A USER'S GUIDE TO THE ALGEBRAIC THEORY OF SURGERY

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A. A. Ranicki, *Algebraic L-theory and topological manifolds*, *Tracts in Math.* **102** (Cambridge Univ. Press, Cambridge, 1992).

Warnings.
1. Wall's L-group $L_n(G)$ for a group G is denoted $L_n(\mathbb{Z}[G])$ in Ranicki's works. For example, $L_n(\mathbb{Z})$ is the n-th L-group of the group ring $\mathbb{Z}[\{e\}]$ of the trivial group; this has been usually denoted $L_n(\{e\})$ or simply $L_n(e)$ in the classical notation.
2. Traditionally, the indexing of an L-theory spectrum is the negative of the usual one. So an Ω-spectrum $\mathbb{L}$ is a sequence of pointed spaces (or Δ-sets) $\{\mathbb{L}_n | n \in \mathbb{Z}\}$ together with homotopy equivalences $\mathbb{L}_{n+1} \to \Omega\mathbb{L}_n$.

1. 4-Periodic Theory

$\mathbb{L}_\bullet(A) = \{\mathbb{L}_n(A) | n \in \mathbb{Z}\}$ is the quadratic $\mathbb{L}$-spectrum of a ring with involution A. The homotopy groups are the quadratic L-groups of A

$$\pi_i(\mathbb{L}_\bullet(A)) = L_i(A) \quad (i \in \mathbb{Z}) \ .$$

In particular, $\mathbb{L}_\bullet(\mathbb{Z})$ denotes the simply-connected surgery spectrum.

If R is a commutative ring and K is a simplicial complex, then there is defined an **assembly map**

$$A : \mathbb{H}_\bullet(K; \mathbb{L}_\bullet(R)) \to \mathbb{L}_\bullet(R[\pi_1(K)]) \ ,$$

which induces the **universal assembly maps** (p.101)

$$A : H_i(K; \mathbb{L}_\bullet(R)) \to L_i(R[\pi_1(K)]) \qquad (i \in \mathbb{Z}).$$

The **quadratic structure spectrum** $\mathbb{S}_\bullet(R, K)$ of (R, K) is defined so that there is a fibration sequence of spectra:

$$\mathbb{H}_\bullet(K; \mathbb{L}_\bullet(R)) \to \mathbb{L}_\bullet(R[\pi_1(K)]) \to \mathbb{S}_\bullet(R, K) \ ,$$

which induces the **algebraic surgery exact sequence**

$$\begin{aligned} \cdots \to H_n(K; \mathbb{L}_\bullet(R)) &\xrightarrow{A} L_n(R[\pi_1(K)]) \\ &\xrightarrow{\partial} \mathcal{S}_n(R, K) \to H_{n-1}(K; \mathbb{L}_\bullet(R)) \to \dots \end{aligned}$$

where $\mathcal{S}_n(R, K)$ denotes the quadratic structre group $\pi_n(\mathbb{S}_\bullet(R, K))$. The groups $H_n(K; \mathbb{L}_\bullet(R))$, $L_n(R[\pi_1(K)])$, $\mathcal{S}_n(R, K)$ are all 4-periodic.

2. Connective L-theory

In general, a spectrum $\mathbb{F}$ is **q-connective** if $\pi_n(\mathbb{F}) = 0$ for $n < q$.

For any ring with involution A, there are **q-connective $\mathbb{L}$-spectrum** $\mathbb{L}_\bullet\langle q\rangle(A)$ (p.157). The homotopy groups are the "q-connective" L-groups:

$$\pi_i(\mathbb{L}_\bullet\langle q\rangle(A)) = \begin{cases} L_i(A) & (\text{if } i \geq q) \\ 0 & (\text{if } i < q) \ . \end{cases}$$

If R is a commutative ring and K is a simplicial complex, then the **q-connective quadratic structure groups** of (R, K) are also defined and denoted $\mathcal{S}_n\langle q\rangle(R, K)$ (p.158). These are the homotopy groups of the **q-connective quadratic structure spectrum** $\mathbb{S}_\bullet\langle q\rangle(R, K)$:

$$\pi_n(\mathbb{S}_\bullet\langle q\rangle(R, K)) = \mathcal{S}_n\langle q\rangle(R, K) \ .$$

