Arithmetic aspects of growth rates of hyperbolic Coxeter groups

Complex Hyperbolic Geometry and Related Topics

January 10th, Okayama University of Science

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Contents

1. Growth rates of hyperbolic Coxeter groups

2. Cocompact 2 and 3-dimensional hyperbolic Coxeter groups

3. Cofinite 3-dimensional hyperbolic Coxeter groups

4. 2-Salem numbers as growth rates of 4-dim. Coxeter groups
1. Growth rates of hyperbolic Coxeter groups

Hyperbolic Coxeter polyhedron $P = \cap_{i=1}^{k} H_{i}^{-} \subset \mathbb{H}^n$
all dihedral angles are $\pi/n$, ($n \in \mathbb{N} \cup \{\infty\}$)
$P$ is represented by its Coxeter diagram

Geometric Coxeter group $(W, S)$
$S = \{r_1, r_2, \cdots, r_k\}$, $W = \langle r_1, r_2, \cdots, r_k \rangle$
Growth function of \((W, S)\)

\[ f_S(t) = \sum_{k \geq 0} a_k t^k = 1 + \#St + \cdots \]
where \(a_k = \#\{g \in W \mid \ell_S(g) = k\}\)

The growth rate of \((W, S)\): \(\tau := \limsup_{k \to \infty} \sqrt[k]{a_k} = 1/R \) (\(R\): the radius of convergence of \(f_S(t)\))

\(\tau > 1\) i.e. of exponential growth (de la Harpe 87?)
Theorem (Steinberg 68)
Let us denote by $(W_T, T)$ the Coxeter subgroup of $(W, S)$ generated by the subset $T \subseteq S$, and let its growth function be $f_T(t)$. Set $\mathcal{F} = \{ T \subseteq S : W_T \text{ is finite} \}$. Then
\[
\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.
\]

Theorem (Solomon 66)
The growth function $f_S(t)$ of an irreducible finite Coxeter group $(G, S)$ can be written as $f_S(t) = \prod_{i=1}^{k} [m_i + 1]$ where $[n] := 1 + t + \cdots + t^{n-1}$ and $\{m_1, m_2, \cdots, m_k\}$ is the set of exponents of $(G, S)$. 
\[
\frac{1}{f_s(t^{-1})} = \tilde{Q}(t)/\tilde{P}(t) \Rightarrow f_s(t) = P(t)/Q(t)
\]

where \(P(t) = t^n\tilde{P}(t), Q(t) = t^n\tilde{Q}(t)\).

Hence \(R = 1/\tau\) is the smallest positive root of \(Q(t)\).

Since \(\tilde{Q}(t)\) is monic, \(\tau > 1\) is an algebraic integer.
<table>
<thead>
<tr>
<th>type of subgroup</th>
<th>growth function</th>
<th>number</th>
</tr>
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<tbody>
<tr>
<td>$A_3$</td>
<td>$[2, 3, 4]$</td>
<td>2</td>
</tr>
<tr>
<td>$A_2 \times A_1$</td>
<td>$[2, 2, 3]$</td>
<td>1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$[2, 3]$</td>
<td>4</td>
</tr>
<tr>
<td>$A_1 \times A_1$</td>
<td>$[2, 2]$</td>
<td>2</td>
</tr>
</tbody>
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\[
\frac{1}{f_{S}(t^{-1})} = \frac{-2}{[2, 3, 4]} + \frac{-1}{[2, 2, 3]} + \frac{4}{[2, 3]} + \frac{2}{[2, 2]} + \frac{-4}{[2]} + 1.
\]

\[
f_{S}(t) = \frac{(t + 1)(t^2 + 1)(t^2 + t + 1)}{(t - 1)(t^3 + t - 1)}.
\]
A real algebraic integer $\tau > 1$ is called:

(1) a *Salem number* if $\tau^{-1}$ is a conjugate of $\tau$ and all conjugates of $\tau$ other than $\tau$ and $\tau^{-1}$ lie on the unit circle. We assume also that there exists a conjugate on the unit circle.

(1') a *“Salem” number* if $\tau^{-1}$ is a conjugate of $\tau$ and all conjugates of $\tau$ other than $\tau$ and $\tau^{-1}$ lie on the unit circle (i.e. quadratic units are also “Salem” number).
QUIZ: Which is Salem or “Salem”? 
A real algebraic integer $\tau > 1$ is called:

(2) a *Pisot number* if all algebraic conjugates of $\tau$ other than $\tau$ lie in the open unit disk.

(3) a *Perron number* if all of whose conjugates have strictly smaller absolute values.
Theorem (Cannon-Wagreich 92, Parry 93)
The growth rates of cocompact 2 and 3-dimensional hyperbolic Coxeter groups are “Salem” numbers.

