On the limit set of complex Kleinian groups

Conference in honor of Professor Shigeyasu Kamiya.
Okayama, Japan,
9-11 January, 2015.

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I) Introduction: Classical Kleinian groups.

Kleinian groups were introduced by Poincaré as the monodromy groups of certain second order differential equations.

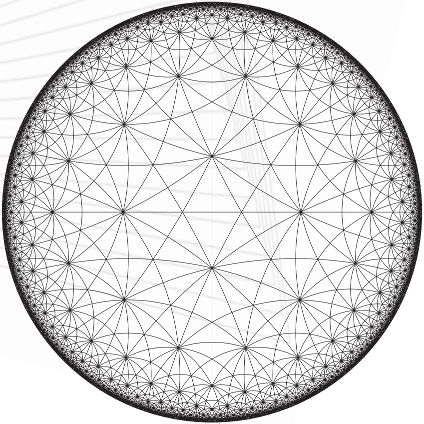
More generally:

Definition

A (classical) Kleinian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ acting on the Riemann sphere $\mathbb{S}^2 \cong \hat{\mathbb{C}} \cong \mathbb{CP}^1$ with non-empty region of discontinuity.

Examples:

- Fuchsian groups (*i.e.*, discrete subgroups of $\mathrm{PU}(1,1)$).
- Fundamental groups of hyperbolic manifolds of dimensions 2, 3.
- Schottky and kissing Schottky groups, etc.



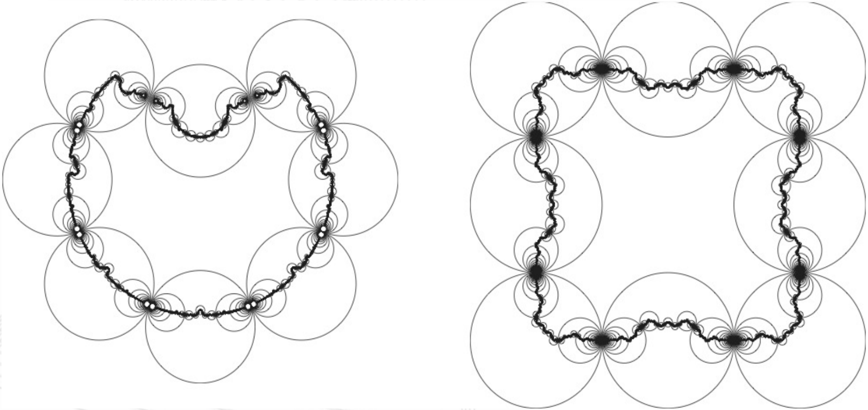


Figura : Kissing Schottky groups

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A basic property

A Kleinian group G splits the sphere in two invariant sets: the limit set $\Lambda(G)$ and the ordinary set $\Omega(G)$. The latter is where the group acts discontinuously, and $\Lambda(G)$ is its complement.

Theorem (Some fundamental properties of limit set)

- *$\Lambda(G)$ is closed and non-empty (unless G is finite).*
- *$\Lambda(G)$ is the set of cluster points of the orbits in $\mathbb{CP}^1$*
- *$\Lambda(G)$ consists of 1, 2 or infinitely many points. (The group is said to be elementary when $\Lambda(G)$ has finite cardinality.)*
- *If G is not elementary, then $\Lambda(G)$ is a nowhere dense perfect set, minimal for the action of G.*

Theorem (Some fundamental properties of ordinary set)

- *$\Omega(G)$ is largest set where action is properly discontinuously.*
- *$\Omega(G)/G$ is a Riemann surface with a projective orbifold structure.*
- *$\Omega(G)$ coincides with equicontinuity set.*
- *The number of connected components of $\Omega(G)$ can be $0, 1, 2$ or ∞.*

Many more properties that I will not mention.

Action on $\Omega(G)$ is "mild". The study of orbit spaces $\Omega(G)/G$ is the paradigm of complex geometry.

