1. Controlled Homotopy Equivalences

We first introduce basic concepts in controlled homotopy theory, and state several elementary facts on them. These are analogues of the standard non-controlled theory. We introduce metric control into the notion of homotopy and homotopy equivalence. We basically follow [4].

Definition. Let X be a metric space. A homotopy $h : K \times [0,1] \to X$ has *diameter* $\leq \delta$, if the image of the path $h(x,-) : [0,1] \to X$ has diameter $\leq \delta$ for every $x \in K$.

Convention. When $h : A \times [0,1] \to B$ is a homotopy, $h_t : A \to A$ will denote the map defined by $h_t(a) = h(a,t)$.

Definition. Fix a "*control map*" $p : K \to X$ from a space K to a metric space X, and let δ be a non-negative number. Take a topological space M (and its subset S). Two maps $f,\ g : M \to K$ are called *δ-close* if $d(p \circ f(x), p \circ g(x)) \leq \delta$ for every $x \in M$. A homotopy $H : M \times [0,1] \to K$ (rel S) is called a *δ-homotopy* (*rel* S) if the homotopy $p \circ H$ has diameter $\leq \delta$. When we need to specify the control map, we call it a *$p^{-1}(\delta)$-homotopy.* A map $f : M \to K$ (between spaces with common subsets S) is said to be a *δ-homotopy equivalence* (*rel* S) if there exists a map $g : K \to M$ together with a $(p \circ f)^{-1}(\delta)$-homotopy (*rel* S)

$$h : M \times [0,1] \to M : gf \simeq 1_M$$

and a $p^{-1}(\delta)$-homotopy (*rel* S)

$$k : K \times [0,1] \to K : gf \simeq 1_K\ .$$

When we want to specify the control map, we call it a *$p^{-1}(\delta)$-homotopy equivalence.*

Proposition 1.1. *Suppose $f : K' \to K$ and $g : K \to K'$ are maps such that $f \circ g : K \to K$ is $p^{-1}(\epsilon)$-homotopic to the identity. If $h_t : R \to K$ is a $p^{-1}(\delta)$-homotopy, then $g \circ h_t : R \to K'$ is a $(p \circ f)^{-1}(\delta + 2\epsilon)$-homotopy.*

Proof. Pick a point $x \in R$, then $\{p \circ f \circ g \circ h_t(x) | t \in [0,1]\}$ is in the closed ϵ neighborhood of $\{p \circ h_t(x) | t \in [0,1]\}$, and hence has diameter $\leq \delta + 2\epsilon$. □

Proposition 1.2. *Suppose $f' : K' \to K$ is a $p^{-1}(\epsilon')$-homotopy equivalence* (*rel* S) *$f'' : K'' \to K'$ is a $(p \circ f)^{-1}(\epsilon'')$-homotopy equivalence* (*rel* S). *Then $f' \circ f'' : K'' \to K$ is a $p^{-1}(\epsilon' + \epsilon'')$-homotopy equivalence* (*rel* S).

Proof. Easy to check. □

As in the standard case, the problem on maps can be converted into the problem on inclusion maps by replacing the target space with the mapping cylinder. The notion of n-connectedness of a pair is important especially when we are dealing with CW complexes. See [8] or [2] for basics on (relative) CW complexes. We use the notation e^r for an open r-cell, and $\bar{e}^r$ for a closed r-cell. When (K, L) is a relative CW complex, K_r will denote its r-skeleton, *i.e.* the union of L and the cells of dimension $\leq r$.

Definition. Let $p : K \to X$ be a control map, L be a subset of K, and n be a non-negative integer. When every map $f : (R, S) \to (K, L)$ from an n-dimensional relative CW complex to (K, L) is δ-homotopic *rel* S to a map into L, the pair (K, L) is said to be *(δ, n)-connected*, or *$(p^{-1}(\delta), n)$-connected* if we need to specify the control map. A pair (K, L) is said to be *$(\delta, 1.5)$-connected* if it is $(\delta, 1)$-connected and, for each map $f : (R, S) \to (K, L)$ from a 2-dimensional relative CW complex, there exists a map $f' : (R, S) \to (K, L)$ such that $f'|S = f|S$ and f' and f are δ-close with respect to p.

Note that we are using a different terminology from that used in [4].

A $(\delta, 2)$-connected pair is $(\delta, 1.5)$-connected, and a $(\delta, 1.5)$-connected pair is $(\delta, 1)$-connected. The $(\delta, 1.5)$-connectedness is a controlled analog of the condition that the inclusion map induces an isomorphism on the fundamental groups.

The (δ, n)-connectivity is preserved by ϵ-homotopy equivalences in the following manner.

