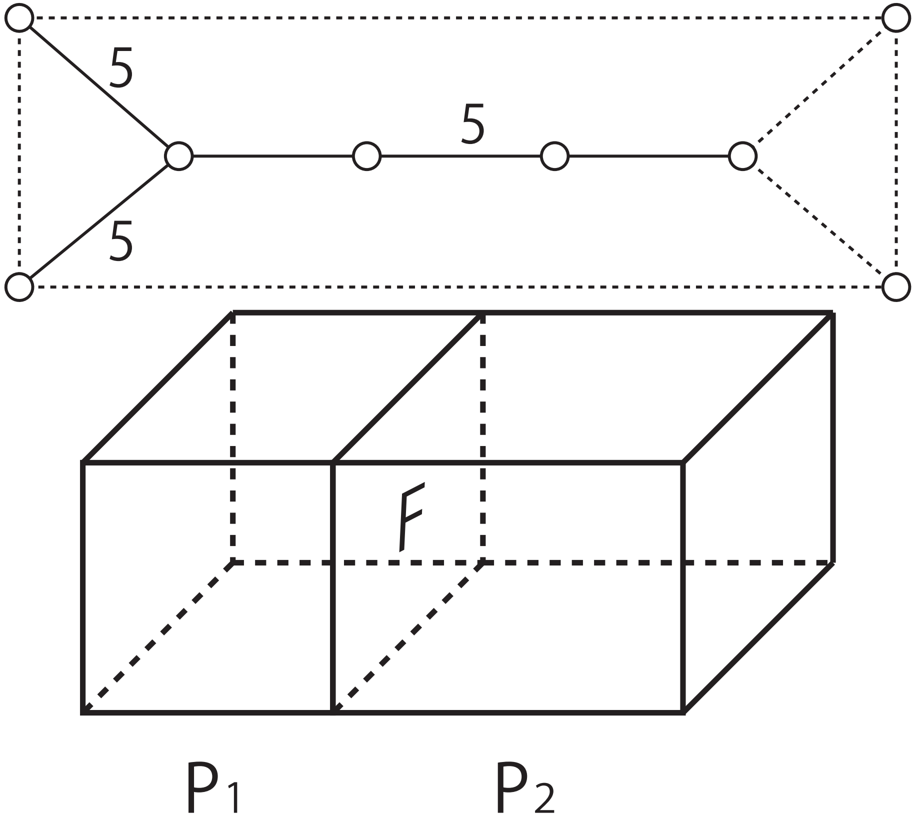
A real algebraic integer $\alpha > 1$ is called a 2-Salem number if it has a real conjugate $\beta > 1$ while other conjugates ω satisfy $|\omega| \leq 1$ and at least one of them is on the unit circle.



Coxeter garlands (T. Zehrt and C. Zehrt 2011)






5
5
5
F
P1
P2

Gluing formula (T. Zehrt and C. Zehrt)

Consider two Coxeter n-polytope P_1 and P_2 having the same orthogonal face F which is a Coxeter (n-1)-polytope, and let their growth functions be $W_1(t), W_2(t)$ and $F(t)$ respectively. Then the growth function $W_1 *_{P_0} W_2(t)$ of the Coxeter polytope obtained by gluing P_1 and P_2 along F is given by

$$\frac{1}{W_1 *_F W_2(t)} = \frac{1}{W_1(t)} + \frac{1}{W_2(t)} + (\frac{t-1}{1+t})\frac{1}{F(t)}$$

Let G_n be the Coxeter polytope constructed from n copies of G by (n-1)- gluings along orthogonal facets of G. Then the growth function of G_n is equal to $[2,2,5,6](t^5+1)/Z_n(t)$ where

$$Z_n(t) = t^{16} - 2(n+1)t^{15} + t^{14} + (n-1)t^{13} + t^{12} + nt^{11} \\ +(n-1)t^{10} + 2t^9 + 2(n-1)t^8 + 2t^7 + (n-1)t^6 \\ +nt^5 + t^4 + (n-1)t^3 + t^2 - 2(n+1)t + 1.$$

They showed that $Z_n(t)$ has 2 reciprocal pairs of positive real zeros and all the other zeros locate on the unit circle. Hence Coxeter garlands have "2-Salem" numbers as their growth rates.