Action on $\Lambda(G)$ is "wild": It is here where dynamics concentrates. Paradigm of holomorphic dynamics

II) Higher dimensional setting

In complex dimension 1 we have isomorphisms:

$$\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Conf}_+(\mathbb{S}^2) \cong \mathrm{Iso}_+(\mathbb{H}^3_{\mathbb{R}})$$

In higher dimensions there is a dichotomy :

• **We can look at** $\mathrm{Conf}_+(\mathbb{S}^n)\,;\, n \geq 2$. Same as $\mathrm{Iso}_+(\mathbb{H}^{n+1}_{\mathbb{R}})$. Rich theory with outstanding developments by many authors. Main-stream for several decades.

• **We can also look at** $\mathrm{PSL}(\mathrm{n}+1,\mathbb{C})$**, holomorphic automorphisms of** $\mathbb{CP}^n$. Rich theory too, still in its childhood in many aspects.

Discrete groups of complex projective transformations have appeared in various settings. For instance, the most natural:

• **Discrete subgroups of $\mathrm{PU}(\mathrm{n}, 1)$, the group of holomorphic isometries of complex hyperbolic space**. This is itself a very rich and interesting area. Interesting work by various authors: **Professor Kamiya has made remarkable contributions.**

• **Subgroups of $\mathrm{Aff}(\mathbb{C}^n)$**. Include fundamental groups of Hopf surfaces and Inoue surfaces, and many more.

• **Discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$ appear also as the monodromy** groups of certain:

- Partial differential equations (M. Yoshida, 1980).
- Higher order ordinary differential equations.
- Ricatti Foliations (B. Scardua).

• Discrete subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$ have appeared **also in interesting papers by M. Kato** on compact complex 3-folds, and by **M. Nori** as a mean to constructing compact complex manifolds (with a projective structure) in higher dimensions.

In 1990 J. Seade & Alberto Verjovsky introduced the following concept. This unifies previous examples:

Definition

A complex Kleinian group is a discrete subgroup of projective transformations that acts on some $\mathbb{P}^n_{\mathbb{C}}$ leaving invariant a non-empty open set where the action is properly discontinuous.

We gave several ways for constructing such groups, besides complex hyperbolic and complex affine groups **(via twistor theory, Schottky groups, suspensions, etc.)** For instance we proved:

Theorem

The fundamental group of every real hyperbolic manifold of dimension 5 acts canonically on $\mathbb{CP}^3$ as a subgroup of $\mathrm{PSL}(4, \mathbb{C})$ and every orbit is dense.

Our focus now is in complex dimension 2.
Some of the main questions we have studied are:

- What open subsets of $\mathbb{CP}^2$ appear as invariant sets where a discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$ acts properly discontinuously with compact quotient. What about the orbit spaces?
- What is the “correct"notion of the limit set? What can we say about its geometry, topology and dynamics? and about its complement?
- What type of complex Kleinian groups one has? (Classification?)
- Is there a Sullivan's dictionary in dimension 2, comparing Kleinian groups and rational maps?

From now on I speak about work done by our team in Mexico, mostly with;

Angel Cano,

W. Barrera and

J. P. Navarrete,

and others.

Also a recent work with John Parker.

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About I: divisible sets.

(Joint with Angel Cano (Geometria Dedicata 2014))

Definition

A subgroup $G \subset \mathrm{PSL}(3,\mathbb{C})$ is quasi-cocompact if there exists a non-empty G-invariant open set $\Omega \subset \mathbb{CP}^2$ on which G acts properly discontinuously with compact quotient Ω/G.

Notice Ω/G is a compact orbifold with a complex projective structure.

The following theorem improves a classical theorem of Kobayashi-Ochiai, Inoue and Klingler for compact complex surfaces with a projective structure.

Theorem (Cano-Seade)

Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a quasi-cocompact group. Then G is either virtually affine or complex hyperbolic. Morever

- *If G is not virtually cyclic, then the open set U is contained in $\Omega_{Kul}(G)$, which is the largest open set on which G acts properly discontinuously.*
- *If G is not virtually the fundamental group of an Inoue Surface or the fundamental group of a primary Kodaira surface then $Eq(G) = \Omega_{Kul}(G)$.*

The full version of this result also provides the complete list of:

- The open sets in $\mathbb{P}^2_{\mathbb{C}}$ that can appear as (maximal) regions where the action is properly discontinuous.
- The classification of all quasi-cocompact groups.
- The corresponding orbifolds.