Proposition 1.3. *If $f : (K', L) \to (K, L)$ is a $p^{-1}(\epsilon)$-homotopy equivalence rel L and (K, L) is $(p^{-1}(\delta), n)$-connected, then (K', L) is $((p \circ f)^{-1}(\delta + 3\epsilon), n)$-connected.*

Proof. Let $r : (R, S) \to (K', L)$ be a map from a relative n-complex. Then there exists a $p^{-1}(\delta)$-homotopy $h_t : R \to K$ *rel* S such that $h_0 = f \circ r$ and $h_1(R) \subset L$. Now $g \circ h_t : R \to K'$ is a $(p \circ f)^{-1}(\delta + 2\epsilon)$-homotopy such that $g \circ h_0 = g \circ f \circ r$ and $g \circ h_1(S) \subset L$. Since there is a $(p \circ f)^{-1}(\epsilon)$-homotopy between r and $g \circ f \circ r$, r is $(p \circ f)^{-1}(\delta + 3\epsilon)$-homotopic to $g \circ h_1$. □

Proposition 1.4 (Controlled Whitehead theorem)**.** *Let $n \geq 1$ be an integer. There exists a $\kappa > 1$ depending on n so that if*

1. *X is a metric space, and*
2. *(K, L) is a (δ, n)-connected n-dimensional relative CW complex with cells of diameter $\leq \delta$ with respect to a control map $p : K \to X$,*

then the inclusion map $L \to K$ is a $\kappa\delta$-homotopy equivalence.

Proof. For each $r = 0, 1, \cdots, n$, we inductively construct a constant $\delta_r > 0$ and a δ_r-homotopy $H : K_r \times [0, 1] \to K$ *rel* L such that

$$H_0 = 1 \quad \text{and} \quad H_1(K_r) \subset L \ .$$

For each 0-cell v of (K, L), there exists a path ρ_v in K from v to a point in L whose image in X has diameter $\leq \delta$. These paths define a δ-homotopy $H : K_0 \times [0, 1] \to K$ *rel* L such that $H(K_0) \subset L$, and so $\delta_0 = \delta$ works.

Assume inductively that we have defined δ_{r-1} and a δ_{r-1}-homotopy $H : K_{r-1} \times [0, 1] \to K$ *rel* L as above. Take an r-cell e^r and let $\theta : D^r \to K_r$ be its characteristic map. Let us construct a constant $\delta_r > 0$ and a δ_r-homotopy $G : D^n \times [0, 1] \to K$ such that

$$G_t|S^{r-1} = H_t \circ \theta|S^{r-1} \ , \quad G_0 = \theta \ , \quad \text{and} \quad G_1(D^r) \subset L \ .$$

Let $\varphi : D^r \times [0, 1] \to D^r \times [0, 1]$ be a homeomorphism so that $\varphi(D^r \times \{0\}) = D^r \times \{0\} \cup S^{r-1} \times [0, 1]$, and define a map $f : (D^r \times \{0\}, S^{r-1} \times \{0\}) \to (K, L)$ by

$$f = H \circ (\theta \times 1) \circ \varphi|D^r \times \{0\} \ .$$

The diameter of the image $p(f(D^r \times \{0\}))$ is $\leq \delta + 2\delta_{r-1}$. By the (δ, n)-connectivity hypothesis, there exists a δ-homotopy $F : D^r \times [0, 1] \to K$ of f such that

$$F(S^{r-1} \times [0, 1] \cup D^r \times \{1\}) \subset L \ .$$

If we set $\delta_r = 3\delta + 2\delta_{r-1}$, then the image $p(F(D^r \times [0, 1]))$ has diameter $\leq \delta_r$, and $F \circ \varphi^{-1}$ is the desired δ_r-homotopy G. G's for the r-cells of (K, L) together induce a δ_r-homotopy $H : K_r \times [0, 1] \to K$ *rel* L.

By induction we can conclude that there is a δ_n-deformation $H : K \times [0, 1] \to K$ *rel* L to L; in particular, the inclusion map $L \to K$ is a δ_n-homotopy equivalence. Thus $\kappa = 4 \cdot 2^n - 3$ works. □

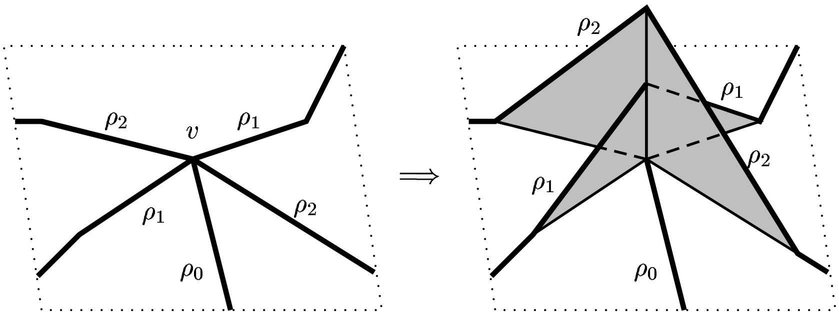
Proposition 1.5 (Geometric connectivity for relative CW complexes)**.** *Let $0 \leq n$ be an integer. There exists a $\kappa > 1$ depending on n so that if*

1. *X is a metric space,*
2. *r is an integer such that $0 \leq r \leq n$, and*

(3) (K, L) *is a* (δ, r)*-connected* n*-dimensional relative CW complex with cells of diameter* $\leq \delta$ *with respect to a control map* $p : K \to X$,

then there exists a $\max\{n, r+2\}$*-dimensional relative CW complex* (X', L) *with no cells of dimension* $\leq r$ *together with a* $\kappa\delta$*-homotopy equivalence* $K' \to K$ *rel* L.