Proposition (Kempner 35, T. Zehrt and C. Zehrt)

For $f \in \mathbf{Z}[t]$ be a palindromic polynomial of even degree $n \geq 2$ with $f(\pm 1) \neq 0$, define $g(u) \in \mathbf{Z}[u]$ by

$$g(u) := (\sqrt{u} - i)^n f(\frac{\sqrt{u} + i}{\sqrt{u} - i}).$$

Then

(1) $f(t)$ has 2k zeros on the unit circle iff $g(u)$ has k positive real zeros.

(2) $f(t)$ has 2ℓ real zeros iff $g(u)$ has ℓ negative real zeros.

Proposition (K. 2013)

Denominator polynomials $Z_n(t)$ are irreducible for any $n \in \mathbf{N}$. Hence Coxeter garlands have 2-Salem numbers as their growth rates.

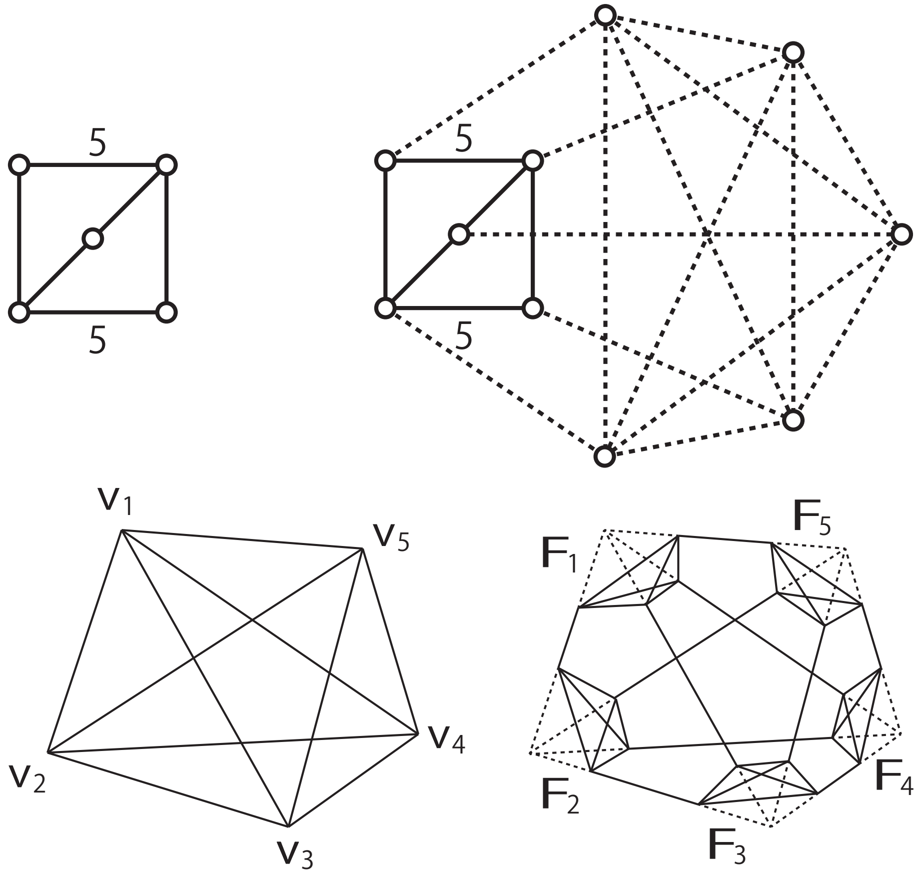
Key idea: $Z_n(i) = 2$ for all $n \in \mathbf{N}$.