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II) The limit set

In dimension 1, limit set is:

a) Complement of the set where action is discontinuous.

b) Complement of the largest set where action is properly discontinuous.

c) Complement of region of equicontinuity.

d) Set of cluster points of orbits.

e) Closure of set of fixed points of loxodromic elements.

In higher dimensions, life is “richer".

The Kulkarni-Navarrete's example:

Let G be the cyclic group generated by the projective transformation:

$$\begin{bmatrix} 2 & & \\ & 1 & \\ & & 2^{-1} \end{bmatrix}$$

This has three fixed points $\{[e_1],[e_2],[e_3]\}$, which determine three invariant lines.

Two of these lines are specially interesting because they are repelling/attracting: $\overleftrightarrow{[e_2],[e_1]}$ and $\overleftrightarrow{[e_2],[e_3]}$.

Easy to show:

- All orbits of points accumulate at $\{[e_1], [e_2], [e_3]\}$ but action is not properly discontinuous on the complement of these points.
- One has two sets:
  $$\mathbb{P}^2 \setminus \left(\overleftrightarrow{[e_2], [e_1]} \cup \{[e_3]\} \right), \qquad \mathbb{P}^2 \setminus \left(\overleftrightarrow{[e_2], [e_3]} \cup [e_1] \right)$$
  which are maximal regions on which G acts properly discontinuously. There is not one such largest set.
- The equicontinuity set is: $\mathbb{P}^2 \setminus \left(\overleftrightarrow{[e_2], [e_1]} \cup \overleftrightarrow{[e_2], [e_3]} \right) = Eq(G)$.
- The discontinuity set of G is: $\mathbb{P}^2 \setminus (\{[e_1], [e_2], [e_3]\})$.

So ..... whom should we call the limit set?

There are other examples with even more “pathologies".

The Kulkarni limit set

We consider discrete subgroups of $\mathrm{PSL(n+1,\mathbb{C})}$ acting on $\mathbb{P}^n_{\mathbb{C}}$.

Definition

- Let $\Lambda(G)$ be the closure of the set cluster points of orbits .
- The Kulkarni's limit set is $\Lambda_{Kul}(G) = \Lambda(G) \cup L_2(G)$, where $L_2(G)$ is the closure of cluster points of the family $\{GK : K \subset \mathbb{P}^n \setminus \Lambda(G) \text{ is compact}\}$.
- The Kulkarni region of discontinuity is $\Omega_{Kul}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda_{Kul}(G)$.

It is easy to see that the action on $\Omega_{Kul}(G)$ is properly discontinuous. Hence: the group is complex Kleinian if $\Omega_{Kul}(G)$ is non-empty.

We'll show this is the correct notion of limit set in complex dimension 2 (Not so in higher dimensions)

Even so, there are examples (as the previous one) which are "exceptional". This corresponds to the "elementary groups", which in some sense are special.

In previous example, the Kulkarni limit set consists of exactly two lines, and its complement coincides with the region of equicontinuity.

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Structure of the limit set

For Kleinian subgroups of $\mathrm{PSL}(2,\mathbb{C})$ the limit set may consist of 1 or 2 points, or else it has infinite cardinality. **Previous example in $\mathbb{CP}^2$ has two lines as limit set**.
Other examples:

Example (a line; parabolic)

Consider the cyclic group generated by the projectivization of the map:

$$\tilde{\gamma} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then 1 is the only eigenvalue. The limit set is

$$\Lambda_{Kul}(G) = \overleftrightarrow{e_1, e_2}.$$

Another example:

Example (a line and a point; loxodromic)

Now let G be the cyclic group generated by the projectivization of the map:

$$\tilde{\gamma} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}, \text{ with } |\alpha| \neq 1 .$$

Then $L_0 = L_1 = L_2$ is the union of the line $\overleftrightarrow{e_1, e_2}$ and the point e_3. Hence $\Lambda_{Kul}(G)$ is now:

$$\Lambda_{Kul}(G) = \overleftrightarrow{e_1, e_2} \cup \{e_3\} .$$

Example (three lines)

Now consider the group generated by the matrix in previous example, together with a new generator:

$$\widetilde{G} = \left\langle \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, |\alpha| \neq 1 \right\rangle$$

Second matrix permutes the invariant lines $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_2, e_3}$ and $\overleftrightarrow{e_3, e_1}$. Hence the limit set is:

$$\Lambda_{Kul}(G) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3} \cup \overleftrightarrow{e_3, e_1}.$$

So we have examples of groups where the limit set has:
i) **one line**; ii) **one line and one point**; iii) **two lines**; iv) **three lines**; and v) **infinitely many lines**.