There is a **q-connective algebraic surgery exact sequence**

$$\begin{aligned} \ldots \xrightarrow{\partial} \mathcal{S}_{n+1}\langle q\rangle(R, K) \to & H_n(K; \mathbb{L}_\bullet\langle q\rangle(R)) \xrightarrow{A} L_n(R[\pi_1(K)]) \\ & \xrightarrow{\partial} \mathcal{S}_n\langle q\rangle(R, K) \to H_{n-1}(K; \mathbb{L}_\bullet\langle q\rangle(R)) \xrightarrow{A} L_{n-1}(R[\pi_1(K)]) \to \ldots \\ \ldots \xrightarrow{\partial} \mathcal{S}_{2q+1}\langle q\rangle(R, K) \to & H_{2q}(K; \mathbb{L}_\bullet\langle q\rangle(R)) \xrightarrow{A} L_{2q}(R[\pi_1(K)]) \\ & \xrightarrow{\partial} \mathcal{S}_{2q}\langle q\rangle(R, K) \to H_{2q-1}(K; \mathbb{L}_\bullet\langle q\rangle(R)) \ , \end{aligned}$$

which is induced by a certain fibration sequence of spectra (p.159). This sequence actually continues to the right, but the next term may not be $L_{2q-1}(R[\pi_1(K)])$.

If $n \geq 2q + 4$, the groups $\mathcal{S}_n\langle q\rangle(R, K)$ and $\mathcal{S}_n\langle q + 1\rangle(R, K)$ are related by exact sequences (p.159) :

$$H_{n-q}(K; L_q(R)) \to \mathcal{S}_n\langle q + 1\rangle(R, K) \to \mathcal{S}_n\langle q\rangle(R, K) \to H_{n-q-1}(K; L_q(R)) \ .$$

If $n \geq \max\{q + \dim K + 1, 2q + 4\}$, then

$$\mathcal{S}_n\langle q\rangle(R, K) = \mathcal{S}_n(R, K) \ .$$

The homology groups $H_*(K; \mathbb{L}_\bullet\langle q + 1\rangle(R))$ and $H_*(K; \mathbb{L}_\bullet\langle q\rangle(R))$ are related by an exact sequence

$$\begin{aligned} & \cdots \to H_{n+1-q}(K; L_q(R)) \to \\ H_n(K; \mathbb{L}_\bullet\langle q + 1\rangle(R)) \to H_n(K; \mathbb{L}_\bullet\langle q\rangle(R)) & \to H_{n-q}(K; L_q(R)) \to \ldots \ , \end{aligned}$$

(pp. 152–153) and there are isomorphisms

$$H_n(K; \mathbb{L}_\bullet\langle q\rangle(R)) \cong H_n(K; \mathbb{L}_\bullet(R)) \qquad (n \geq \dim K + q) \ .$$

3. Important special cases

Ranicki uses the notation

$$\mathbb{L}_\bullet\langle q\rangle = \mathbb{L}_\bullet\langle q\rangle(\mathbb{Z})$$

for the q-connective $\mathbb{L}$-spectrum of $\mathbb{Z}$ ($q \in \mathbb{Z}$). The 0-connective and the 1-connective ones are especially important; the following notation for these is used:

$$\overline{\mathbb{L}}_\bullet = \mathbb{L}_\bullet\langle 0\rangle(\mathbb{Z}) \ , \qquad \mathbb{L}_\bullet = \mathbb{L}_\bullet\langle 1\rangle(\mathbb{Z}) \ ,$$

These are the 0-connective and the 1-connective simply-connected surgery spectra which appear in various surgery exact sequences. For a simplicial complex K, we have

$$\begin{aligned} H_i(K; \overline{\mathbb{L}}_\bullet) &= H_i(K; \mathbb{L}_\bullet(\mathbb{Z})) \qquad (i \geq \dim K) \ , \\ H_i(K; \mathbb{L}_\bullet) &= H_i(K; \mathbb{L}_\bullet(\mathbb{Z})) \qquad (i \geq \dim K + 1) \ . \end{aligned}$$

So the 0-connective and the 1-connective spectra define the same homology group $H_i(K;\overline{\mathbb{L}}_\bullet) = H_i(K;\mathbb{L}_\bullet)$ for $i > \dim K$. Their difference can be studied using the exact sequence

$$H_{i+1}(K) \to H_i(K;\mathbb{L}_\bullet) \to H_i(K;\overline{\mathbb{L}}_\bullet) \to H_i(K) ,$$

where $H_*(K)$ are the ordinary homology groups with coefficients in $\mathbb{Z} = L_0(\mathbb{Z}[\{1\}])$.

