A USER’S GUIDE TO THE ALGEBRAIC THEORY OF SURGERY

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Warnings.
1. Wall’s \( L_n(G) \) for a group \( G \) is denoted \( L_n(\mathbb{Z}[G]) \) in Ranicki’s works. For example, \( L_n(\mathbb{Z}) \) is the \( n \)-th \( L \)-group of the group ring \( \mathbb{Z}[[e]] \) of the trivial group; this has been usually denoted \( L_n(\{e\}) \) or simply \( L_n(\epsilon) \) in the classical notation.
2. Traditionally, the indexing of an \( L \)-theory spectrum is the negative of the usual one. So an \( \Omega \)-spectrum \( L \) is a sequence of pointed spaces (or \( \Delta \)-sets) \( \{L_n|n \in \mathbb{Z}\} \) together with homotopy equivalences \( L_{n+1} \rightarrow \Omega L_n \).

1. 4-Periodic Theory

\[ \mathbb{L}_n(A) = \{L_n(A)|n \in \mathbb{Z}\} \]

is the quadratic \( L \)-spectrum of a ring with involution \( A \). The homotopy groups are the quadratic \( L \)-groups of \( A \)

\[ \pi_i(L_\bullet(A)) = L_i(A) \quad (i \in \mathbb{Z}) \]

In particular, \( \mathbb{L}_\bullet(\mathbb{Z}) \) denotes the simply-connected surgery spectrum.

If \( R \) is a commutative ring and \( K \) is a simplicial complex, then there is defined an assembly map

\[ A : H_\bullet(K; \mathbb{L}_\bullet(R)) \rightarrow L_\bullet(R[\pi_1(K)]) , \]

which induces the universal assembly maps (p.101)

\[ A : H_i(K; \mathbb{L}_\bullet(R)) \rightarrow L_i(R[\pi_1(K)]) \quad (i \in \mathbb{Z}). \]

The quadratic structure spectrum \( S_\bullet(R, K) \) of \( (R, K) \) is defined so that there is a fibration sequence of spectra:

\[ H_\bullet(K; \mathbb{L}_\bullet(R)) \rightarrow \mathbb{L}_\bullet(R[\pi_1(K)]) \rightarrow S_\bullet(R, K) , \]

which induces the algebraic surgery exact sequence

\[ \cdots \rightarrow H_n(K; \mathbb{L}_\bullet(R)) \xrightarrow{A} L_n(R[\pi_1(K)]) \]

\[ \xrightarrow{\partial} S_n(R, K) \rightarrow H_{n-1}(K; \mathbb{L}_\bullet(R)) \rightarrow \cdots \]

where \( S_n(R, K) \) denotes the quadratic structre group \( \pi_n(S_\bullet(R, K)) \). The groups \( H_n(K; \mathbb{L}_\bullet(R)), L_n(R[\pi_1(K)]), S_n(R, K) \) are all 4-periodic.
2. Connective $L$-theory

In general, a spectrum $F$ is $q$-connective if $\pi_n(F) = 0$ for $n < q$.

For any ring with involution $A$, there are $q$-connective $L$-spectrum $L_\bullet(q)(A)$ (p.157). The homotopy groups are the “$q$-connective” $L$-groups:

$$\pi_i(L_\bullet(q)(A)) = \begin{cases} L_i(A) & \text{(if } i \geq q) \\ 0 & \text{(if } i < q) \end{cases}.$$

If $R$ is a commutative ring and $K$ is a simplicial complex, then the $q$-connective quadratic structure groups of $(R, K)$ are also defined and denoted $S_n(q)(R, K)$ (p.158). These are the homotopy groups of the $q$-connective quadratic structure spectrum $S_\bullet(q)(R, K)$:

$$\pi_n(S_\bullet(q)(R, K)) = S_n(q)(R, K).$$

There is a $q$-connective algebraic surgery exact sequence

$$\ldots \to S_{n+1}(q)(R, K) \to H_n(K; L_\bullet(q)(R)) \xrightarrow{\Delta} L_n(R[\pi_1(K)])$$

$$\to S_n(q)(R, K) \to H_{n-1}(K; L_\bullet(q)(R)) \xrightarrow{\Delta} L_{n-1}(R[\pi_1(K)]) \to \ldots$$

$$\ldots \to S_{2q+1}(q)(R, K) \to H_{2q}(K; L_\bullet(q)(R)) \xrightarrow{\Delta} L_{2q}(R[\pi_1(K)])$$

$$\to S_{2q}(q)(R, K) \to H_{2q-1}(K; L_\bullet(q)(R)),$$

which is induced by a certain fibration sequence of spectra (p.159). This sequence actually continues to the right, but the next term may not be $L_{2q-1}(R[\pi_1(K)])$.