Theorem (Cano-Seade)

Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a complex Kleinian group. Then $\Lambda_{Kul}(G)$ contains at least one projective line.

Theorem (Barrera-Cano-Navarrete)

Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete group, then:

- *The number of lines in $\Lambda_{Kul}(G)$ is 1, 2, 3 or infinite.*
- *The number of lines in general position contained in $\Lambda_{Kul}(G)$ is 1, 2, 3, 4 or infinite. (All these cases actually take place.)*
- *The number of isolated points in $\Lambda_{Kul}(G)$ is at most one, and that can only happen when the group is virtually cyclic.*

PROBLEM: Understand topology and geometry of the discontinuity region and the limit set $\Lambda_{Kul}(G)$, and the dynamics on the latter.

Not easy. Have partial results, actually just the first steps. A lot to be done.

To deepen our study, let us focus in the particularly rich class of **comply hyperbolic groups.**

III) Complex hyperbolic groups

Consider discrete $G \subset \mathrm{PU}(\mathrm{n}, 1)$.

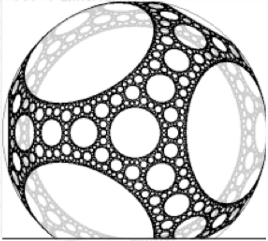
Acts on $\mathbb{CP}^n$ preserving the ball $\mathbb{H}^n_{\mathbb{C}}$ consisting of points in $\mathbb{CP}^n$ whose homogeneous coordinates $(z_1, ..., z_{n+1})$ satisfy:

$$|z_1|^2 + |z_1|^2 + \cdots + |z_n|^2 < |z_{n+1}|^2$$

In this setting there is another useful concept of a limit set, following classical (Poincaré's) definition:

Definition

The Chen-Greenberg limit set $\Lambda_{CG}(G)$ is the set of accumulation points of orbits in $\mathbb{H}^n_{\mathbb{C}}$.



As in the classical case one has $\Lambda_{CG}(G) \subset \partial \mathbb{H}^n_{\mathbb{C}} \cong \mathbb{S}^{2n-1}$, since action on $\mathbb{H}^n_{\mathbb{C}}$ is by isometries.

This definition of a limit set is good if we look at action only on $\mathbb{H}^n_{\mathbb{C}}$: It has all properties of usual limit set for Kleinian groups,

• **Yet, this takes no notice of the action on the complement $\mathbb{CP}^n \setminus \mathbb{H}^n_{\mathbb{C}}$.**

The next result determines relation between Chen-Greenberg and Kulkarni limit sets.

This provides some global info about action on $\mathbb{CP}^2 \setminus \mathbb{H}^2_{\mathbb{C}}$ out from info about the action on the ball $\mathbb{H}^2_{\mathbb{C}}$

For each $x \in \partial\mathbb{H}^2_{\mathbb{C}}$, let $\mathcal{L}_x$ be the unique complex projective line in $\mathbb{CP}^2$ tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at x.

Theorem (Navarrete 2006)

The Kulkarni limit set is

$$\Lambda_{Kul}(G) = \bigcup_{x \in \Lambda_{CG}(G)} \mathcal{L}_x$$

Hence, if G is non-elementary, then $\Lambda_{Kul}(G)$ has ∞-many lines.

Want to deepen understanding: Restrict to a particularly interesting special case: **joint work with Angel Cano and John Parker.**

Start with an easy example:

Example ($\mathbb{C}$-Fuchsian groups)

Let $G \subset \mathrm{U}(1,1) \cong \mathrm{Iso}_+(\mathbb{H}^2_{\mathbb{R}})$ cofinite; and consider inclusion:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \right] \in \mathrm{PU}(2,1).$$

Leaves invariant sphere $\partial\mathbb{H}^2_{\mathbb{C}}$ and also leaves invariant the projective line $\mathcal{L}_\infty = \{[0 : z_2 : z_3] : (z_2, z_3) \in \mathbb{C}^2 \setminus \{(0,0)\}\}$. Hence it leaves invariant the circle

$$\partial\mathbb{H}^1_{\mathbb{C}} = \partial\mathbb{H}^2_{\mathbb{C}} \cap \mathcal{L}_\infty = \{[0 : e^{i\phi} : 1] : \phi \in [0, 2\pi)\}.$$

This circle turns out to be **the Chen-Greenberg limit set.**

Every line tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at a point in $\partial\mathbb{H}^2_{\mathbb{C}} \cap \mathcal{L}_\infty$ passes thru $[1:0:0]$, the projective dual of line $\mathcal{L}_\infty$.