Theorem (Floyd 92)
The growth rates of cofinite 2-dimensional hyperbolic Coxeter groups are Pisot numbers.

Theorem (K. and Umemoto 2012)
The growth rates of cofinite 3-dimensional hyperbolic Coxeter groups with 4 and 5 generators (i.e. simplexes, pyramids and prisms) are Perron numbers.
Remark

(1) Kellerhals and Perren (2011) observed numerically that many cocompact 4-dimensional hyperbolic Coxeter groups (including 5 and 6 generated groups) have Perron numbers as their growth rates.

(2) Kolpakov (2012) studied cofinite 3-dimensional hyperbolic Coxeter groups whose growth rated are Pisot numbers.
(3) Kellerhals (2011?) conjectured that every hyperbolic $(W, S)$ has a Perron number as its growth rate. It seems to be a delicate problem heavily depending on the system of generators $S$:

An example of Machì:

$G = \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z}$, $S = \{a, b^\pm\}$. Then

$$f_S(t) = \frac{(1 + t)(1 + 2t)}{(1 - t)(1 - 2t^2)}.$$
2. Cocompact 2 and 3-dimensional hyperbolic Coxeter groups

Theorem (Cannon-Wagreich 92, Parry 93)
The growth rates of cocompact 2 and 3-dimensional hyperbolic Coxeter groups are “Salem” numbers.
Proposition (Parry 93)

Let $c_2, \cdots, c_N \in \mathbb{N} \cup \{0\}$ be such that $\sum_{n=2}^{N} \frac{n-1}{n} c_n > 2$. Let $R(x)$ be the rational function

$$R(x) = \frac{x + 1}{x - 1} + \sum_{n=2}^{N} c_n \frac{x - x^n}{(x - 1)(x^n - 1)} = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are relatively prime $\mathbb{Z}$-polynomials. Then $P(x)$ is a product of distinct irreducible cyclotomic polynomials with exactly one “Salem” polynomial.
Salem or “Salem”? (K. 2013)

\text{dim} = 2: \text{pentagon with angles } \pi/2, \pi/4, \pi/4, \pi/4, \pi/4

\[ 1/f_S(x^{-1}) = 1 + \frac{x - x^2}{(x + 1)(x^2 - 1)} + \frac{4(x - x^4)}{(x + 1)(x^4 - 1)} \]
\[ = \frac{(x^2 - 4x + 1)(x^2 + x + 1)}{(x + 1)^2(x^2 + 1)} \]

\[ f_S(x) = \frac{(x + 1)^2(x^2 + 1)}{(x^2 - 4x + 1)(x^2 + x + 1)} \]
dim=3: Lambert cube

\[
\frac{1}{f_S(x^{-1})} = 1 - \frac{6}{x + 1} + \frac{9}{(x + 1)^2} + \frac{3}{6} \frac{(x + 1)(x^3 + x^2 + x + 1)}{(x + 1)^3} - \frac{(x + 1)^2(x^3 + x^2 + x + 1)}{(x - 1)(x^2 - 3x + 1)(x^2 + x + 1)}
\]

\[
f_S(x) = \frac{(x + 1)^3(x^2 + 1)}{(x - 1)(x^2 - 3x + 1)(x^2 + x + 1)}
\]
3. Cofinite 3-dimensional hyperbolic Coxeter groups

Classification of cofinite 3-dim. Coxeter simplexes (Lannér 50)
• \((t - 1)(t^4 + t^3 + t^2 + t - 1)\)

• \((t - 1)(3t^2 + t - 1)\)

• \((t - 1)(t^7 + t^6 + 2t^5 + 2t^4 + t^3 + t^2 - 1)\)

• \((t - 1)(t^9 + t^7 + t^6 + t^4 + t^2 + t - 1)\)

• \((t - 1)(2t^5 + t^4 + t^2 + t - 1)\)

• \((t - 1)(t^7 + t^6 + t^5 + t^4 + t^3 - 1)\)
Classification of cofinite 3-dim. Coxeter pyramids (Tumarkin 2004)

\[ k = 2, 3, 4; \quad m = 2, 3, 4; \]
\[ l = 3, 4; \quad n = 3, 4. \]

\[ k = 5, 6; \quad m = 2, 3; \]
\[ l = 2, 3, 4, 5, 6. \]
\begin{itemize}
  \item \((k, l, m, n) = (2, 3, 2, 3) : (t-1)(t^5 + 2t^4 + 2t^3 + t^2 - 1)\)
  \item \((2, 3, 2, 4) : (t-1)(t^7 + t^6 + 2t^5 + t^4 + 2t^3 + t - 1)\)
  \item \((2, 3, 3, 3) : (t-1)(t^4 + 2t^3 + t^2 + t - 1)\)
  \item \((2, 3, 3, 4) : (t-1)(t^7 + 2t^6 + 2t^5 + 2t^4 + 2t^3 + t^2 + t - 1)\)
  \item \((2, 3, 4, 4) : (t-1)(t^5 + t^4 + t^3 + 2t - 1)\)
  \item \((2, 4, 2, 4) : (t-1)(t^4 + 2t^3 + t^2 + t - 1)\)
\end{itemize}
Proposition (K. and Umemoto 2012)
Consider the $Z$-polynomial of degree $n \geq 2$