When M is a compact n-dimensional topological manifold with $n \geq 5$, the topological manifold surgery exact sequence for M

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathcal{S}(M \ rel\ \partial) \to [M, \partial M; G/TOP, *] \to L_n(\mathbb{Z}[\pi_1(M)])$$

can be identified with the appropriate part of the 1-connective algebraic surgery exact sequence

$$\begin{aligned} \ldots \xrightarrow{\partial} \mathcal{S}_{i+1}(M) \to H_i(M;\mathbb{L}_\bullet) \xrightarrow{A} L_i(\mathbb{Z}[\pi_1(M)]) \xrightarrow{\partial} \mathcal{S}_i(M) \to \ldots \\ \ldots \xrightarrow{\partial} \mathcal{S}_2(M) \to H_1(M;\mathbb{L}_\bullet) \end{aligned}$$

under the identifications

$$\begin{aligned} &\mathcal{S}(M \times D^j \ rel\ \partial) = \mathcal{S}_{n+j+1}(M) , \\ &[M \times D^j, \partial(M \times D^j); G/TOP, *] = H_{n+j}(M;\mathbb{L}_\bullet) . \end{aligned}$$

Suppose B is a finite simplicial complex and a dimension $n \geq 4$ is given. Then, for sufficiently small $\epsilon \gg \delta > 0$ and for any UV^1-map $p : M \to B$ from an n-dimensional compact topological manifold M, there is a controlled surgery exact sequence :

$$\cdots \to H_{n+1}(B, \overline{\mathbb{L}}_\bullet) \to \mathcal{S}_{\epsilon,\delta}(M \ rel\ \partial; p) \to [M, \partial M; G/TOP, *] \to H_n(B;\overline{\mathbb{L}}_\bullet) .$$

Note that there is an identification

$$[M, \partial M; G/TOP, *] = H_n(M;\mathbb{L}_\bullet) ,$$

and also that we can replace the $\overline{\mathbb{L}}_\bullet$-homology groups $H_n(B;\overline{\mathbb{L}}_\bullet)$ by the $\mathbb{L}$-homology groups $H_i(B;\mathbb{L}_\bullet(\mathbb{Z}))$, since they are the same for $i \geq n \geq \dim B$.

When M is a compact n-dimensional ANR homology manifold with $n \geq 5$, the homology manifold surgery exact sequence for M

$$\begin{aligned} \cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathcal{S}^H(M) \\ \to [M, L_0(\mathbb{Z}) \times G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)]) \end{aligned}$$

can be identified with the appropriate part of the 0-connective algebraic surgery exact sequence

$$\begin{aligned} \ldots \xrightarrow{\partial} \overline{\mathcal{S}}_{i+1}(M) \to H_i(M;\overline{\mathbb{L}}_\bullet) \xrightarrow{A} L_i(\mathbb{Z}[\pi_1(M)]) \xrightarrow{\partial} \overline{\mathcal{S}}_i(M) \to \ldots \\ \ldots \xrightarrow{\partial} \overline{\mathcal{S}}_0(M) \to H_{-1}(M;\overline{\mathbb{L}}_\bullet) \end{aligned}$$

under the identifications

$$\begin{aligned} &\mathcal{S}^H(M \times D^j \ rel\ \partial) = \overline{\mathcal{S}}_{n+j+1}(M) , \\ &[M \times D^j, M \times S^{j-1}; L_0(\mathbb{Z}) \times G/TOP, *] = H_{n+j}(M;\overline{\mathbb{L}}_\bullet) . \end{aligned}$$