Thus Navarrrete's theorem implies:
$\Lambda_{Kul}(G)$ is a cone, union of all projective lines passing thru $[1:0:0]$ and a point in circle $\partial\mathbb{H}^1_{\mathbb{C}}$.

$\Rightarrow \Lambda_{Kul}(G) \setminus \{[1:0:0]\}$ is a solid torus $\mathbb{S}^1 \times \mathbb{C}$.

Complement Ω_{Kul} is a trivial fibre bundle over the projective line $\mathcal{L}_\infty$ minus the equator $\partial\mathbb{H}^1_{\mathbb{C}}$, with fibre $\mathbb{C}$.

Projection $\Omega_{Kul} \longrightarrow \mathcal{L}_\infty \setminus \partial\mathbb{H}^1_{\mathbb{C}}$ is easy to describe:

If $x \in \Omega_{Kul}$ let ℓ_x be projective line determined by x and $\{[1:0:0]\}$. Then $x \mapsto (\ell_x \cap \mathcal{L}_\infty)$

Hence $\Omega_{Kul} \cong$ two copies of $D^2 \times \mathbb{C}$, where $D^2 =$ open 2-disc.

A special type of a "suspension group": Easy to describe dynamics on $\mathbb{CP}^2$ out from the action on $\mathcal{L}_\infty \cong \mathbb{CP}^1$.

Example ($\mathbb{R}$-Fuchsian groups)

Now consider same $G \subset \mathrm{Iso}_+(\mathbb{H}^2_{\mathbb{R}})$, now regarded as $\mathrm{SO}_+(2,1)$, and its inclusion in $\mathrm{PU}(2,1)$ via its embedding in $\mathrm{SU}(2,1)$.

Action on $\mathbb{CP}^2$ leaves invariant:

a) $\mathbb{H}^2_{\mathbb{C}}$ and its boundary $\partial\mathbb{H}^2_{\mathbb{C}} = \mathbb{S}^3$

b) $\mathcal{P}_{\mathfrak{R}} = \mathbb{P}(\{(x,y,z \in \mathbb{C}^3 | x,y,z \in \mathbb{R}\}) \cong \mathbb{RP}^2$, and

c) the intersections: $\mathbb{H}^2_{\mathbb{C}} \cap \mathcal{P}_{\mathfrak{R}} = \mathbb{H}^2_{\mathbb{R}}$;
$\partial\mathbb{H}^2_{\mathbb{C}} \cap \mathcal{P}_{\mathfrak{R}} = \partial\mathbb{H}^2_{\mathbb{R}} = \mathbb{RP}^1 \cong \mathbb{S}^1$

Now the Chen-Greenberg limit set is $\Lambda_{CG}(G) = \partial\mathbb{H}^2_{\mathbb{R}} \cong \mathbb{RP}^1$.

Want to look at Kulkarni limit set $\Lambda_{Kul}(G)$.

This is the union of all projective lines tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at points in $\Lambda_{CG}(G) = \partial\mathbb{H}^2_{\mathbb{R}}$.

Not as simple as $\mathbb{C}$-Fuchsian case: In that case all lines were confocal.

Now each point in interior of Möbius strip $\mathcal{M} := \mathcal{P}_{\mathfrak{R}} \setminus \mathbb{H}^2_{\mathbb{R}}$ is the meeting point of exactly two lines in $\Lambda_{Kul}(G)$.
Hence $\mathcal{M} \subset \Lambda_{Kul}(G)$.

What else we have in $\Lambda_{Kul}(G)$?