\[ g(t) = \sum_{k=1}^{n} a_k t^k - 1, \]

where $a_k$ is a non-negative integer. We also assume that the greatest common divisor of \( \{k \in \mathbb{N} \mid a_k \neq 0\} \) is 1. Then there is a real number $r_0$, $0 < r_0 < 1$ which is the unique zero of $g(t)$ having the smallest absolute value of all zeros of $g(t)$. 

22
Classification of cofinite 3-dim. Coxeter prisms (Kaplinskaya 74)

$P$: a non-compact hyp. Coxeter prism with $k \geq 7$. 
Then the growth function \( f_{P_1}(t) \) of \( P_1 \) of the non-compact straight hyperbolic Coxeter prism \( P_1 \) with Coxeter diagram

\[ \begin{array}{c}
\text{0} & \text{---} & \text{0} & \text{---} & \text{0} \\
\text{k} & \text{6} & \text{0} & \text{0} & \text{0}
\end{array} \]

can be calculated as

\[ \frac{(t + 1)^3(t^2 - t + 1)(t^2 + t + 1)(t^{k-1} + \cdots + t + 1)}{(t - 1)Q_1(t)} \]

where \( Q_1(t) = 2t^{k+4} + 3t^{k+3} + 4t^{k+2} + 5t^{k+1} + 6t^k + \cdots + 6t^6 + 5t^5 + 3t^4 + 2t^3 + t^2 - 1, \)
while the growth function $f_{P_2}(t)$ of the compact straight hyperbolic Coxeter prism $P_2$ with Coxeter diagram

$$0 \quad \cdots \quad 0 \quad 0 \quad 0$$

is equal to

$$\frac{(t + 1)^3(t^2 + 1)(t^2 + t + 1)(t^{k-1} + \cdots + t + 1)}{(t - 1)Q_2(t)}.$$

where $Q_2(t) = -t^{k+5} - t^{k+4} + 2t^{k+2} + 4t^{k+1} + 5t^k + \cdots + 5t^5 + 4t^4 + 2t^3 - t - 1.$
Now $P$ is the “amalgam” of $P_1$ and $P_2$ along $T$, the growth function $f_P(t)$ of $P$ satisfies

$$\frac{1}{f_P(t)} = \frac{1}{f_{P_1}(t)} + \frac{1}{f_{P_2}(t)} - (\frac{1-t}{1+t})}\frac{1}{f_T(t)}$$

where $f_T(t)$ is the growth function of the hyperbolic triangle $T$ with Coxeter diagram

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    k
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$$\frac{(t+1)^2(t^2 + t + 1)(t^{k-1} + \cdots + t + 1)}{t^{k+3} + t^{k+2} - t^k - \cdots - t^3 + t + 1}.$$
As a conclusion, $f_P(t)$ of the prism $P$ can be written as

$$
\frac{(t + 1)^3(t + 1)^2(t^2 - t + 1)(t^2 + t + 1)(t^{k-1} + \cdots + t + 1)}{(t - 1)Q(t)}
$$

where

$$Q(t) = 2t^k + 6 + 4t^{k+5} + 7t^{k+4} + 10t^{k+3} + 12t^{k+2} + 14t^{k+1} + 15t^k + \cdots + 14t^7 + 12t^6 + 9t^5 + 6t^4 + 3t^3 + t^2 - 1.$$ 

Theorem (K. and Umemoto 2012)
The growth rates of cofinite 3-dimensional hyperbolic Coxeter groups with 4 and 5 generators (i.e. simplexxes, pyramids and prisms) are Perron numbers.
4. 2-Salem numbers as growth rates of 4-dim. Coxeter groups