If $n \geq 2q + 4$, the groups $S_n(q)(R, K)$ and $S_n(q + 1)(R, K)$ are related by exact sequences (p.159):

$$H_{n-q}(K; L_q(R)) \to S_n(q + 1)(R, K) \to S_n(q)(R, K) \to H_{n-q-1}(K; L_q(R)).$$

If $n \geq \max\{q + \dim K + 1, 2q + 4\}$, then

$$S_n(q)(R, K) = S_n(R, K).$$

The homology groups $H_\bullet(K; L_\bullet(q + 1)(R))$ and $H_\bullet(K; L_\bullet(q)(R))$ are related by an exact sequence

$$\ldots \to H_{n+1-q}(K; L_q(R)) \to H_n(K; L_\bullet(q + 1)(R)) \to H_n(K; L_\bullet(q)(R)) \to H_{n-q}(K; L_q(R)) \to \ldots,$$

(pp. 152–153) and there are isomorphisms

$$H_n(K; L_\bullet(q)(R)) \cong H_n(K; L_\bullet(R)) \quad (n \geq \dim K + q).$$

3. Important special cases

Ranicki uses the notation

$$L_\bullet(q) = L_\bullet(q)(\mathbb{Z})$$

for the $q$-connective $L$-spectrum of $\mathbb{Z}$ ($q \in \mathbb{Z}$). The 0-connective and the 1-connective ones are especially important; the following notation for these is used:

$$\overline{L}_\bullet = L_\bullet(0)(\mathbb{Z}), \quad L_\bullet = L_\bullet(1)(\mathbb{Z}).$$

These are the 0-connective and the 1-connective simply-connected surgery spectra which appear in various surgery exact sequences. For a simplicial complex $K$, we have

$$H_i(K; \overline{L}_\bullet) = H_i(K; L_\bullet(\mathbb{Z})) \quad (i \geq \dim K),$$

$$H_i(K; L_\bullet) = H_i(K; L_\bullet(\mathbb{Z})) \quad (i \geq \dim K + 1).$$
So the 0-connective and the 1-connective spectra define the same homology group
\(H_i(K; \mathbb{L}_*) = H_i(K; \mathbb{L}_*)\) for \(i > \text{dim} K\). Their difference can be studied using the exact sequence

\[ H_{i+1}(K) \to H_i(K; \mathbb{L}_*) \to H_i(K; \mathbb{L}_*) \to H_i(K), \]

where \(H_*(K)\) are the ordinary homology groups with coefficients in \(\mathbb{Z} = L_0(\mathbb{Z}[[1]])\).

When \(M\) is a compact \(n\)-dimensional topological manifold with \(n \geq 5\), the topological manifold surgery exact sequence for \(M\)

\[ \cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to S(M \times D^3, \partial) \to [M, \partial M; G/TOP, *] \to L_n(\mathbb{Z}[\pi_1(M)]) \]

can be identified with the appropriate part of the 1-connective algebraic surgery exact sequence

\[ \cdots \xrightarrow{\partial} S_{i+1}(M) \to H_i(M; \mathbb{L}_*) \xrightarrow{\Delta} L_i(\mathbb{Z}[\pi_1(M)]) \xrightarrow{\partial} S_i(M) \to \cdots \]

under the identifications

\[ S(M \times D^j, \partial) = S_{n+j+1}(M), \]
\[ [M \times D^j, \partial(M \times D^j); G/TOP, *] = H_{n+j}(M; \mathbb{L}_*). \]

Suppose \(B\) is a finite simplicial complex and a dimension \(n \geq 4\) is given. Then, for sufficiently small \(\epsilon \gg \delta > 0\) and for any \(UV^1\)-map \(p : M \to B\) from an \(n\)-dimensional compact topological manifold \(M\), there is a controlled surgery exact sequence:

\[ \cdots \to H_{n+1}(B; \mathbb{L}_*) \to S_{i, \epsilon}(M \times D^j; \partial; p) \to [M, \partial M; G/TOP, *] \to H_{n}(B; \mathbb{L}_*). \]

Note that there is an identification

\[ [M, \partial M; G/TOP, *] = H_n(M; \mathbb{L}_*), \]

and also that we can replace the \(\mathbb{L}_*\)-homology groups \(H_n(B; \mathbb{L}_*)\) by the \(\mathbb{L}\)-homology groups \(H_*(B; \mathbb{L}_*(\mathbb{Z}))\), since they are the same for \(i \geq n \geq \text{dim} B\).