Need to determine when a projective line in $\mathbb{CP}^2$ is tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at a point in $\partial\mathbb{H}^2_{\mathbb{R}}$.

i.e. Want to characterise 2-planes in $\mathbb{C}^{2,1}$ which give rise to projective lines tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$, particularly –but not only– at points in $\partial\mathbb{H}^2_{\mathbb{R}}$.

Use Hermitian cross product $\boxtimes : \mathbb{C}^{2,1} \times \mathbb{C}^{2,1} \to \mathbb{C}^{2,1}$ (Bill Goldman)

Can be defined by:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \boxtimes \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \overline{z}_3\overline{w}_2 - \overline{z}_2\overline{w}_3 \\ \overline{z}_1\overline{w}_3 - \overline{z}_3\overline{w}_1 \\ \overline{z}_1\overline{w}_2 - \overline{z}_2\overline{w}_1 \end{pmatrix}.$$

For every $\lambda,\ \mu \in \mathbb{C}^*$ and for every $\mathbf{z}$, $\mathbf{w} \in \mathbb{C}^{2,1}$ one has:

$$(\lambda \mathbf{z}) \boxtimes (\mu \mathbf{w}) = \overline{\lambda}\,\overline{\mu}\,(\mathbf{z} \boxtimes \mathbf{w})\,.$$

Alternating form, bilinear, except that scalars act via their complex conjugate.

If $\mathbf{z}$ and $\mathbf{w}$ are L. D. $\Rightarrow \boxtimes = (0, 0, 0)$

If $\mathbf{z}$ and $\mathbf{w}$ are L. I. $\Rightarrow \boxtimes$ is a vector orthogonal to both $\mathbf{z}$ and $\mathbf{w}$.

Theorem

1. Λ_{Kul} *is a 3-dimensional semi-algebraic set defined by:* $\langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle = 0$ *and* $|z_1|^2 + |z_2|^2 \geq |z_3|^2$.
2. *Its singular set is the Möbius strip* $\mathcal{M} := \mathcal{P}_{\mathfrak{R}} \setminus \mathbb{H}^2_{\mathbb{R}}$. *Away from it, is a smooth 3-manifold:* $\Lambda_{Kul} \setminus \mathcal{M}$ *actually is a fibre bundle over* $\partial\mathbb{H}^2_{\mathbb{R}}$.
   (with fibre at each $x \in \partial\mathbb{H}^2_{\mathbb{R}}$ *the corresponding sphere* $\mathcal{L}_x$ *–tangent to* $\partial\mathbb{H}^2_{\mathbb{C}}$ *at* x*– minus the circle* $C_x := \mathcal{L}_x \cap \mathcal{M}$*.)*
3. *Thence* $\Lambda_{Kul} \setminus \mathcal{M}$ *is diffeomorphic to a disjoint union of two solid tori* $\mathbb{S}^1 \times \mathbb{R}^2$.

Thus Λ_{Kul} splits $\mathbb{CP}^2$ into “pieces"that form the Kulkarni region of discontinuity Ω_{Kul}.

Now we want to describe Ω_{Kul}.

Recall one has a partition $\mathbb{C}^{2,1} = V_- \cup V_0 \cup V_+$, where these are the sets of negative, null and positive vectors for the usual quadratic form in $\mathbb{C}^{2,1}$.

Similarly, **consider function $f(\mathbf{z}) = \langle i\mathbf{z} \boxtimes \bar{\mathbf{z}}, i\mathbf{z} \boxtimes \bar{\mathbf{z}} \rangle$, one has a partition of $\mathbb{C}^{2,1}$ into positive, null and negative vectors for f:**

$$
\begin{aligned}
U_+ &= \{\mathbf{z} \in \mathbb{C}^3 \ : \ f(\mathbf{z}) > 0\} = \{\mathbf{z} \in \mathbb{C}^3 \ : i\mathbf{z} \boxtimes \bar{\mathbf{z}} \in V_+\}, \\
U_0 &= \{\mathbf{z} \in \mathbb{C}^3 \setminus \{\mathbf{0}\} \ : \ f(\mathbf{z}) = 0\} \\
&= \{\mathbf{z} \in \mathbb{C}^3 \setminus \{\mathbf{0}\} : i\mathbf{z} \boxtimes \bar{\mathbf{z}} \in V_0 \ \text{ or } \ i\mathbf{z} \boxtimes \bar{\mathbf{z}} = \mathbf{0}\}; \\
U_- &= \{\mathbf{z} \in \mathbb{C}^3 \ : \ f(\mathbf{z}) < 0\} = \{\mathbf{z} \in \mathbb{C}^3 \ : i\mathbf{z} \boxtimes \bar{\mathbf{z}} \in V_-\}.
\end{aligned}
$$