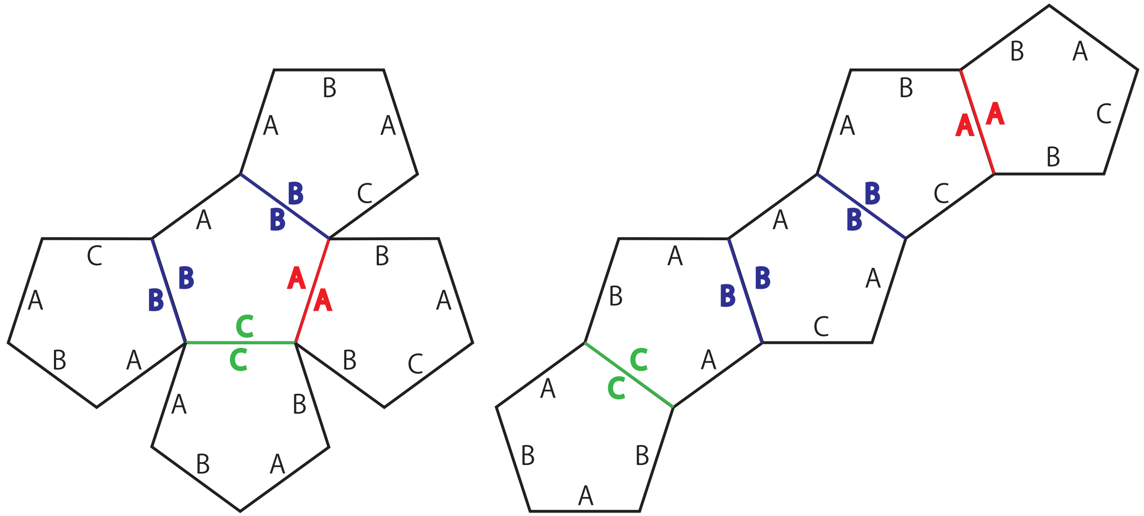
Suppose that $Z_n(t)$ is reducible in $\mathbf{Z}[t]$ as

$$(t^2 + pt + 1)(t^{14} + \cdots + 1).$$

Then $Z_n(i) = pi(a + bi) = 2$ implies that $p = -2$ or $p = -1$ which means $t = 1$ or $t = \frac{1\pm\sqrt{3}i}{2}$ must be a solution of $Z_n(t)$, but $Z_n(1) = 4n, Z_n(\frac{1\pm\sqrt{3}i}{2}) = (1 \mp \sqrt{3})(n + 1)$, a contradiction.

Coxeter dominoes (Yuriko Umemoto 2013)





Let $D_{\ell,m,n}$ be the Coxeter polytope constructed from $n+1$ copies of D by ℓ, m and $\ell-m$-times gluings along orthogonal facets of types A, B and C. Then the growth function of $D_{\ell,m,n}$ is equal to $[2,4,6,10]/Q_{\ell,m,n}(t)$ where

$$
\begin{aligned}
Q_{\ell,m,n}(t) &= t^{18} - (4n+6)t^{17} + (2n-m+3)t^{16} \\
&+ (3n-m+\ell+5)t^{15} - (n-4m+1)t^{14} - (n-4m+1)t^{13} \\
&+ (8n-4m+\ell+9)t^{12} + (5m-\ell)t^{11} + (10n-5m+\ell+11)t^{10} \\
&- (2n-6m+2)t^{9} + (10n-5m+\ell+11)t^{8} + (5m-\ell)t^{7} \\
&+ (8n-4m+\ell+9)t^{6} - (n-4m+1)t^{5} - (n-4m+1)t^{4} \\
&+ (3n-m+\ell+5)t^{3} + (2n-m+3)t^{2} - (4n+6)t + 1
\end{aligned}
$$

She showed that the zeros of $Q_{\ell,m,n}(t)$ are 2 reciprocal pairs of positive real zeros and the others locating on the unit circle. Hence Coxeter dominoes also have "2-Salem" numbers as their growth rates.

Theorem (Umemoto 2013)

For any $n \equiv 1 \mod 3$, Denominator polynomials $Q_{n,0,n}(t)$ and $Q_{0,n,n}(t)$ are irreducible. Hence these Coxeter dominoes have 2-Salem numbers as their growth rates.

Final reamrks

1. In general cocompact 4-dim hyp. Coxeter groups have not 2-Salem numbers as their growth rates.

2. There are notions of j-Salem or j-Pisot numbers (due to Samet and Kerada)