Now we discuss periodicity (pp. 289–290). Using the "double skew-suspension" isomorphism

$$H_i(K;\overline{\mathbb{L}}_\bullet) \xrightarrow{\cong} H_{i+4}(K;\mathbb{L}_\bullet\langle 4\rangle)$$

and the isomorphisms in the sequences

$$0 \to H_{i+4}(K;\mathbb{L}_\bullet\langle k+1\rangle) \xrightarrow{\cong} H_{i+4}(K;\mathbb{L}_\bullet\langle k\rangle) \to 0\ ,$$

where $i \geq \dim K - 1$ and $k = 0, 1, 2, 3$, we have periodicity for the $\overline{\mathbb{L}}_\bullet$-homology:

$$H_i(K;\overline{\mathbb{L}}_\bullet) \xrightarrow{\cong} H_{i+4}(K;\overline{\mathbb{L}}_\bullet) \qquad (i \geq \dim K - 1)\ .$$

For the 0-connective and the 1-connective structure groups of $(\mathbb{Z}, K)$, we use the following notation

$$\overline{\mathcal{S}}_i(K) = \mathcal{S}_i\langle 0\rangle(\mathbb{Z}, K)\ , \qquad \mathcal{S}_i(K) = \mathcal{S}_i\langle 1\rangle(\mathbb{Z}, K)\ .$$

Recall that, if $i \geq 4$, these are related by an exact sequence

$$H_i(K; L_0(\mathbb{Z})) \to \mathcal{S}_i(K) \to \overline{\mathcal{S}}_i(K) \to H_{i-1}(K; L_0(\mathbb{Z}))\ .$$

Therefore, if $i \geq \dim K + 2$, then

$$\mathcal{S}_i(K) = \overline{\mathcal{S}}_i(K)\ .$$

Also note that, for $i \geq 2$, there is an exact sequence

$$0 = H_i(K; L_{-1}(\mathbb{Z})) \to \overline{\mathcal{S}}_i(K) \xrightarrow{\cong} \mathcal{S}_i\langle -1\rangle(\mathbb{Z}, K) \to H_{i-1}(K; L_{-1}(\mathbb{Z})) = 0\ .$$

Therefore, for $i \geq \max\{\dim K, 2\}$, we have

$$\overline{\mathcal{S}}_i(K) = \mathcal{S}_i\langle -1\rangle(\mathbb{Z}, K) = \mathcal{S}_i(\mathbb{Z}, K) = \mathcal{S}_{i+4}(\mathbb{Z}, K) = \overline{\mathcal{S}}_{i+4}(K)\ .$$

This can be also observed by using the algebraic surgery exact sequences and the 4-periodicities of L-groups and $\overline{\mathbb{L}}_\bullet$-homologies.

On the other hand, for $i \geq \dim K + 2$, we have

$$\mathcal{S}_i(K) = \mathcal{S}_i(\mathbb{Z}, K) = \mathcal{S}_{i+4}(K)\ .$$

4. A sample usage

In [*Surgery groups of knot and link complements*, Bull. London Math. Soc. **29** (1997) 400 – 406], Arvinda, Farrell and Roushon calculated the L-groups of knot and link complements. In the process, they proved the following:

Theorem. *Let K be a knot or a non-split link, and $E(K)$ denote its exterior; then the assembly map $A : H_i(E(K); \mathbb{L}_\bullet(\mathbb{Z})) \to L_i(\pi_1(E(K)))$ is an isomorphism for every $i \in \mathbb{Z}$.*

By a work of Leeb, $S^3 - K$ has a complete Riemannian metric of nonpositive curvature when K is a knot or a non-split link. The double $D(K)$ of $E(K)$ inherits a metric of non-positive curvature. Then the topological rigidity result of Farrell and Jones can be applied to $D(K)$, and we obtain

$$\mathcal{S}(D(K) \times D^n \text{ rel } \partial) = \{*\} \quad (n \geq 2).$$

This implies the vanishing of the algebric structure sets:

$$\mathcal{S}_{n+4}(D(K)) \cong \mathcal{S}(D(K) \times D^n \text{ rel } \partial)$$

for $n \geq 2$. Since $E(K)$ is a retract of $D(K)$, the algebraic structure sets $\mathcal{S}_i(E(K))$ are all trivial for $i \geq 6$. In these dimensions, these are the same as $\mathcal{S}_i(\mathbb{Z}, E(K))$. By the 4-periodicity, $\mathcal{S}_i(\mathbb{Z}, E(K)) = 0$ for every $i \in \mathbb{Z}$. Now the result follows from the 4-periodic algebraic surgery exact sequence.