Get induced partition

$$\mathbb{CP}^2 = \mathbb{P}U_+ \cup \mathbb{P}U_0 \cup \mathbb{P}U_-$$

One has:

Theorem

The three sets $\mathbb{P}U_+$, $\mathbb{P}U_0$, $\mathbb{P}U_-$ are $\mathrm{SO}_+(2,1)$-invariant (and so are $\mathbb{H}^2_{\mathbb{C}}$ and the real projective space $\mathcal{P}_{\Re}$) and:

(i) *Limit set is:* $\Lambda_{Kul} = \mathbb{P}U_0 \setminus \mathbb{H}^2_{\mathbb{R}} = \mathbb{P}U_0 \setminus \mathbb{P}V_-$.

(ii) *Kulkarni discontinuity region is:* $\Omega_{Kul} = (\mathbb{P}U_+ \cup \mathbb{H}^2_{\mathbb{R}}) \cup \mathbb{P}U_-$

Now say more about Ω_{Kul}.

Set: $\mathbb{P}U_+ \cup \mathbb{H}^2_{\mathbb{R}} := \Omega_+$ and $\mathbb{P}U_- := \Omega_-$
Thus $\mathbb{H}^2_{\mathbb{C}} \subset \Omega_+$ and $\Omega_{Kul} = \Omega_+ \cup \Omega_-$

Theorem

1. $\exists$ *natural projection map* $\Pi : \Omega_+ \cup \Omega_- \to \mathbb{H}^2_{\mathbb{R}}$, *an* $\mathrm{SO}_+(2,1)$*-equivariant fibre bundle over* $\mathbb{H}^2_{\mathbb{R}}$ *with fibre three disjoint 2-discs (*$\Rightarrow \Omega_{Kul} \cong$ *3-copies of* $\mathbb{B}^4$*)*
2. *Fibre over* $o := [0:0:1]$ *in* Ω_+ *is the Lagrangian 2-plane "orthogonal to* $\mathbb{H}^2_{\mathbb{R}}$ *within* $\mathbb{H}^2_{\mathbb{C}}$*".*
3. *Fibre over* $o := [0:0:1]$ *in* Ω^1_-, Ω^2_- *are the hemispheres* D^1_o *and* D^2_o *determined by equator* $\Im(z_1\bar{z}_2) = 0$ *in line* $S_o := \left\{[z_1 : z_2 : 0] : (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}\right\}$*, which is the projective dual of* $[0:0:1]$*.*
4. *These fibres* L_o*,* D^1_o*,* D^2_o *have common boundary the circle*
$$C_o = \partial L^+_o = \left\{[iy_1 : iy_2 : 0] \;:\; (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}\right\}$$

We get:

Corollary

- *For every lattice $G \subset SO_+(2,1)$ acting on $\mathbb{CP}^2$ via the inclusion $SO_+(2,1) \to SU(2,1)$, the Kulkarni set Ω_{Kul} consists of three open disjoint 4-balls $\mathbb{B}^4$, each of these being $SO_+(2,1)$-invariant.*
- *This is a complete Kobayashi hyperbolic space, which coincides with the equicontinuity region of G and it is the largest set where G-action is properly discontinuous.*

REMARKS

1) We know (Gusevskii-Parker) that there are embeddings of the modular group in $\mathrm{PU}(2,1)$ as a real Fuchsian and as a $\mathbb{C}$-Fuchsian groups that can be connected by a family of quasi-Fuchsian groups. It would be interesting to describe what happens with the limit set and discontinuity region in $\mathbb{CP}^2$ in this deformation.

2) Related to following general problem: If G is cofinite in $\mathrm{PU}(2,1)$, then its action on $\mathbb{CP}^2 \setminus \mathbb{H}^2_{\mathbb{C}}$ is minimal.

