Knots and 4-dimensional topological surgery

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$M^n$ : an $n$-dim TOP manifold, conn. ori. closed

$\pi \equiv \pi_1(M)$
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$$\pi = \pi_1(M)$$

\{surgery problems\} $\longrightarrow L_n(\pi)$

$$(f : N^n \rightarrow M^n, b) \longmapsto \theta(f, b)$$
\( \theta(f, b) : \) surgery obstruction
\[ \theta(f, b) : \text{surgery obstruction} \]

\[ \theta(f, b) = 0 \]

if can do surgery to get a htpy eq.
The converse is true . . .
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• if $n \geq 5$
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- if $n = 4$ and $\pi = 1$
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- if $n = 4$ and $\pi = 1$
- if $n = 4$ and $\pi$ is good
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- if $n \geq 5$
- if $n = 4$ and $\pi = 1$
- if $n = 4$ and $\pi$ is good

  e.g. $1, \mathbb{Z}^n$, subexponential groups

  [Freedman-Quinn, Krushkal-Q, . . .]
There are other results that depend on topology of $M$.

- Krushkal-Lee (2002),
  $\pi$: free so probably not good
an example due to H-R

\[ K \subset S^3 : \text{a knot} \]

\[ E(K) = S^3 - \dot{N}(K) \]

\[ M(K) = \partial(E(K) \times D^2) \]
an example due to H-R

\[ K \subset S^3 : \text{a knot} \]

\[ E(K) = S^3 - \hat{N}(K) \]

\[ M(K) = \partial(E(K) \times D^2) \]

OK for \( M(K) \), when \( K \) is a torus knot.
Theorem

TOP surgery obstruction theory works for $M(K)$ for any knot $K$. 
properties of $E(K)$ and $S^3 - K$

- homology $S^1$'s
- aspherical
- $S^3 - K$ has a complete non-positively curved metric.

[Leeb 1995]
properties of $M(K)$

- $\pi_1(M(K)) = \pi_1(E(K))$
- not aspherical
the idea of H-R

Construct a 2-dim spine $B$ of $E(K)$ and a projection $q : E(K) \rightarrow B$, so that each $q^{-1}(x)$ is a wedge of intervals along one end.
Restrict the map

\[ E(K) \times D^2 \xrightarrow{\text{proj.}} E(K) \xrightarrow{q} B \]

to \( \partial \) and get the control map

\[ p : M(K) \to B. \]
The point inverses of the control map $p : M(K) \to B$ are all simply-connected.
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\[ \implies \text{a controlled surgery exact sequence for } p \]

[Pedersen-Quinn-Ranicki (2003)]
\( \varepsilon > \delta > 0: \) sufficiently small

\[ \mathcal{N} = \{ \text{surgery problems to } M(K) \} / \sim \]

\[ \mathcal{S}(M(K)) = \{ \text{htpy eq.'s to } M(K) \} / \sim \]

\[
\begin{array}{ccc}
S_{\varepsilon,\delta}(M(K)) & \longrightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
S(M(K)) & \longrightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
\mathcal{S}(M(K)) & \longrightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
H_4(B; \mathbb{L}) & \longrightarrow & L_4(\pi)
\end{array}
\]
The first row is exact \([P-Q-R]\).

Want to show the second row is also exact.

\[
\begin{array}{ccc}
\mathcal{S}_{\epsilon,\delta}(M(K)) & \longrightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
\mathcal{S}(M(K)) & \longrightarrow & \mathcal{N} \\
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccc}
\mathcal{N} & \longrightarrow & H_4(B; \mathbb{L}) \\
\downarrow & & \downarrow A \\
\mathcal{N} & \longrightarrow & L_4(\pi)
\end{array}
\]
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\begin{array}{ccc}
S_{\epsilon, \delta}(M(K)) & \longrightarrow & \mathcal{N} & \longrightarrow & H_4(B; \mathbb{L}) \\
& \downarrow & & \downarrow & \\
S(M(K)) & \longrightarrow & \mathcal{N} & \longrightarrow & L_4(\pi)
\end{array}
\]

Claim: \(A\) is injective.
\[
\begin{align*}
H_4(B; \mathbb{L}) &\xrightarrow{\phi_*} H_4(S^1; \mathbb{L}) \\
L_4(\pi_1(B)) &\xrightarrow{\phi_*} L_4(\pi_1(S^1))
\end{align*}
\]

\[\phi : B \rightarrow S^1: \text{ a homology equivalence} \]
\[
\begin{align*}
H_4(B; \mathbb{L}) & \xrightarrow{\phi_* \cong} H_4(S^1; \mathbb{L}) \\
\downarrow A & & \downarrow A \\
L_4(\pi_1(B)) & \xrightarrow{\phi_*} L_4(\pi_1(S^1))
\end{align*}
\]

\(\phi : B \to S^1\): a homology equivalence

\(\Rightarrow\) top row is an isomorphism
Bottom row is an isomorphism. [Arvinda-Farrell-Roushon, 1997]

This uses the metric on $S^3 - K \simeq B$. 
The assembly map $A$ for $S^1$ is an isomorphism. [Browder, 1966]
The assembly map \( A \) for \( B \) is also an isomorphism. \( \Rightarrow \) exactness follows
Construction of the Spine $B$:

Figure Eight Knot Case
the ideal triangulation of the complement:
dual spine of an ideal 1-simplex
dual spine of an ideal 1-simplex
dual spine of an ideal 2-simplex

\[ Q \]
dual spine of an ideal 2-simplex
dual spine of an ideal 3-simplex
Construction of the Spine \( B \):

Trefoil Knot Case
have a decomposition into ideal cells.

can similarly consider the dual spine.
Construction of the Spine $B$: General Case

We use a simplified but weaker method of D. Thurston to construct a decomposition of the knot complement, and use its dual spine as $B$. 
Identify $S^3$ with $S^2 \times (-\infty, \infty) \cup \{\pm \infty\}$, and consider a knot projection to $S^2 \times 0$, with $n$ crossings.
This divides $S^2 \times 0$ into several regions.
$S^2 \times (-\infty, \infty)$

$K$

$S^2 \times 0$

Pick a point from each region.
Connect the points as indicated above.
$S^2 \times (\infty, \infty)$

$S^2 \times 0$ decomposes into $4n$-many quadrangles $R_i$. 
Roughly speaking $R_i \times (-\infty, \infty) - K$ are the desired cells.
Unfortunately their union is not $S^3 - K$, but

$$S^3 - \{ \pm \infty \} - K.$$
So pick a point on $K$ and dig tunnels to $\pm \infty$. This affects four cells.
This gives a decomposition into ideal cells.

Now use the dual spine.

[Browder] Manifolds with $\pi_1 = \mathbb{Z}$, Bull. Amer. Math. Soc. 72 (1966) 238 – 244