Definition (Samet 52, Kerada 95)
A real algebraic integer $\alpha > 1$ is called a 2-Salem number if it has a real conjugate $\beta > 1$ while other conjugates $\omega$ satisfy $|\omega| \leq 1$ and at least one of them is on the unit circle.
Coxeter garlands (T. Zehrt and C. Zehrt 2011)
Gluing formula (T. Zehrt and C. Zehrt)
Consider two Coxeter n-polytope $P_1$ and $P_2$ having the same orthogonal face $F$ which is a Coxeter (n-1)-polytope, and let their growth functions be $W_1(t), W_2(t)$ and $F(t)$ respectively. Then the growth function $W_1 *_{P_0} W_2(t)$ of the Coxeter polytope obtained by gluing $P_1$ and $P_2$ along $F$ is given by

$$\frac{1}{W_1 *_{F} W_2(t)} = \frac{1}{W_1(t)} + \frac{1}{W_2(t)} + \left(\frac{t-1}{1+t}\right) \frac{1}{F(t)}$$
Let $G_n$ be the Coxeter polytope constructed from $n$ copies of $G$ by (n-1)- gluings along orthogonal facets of $G$. Then the growth function of $G_n$ is equal to 

$$[2, 2, 5, 6](t^5 + 1)/Z_n(t)$$

where

$$Z_n(t) = t^{16} - 2(n + 1)t^{15} + t^{14} + (n - 1)t^{13} + t^{12} + nt^{11} + (n - 1)t^{10} + 2t^9 + 2(n - 1)t^8 + 2t^7 + (n - 1)t^6 + nt^5 + t^4 + (n - 1)t^3 + t^2 - 2(n + 1)t + 1.$$ 

They showed that $Z_n(t)$ has 2 reciprocal pairs of positive real zeros and all the other zeros locate on the unit circle. Hence Coxeter garlands have “2-Salem” numbers as their growth rates.
Proposition (Kempner 35, T. Zehrt and C. Zehrt)

For \( f \in \mathbb{Z}[t] \) be a palindromic polynomial of even degree \( n \geq 2 \) with \( f(\pm 1) \neq 0 \), define \( g(u) \in \mathbb{Z}[u] \) by

\[
g(u) := (\sqrt{u} - i)^n f\left(\frac{\sqrt{u} + i}{\sqrt{u} - i}\right).
\]

Then

(1) \( f(t) \) has 2k zeros on the unit circle iff \( g(u) \) has k positive real zeros.

(2) \( f(t) \) has 2\( \ell \) real zeros iff \( g(u) \) has \( \ell \) negative real zeros.
Proposition (K. 2013)
Denominator polynomials $Z_n(t)$ are irreducible for any $n \in \mathbb{N}$. Hence Coxeter garlands have 2-Salem numbers as their growth rates.

**Key idea:** $Z_n(i) = 2$ for all $n \in \mathbb{N}$.

Suppose that $Z_n(t)$ is reducible in $\mathbb{Z}[t]$ as

$$(t^2 + pt + 1)(t^{14} + \cdots + 1).$$

Then $Z_n(i) = pi(a + bi) = 2$ implies that $p = -2$ or $p = -1$ which means $t = 1$ or $t = \frac{1 \pm \sqrt{3}i}{2}$ must be a solution of $Z_n(t)$, but $Z_n(1) = 4n$, $Z_n(\frac{1 \pm \sqrt{3}i}{2}) = (1 \mp \sqrt{3})(n + 1)$, a contradiction.
Coxeter dominoes (Yuriko Umemoto 2013)
Let $D_{\ell,m,n}$ be the Coxeter polytope constructed from $n + 1$ copies of $D$ by $\ell, m$ and $\ell - m$-times gluings along orthogonal facets of types A, B and C. Then the growth function of $D_{\ell,m,n}$ is equal to $[2, 4, 6, 10]/Q_{\ell,m,n}(t)$ where
\[ Q_{\ell,m,n}(t) = t^{18} - (4n + 6)t^{17} + (2n - m + 3)t^{16} \\
+ (3n - m + \ell + 5)t^{15} - (n - 4m + 1)t^{14} - (n - 4m + 1)t^{13} \\
+ (8n - 4m + \ell + 9)t^{12} + (5m - \ell)t^{11} + (10n - 5m + \ell + 11)t^{10} \\
- (2n - 6m + 2)t^9 + (10n - 5m + \ell + 11)t^8 + (5m - \ell)t^7 \\
+ (8n - 4m + \ell + 9)t^6 - (n - 4m + 1)t^5 - (n - 4m + 1)t^4 \\
+ (3n - m + \ell + 5)t^3 + (2n - m + 3)t^2 - (4n + 6)t + 1 \]

She showed that the zeros of \( Q_{\ell,m,n}(t) \) are 2 reciprocal pairs of positive real zeros and the others locating on the unit circle. Hence Coxeter dominoes also have “2-Salem” numbers as their growth rates.
Theorem (Umemoto 2013)
For any $n \equiv 1 \mod 3$, Denominator polynomials $Q_{n,0,n}(t)$ and $Q_{0,n,n}(t)$ are irreducible. Hence these Coxeter dominoes have 2-Salem numbers as their growth rates.

Final remarks
1. In general cocompact 4-dim hyp. Coxeter groups have not 2-Salem numbers as their growth rates.

2. There are notions of j-Salem or j-Pisot numbers (due to Samet and Kerada)