Assume otherwise that G (is non-elementary and) **acts on $\partial\mathbb{H}^2_{\mathbb{C}}$ with non-empty region of discontinuity.** Many questions, e.g.:

i) What is the largest set in $\mathbb{CP}^2$ where action is properly discontinuous? How many connected components it has?

ii) If we are given a fundamental domain for its action in $\mathbb{H}^2_{\mathbb{C}}$. Can we construct a fundamental domain for its action in $\mathbb{CP}^2$? Or at least on the connected component that contains the ball $\mathbb{H}^2_{\mathbb{C}}$?

3) Recall (classical) that if $G \subset \mathrm{PSL}(2,\mathbb{C})$ is discrete, then its region of discontinuity in $\mathbb{CP}^1$ can have 1, 2 or ∞-many connected components. What can we say for groups acting on $\mathbb{CP}^2$?

- **The $\mathbb{R}$-Fuchsian groups above are 1st examples we know where Ω_{Kul} has exactly 3-components.**

  Now we know examples where the number of connected components in Ω_{Kul} is $0, 1, 2, 3, 4, \infty$.

- Q: Are these the only possibilities ?
- These are some of the very many questions we cannot yet answer. Vast field of research waiting to be explored

Now we move to groups which may not be complex-hyperbolic.
Want to go deeper into the dynamics of the limit set.
Need first some words about:

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Classification of elements

Just as in dimension 1, one has the following trichotomy (following Navarrete et al): An element of $\mathrm{PSL}(3,\mathbb{C})$ is

- **Elliptic** if it has a lifting to $\mathrm{SL}(3,\mathbb{C})$ which is diagonalizable and all eigenvalues have norm 1.
- **Parabollic** if it has a lifting to $\mathrm{SL}(3,\mathbb{C})$ which is not diagonalizable and all eigenvalues have norm 1.
- **Loxodromic** otherwise.

It can be shown that this classification can be given in terms of the limit set (Cano-Loeza): **Elliptic iff** $\Lambda_{Kul} = \emptyset$ or $\mathbb{CP}^2$; **parabolic iff** Λ_{Kul} is connected; **loxodromic iff** Λ_{Kul} has two connected components (can be two lines or a line and a point).

In particular, **each loxodromic element has at least one attracting or repelling line**.

Theorem (Barrera-Cano-Navarrete-Seade)

Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be such that the number of lines in general position in $\Lambda_{Kul}(G)$ is at least 3. Then :

1. $\Omega_{Kul}(G) := \mathbb{CP}^2 \setminus \Lambda_{Kul}(G)$ *is the largest open set on which G acts properly discontinuously, and it coincides with the equicontinuity region.*
2. *The limit set $\Lambda_{Kul}(G)$ is the closure of the set of repelling-attracting lines of loxodromic elements.*

We can also say what happens if the number of lines lying in general position in $\Lambda_{Kul}(G)$ is ≤ 2.

The essentially new part in this theorem is the second statement. The prove is not at all easy.

Key point is proving that if the limit set has at least three lines in general position, then the group must contain loxodromic elements.

Since elliptic elements are all of finite order and therefore they have no effect on the limit set, key-step is studying purely parabolic groups:

Theorem

If G is a purely parabolic group, then its Kulkarni limit set is either a line or a cone of lines with a common vertex p and base a circle $\mathbb{S}^1$ contained in a projective line. So it has at most two lines in general position.

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Sullivan Dictionary dimension 2

Rational function acting on $\mathbb{P}^2_{\mathbb{C}}$ with degree at least 2	Finitely generated non elementary projective group
Julia set	Limit set
Fatou set	Ordinary set
Julia set is closed and invariant	Limit set is closed and invariant
Julia set is connected	Limit set is connected,
Julia set is non empty	Limit set is non-empty
?	Limit set is the closure of invariant attractive lines of loxodromic elements
Fatou set is a complete Kobayashi hyperbolic space	Ordinary set is a complete Kobayashi hyperbolic space
Fatou set is a Stein manifold	Ordinary set is a Stein manifold
Fatou set is pseudoconvex	Ordinary set is pseudoconvex

For more on the foundations of the subject see our monograph (Progress in Maths. vol. 303, Birkhauser, 2012)

Thank you very much for your attention

ありがとうございます

and .....

Congratulations Professor KAMIYA

and

my very best wishes for this new beginning!!!