When \(M\) is a compact \(n\)-dimensional ANR homology manifold with \(n \geq 5\), the homology manifold surgery exact sequence for \(M\)

\[ \cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to S^H(M) \to [M, L_0(\mathbb{Z}) \times G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)]) \]

can be identified with the appropriate part of the 0-connective algebraic surgery exact sequence

\[ \cdots \xrightarrow{\partial} S_{i+1}(M) \to H_i(M; \mathbb{L}_*) \xrightarrow{\Delta} L_i(\mathbb{Z}[\pi_1(M)]) \xrightarrow{\partial} S_i(M) \to \cdots \]

under the identifications

\[ S^H(M \times D^j, \partial) = S_{n+j+1}(M), \]
\[ [M \times D^j, M \times S^j; L_0(\mathbb{Z}) \times G/TOP, *] = H_{n+j}(M; \mathbb{L}_*). \]
Now we discuss periodicity (pp. 289–290). Using the “double skew-suspension” isomorphism

\[ H_i(K; \mathbb{L}_*) \cong H_{i+4}(K; \mathbb{L}_*\langle 4 \rangle) \]

and the isomorphisms in the sequences

\[ 0 \rightarrow H_{i+4}(K; \mathbb{L}_*\langle k + 1 \rangle) \cong H_{i+4}(K; \mathbb{L}_*\langle k \rangle) \rightarrow 0, \]

where \( i \geq \text{dim } K - 1 \) and \( k = 0, 1, 2, 3 \), we have periodicity for the \( \mathbb{L}_*\)-homology:

\[ H_i(K; \mathbb{L}_*) \cong H_{i+4}(K; \mathbb{L}_*) \quad (i \geq \text{dim } K - 1). \]

For the 0-connective and the 1-connective structure groups of \((\mathbb{Z}, K)\), we use the following notation

\[ \mathcal{S}_i(K) = S_i(0)(\mathbb{Z}, K), \quad S_i(K) = S_i(1)(\mathbb{Z}, K). \]

Recall that, if \( i \geq 4 \), these are related by an exact sequence

\[ H_i(K; L_0(\mathbb{Z})) \rightarrow S_i(K) \rightarrow \mathcal{S}_i(K) \rightarrow H_{i-1}(K; L_0(\mathbb{Z})). \]

Therefore, if \( i \geq \text{dim } K + 2 \), then

\[ S_i(K) = \mathcal{S}_i(K). \]

Also note that, for \( i \geq 2 \), there is an exact sequence

\[ 0 = H_i(K; L_{-1}(\mathbb{Z})) \rightarrow \mathcal{S}_i(K) \cong S_i(-1)(\mathbb{Z}, K) \rightarrow H_{i-1}(K; L_{-1}(\mathbb{Z})) = 0. \]

Therefore, for \( i \geq \max\{\text{dim } K, 2\} \), we have

\[ \mathcal{S}_i(K) = S_i(-1)(\mathbb{Z}, K) = S_i(\mathbb{Z}, K) = S_{i+4}(\mathbb{Z}, K) = \mathcal{S}_{i+4}(K). \]

This can be also observed by using the algebraic surgery exact sequences and the 4-periodicities of \( L \)-groups and \( \mathbb{L}_*\)-homologies.

On the other hand, for \( i \geq \text{dim } K + 2 \), we have

\[ S_i(K) = S_i(\mathbb{Z}, K) = S_{i+4}(K). \]

4. A SAMPLE USAGE

In [Surgery groups of knot and link complements, Bull. London Math. Soc. 29 (1997) 400 – 406], Arvinda, Farrell and Roushon calculated the \( L \)-groups of knot and link complements. In the process, they proved the following:

**Theorem.** Let \( K \) be a knot or a non-split link, and \( E(K) \) denote its exterior; then the assembly map \( A: H_i(E(K); \mathbb{L}_*\langle \mathbb{Z} \rangle) \rightarrow L_i(\pi_1(E(K))) \) is an isomorphism for every \( i \in \mathbb{Z} \).

By a work of Leeb, \( S^3 - K \) has a complete Riemannian metric of nonpositive curvature when \( K \) is a knot or a non-split link. The double \( D(K) \) of \( E(K) \) inherits a metric of non-positive curvature. Then the topological rigidity result of Farrell and Jones can be applied to \( D(K) \), and we obtain

\[ S(D(K) \times D^n \text{ rel } \partial) = \{ \ast \} \quad (n \geq 2). \]

This implies the vanishing of the algebraic structure sets:

\[ S_{n+4}(D(K)) \cong S(D(K) \times D^n \text{ rel } \partial) \]

for \( n \geq 2 \). Since \( E(K) \) is a retract of \( D(K) \), the algebraic structure sets \( S_i(E(K)) \) are all trivial for \( i \geq 6 \). In these dimensions, these are the same as \( S_i(\mathbb{Z}, E(K)) \). By the 4-periodicity, \( S_i(\mathbb{Z}, E(K)) = 0 \) for every \( i \in \mathbb{Z} \). Now the result follows from the 4-periodic algebraic surgery exact sequence.