Proof. We first eliminate the 0-cells. We add several cells of dimension ≤ 2 to K to produce another $\max\{n, 2\}$-dimensional complex which is 0-homotopy equivalent *rel* L to K. For each 0-cell v in $K - L$, there exists a path ρ_v in K from v to a point in L whose image in X has diameter $\leq \delta$. Homotop such ρ_v off the interiors of the cells of dimension ≥ 2. Now its image in X has diameter $\leq 2n\delta$. We may assume that each ρ_v is a sequence of consecutive non-loop 1-cells from v to a point in L and that ρ_v does not pass through the same 0-cell more than once. The paths $\{\rho_v\}$ may have non-trivial intersections in $K - L$. Pick a 0-cell v and suppose there are m paths $\rho_1, \ldots, \rho_m$ passing through v among them in addition to the path $\rho_0 = \rho_v$ starting from v.



Let e be a 1-cell of K connecting v and another 0-cell v' which appears in the path ρ_i, and let $\theta : [-1, 1] \to K$ be the characteristic map for e with $\theta(-1) = v'$ and $\theta(1) = v$. Let Δ_1 and Δ_2 be the subsets of $[-1, 1] \times [-1, 1]$ defined by $\Delta_1 = \{(t, u) | t = u\} \cong D^1$ and $\Delta_2 = \{(t, u) | t \leq u\} \cong D^2$. In $K \times [0, m]$, add to K new 0-cells $(v, 1), \ldots, (v, m)$, new 1-cells $v \times [0, 1], \ldots, v \times [m-1, m]$ and more new 1-cells of the form $(\theta \times 1)(\Delta_1)$, and new $2m$ 2-cells of the form $(\theta \times 1)(\Delta_2)$ to obtain a new n-dimensional relative complex (K', L). We can now replace the path ρ_i by a new path passing through (v, i) instead of v for each $i = 1, \cdots, m$ so that v is no longer an intersection point of the paths. The projection $K \times [0, 1] \to K$ restricts to a map $K' \to K$, which is obviously a 0-homotopy equivalence *rel* L with respect to p. Compose this with p to obtain a control map for K'. By continuing this for the other 0-cells in $K - L$, we obtain an n-dimensional complex $\hat{K}$ which is 0-homotopy equivalent *rel* L to the original K. The paths ρ_v's do not have intersection in $\hat{K} - L$, and their images have diameter $\leq 2n\delta$. Each 0-cell of $(\hat{K}, L)$ is contained in exactly one of the paths ρ_v's.

Now we shrink each path ρ_v to a point to obtain a new complex $(K^{(1)}, L)$ with no 0-cells. Actually there is a strong deformation retraction $H : (L \cup P) \times [0, 1] \to L \cup P$ to L of diameter $\leq 2n\delta$, where P denotes the union of the paths ρ_v's. This extends to a homotopy $H : \hat{K} \times [0, 1] \to \hat{K}$ *rel* L because of the NDR-property of $L \cup P$ in $\hat{K}$. In order to estimate the diameter of the homotopy, we describe the construction explicitly. Suppose H has been extended to $\hat{K}_{i-1} \times [0, 1] \to \hat{K}$. For each i-cell e of $(\hat{K}, L)$ with characteristic map $\theta : D^i \to \hat{K}$, define a homotopy $G : D^i \times [0, 1] \to \hat{K}$ by

$$G(x, t) = \begin{cases} H(\theta(\frac{x}{\|x\|}), 2 - \frac{2-t}{\|x\|}) & \text{if } \|x\| > 1 - t/2 \ , \\ \theta(\frac{2-t}{2} x) & \text{if } \|x\| \leq 1 - t/2 \ . \end{cases}$$

This induces an extension of H over $e \times [0,1]$, and the induction step proceeds, and defines a $3n\delta$-homotopy $H : \hat{K} \times [0,1] \to \hat{K}$ *rel* L. Also note that H induces a homotopy $H' : K^{(1)} \times [0,1] \to K^{(1)}$. Let $g^{(1)} : \hat{K} \to K^{(1)}$ denote the quotient map. Define a map $f^{(1)} : K^{(1)} \to \hat{K}$ as follows:

$$f^{(1)}(x) = \begin{cases} H_1(x) & \text{if } x \in K^{(1)} - L = \hat{K} - P \ , \\ x & \text{if } x \in L \ . \end{cases}$$

Then $f^{(1)} \circ g^{(1)} = H_1 \simeq 1_{\hat{K}}$ *rel* L, and $g^{(1)} \circ f^{(1)} = H_1' \simeq 1_{K^{(1)}}$ *rel* L. So $f^{(1)} : K^{(1)} \to \hat{K}$ is a $3n\delta$-homotopy equivalence *rel* L. Define $p^{(1)} : K^{(1)} \to \hat{X}$ by $p \circ f^{(1)}$. The cells of $(K^{(1)}, L)$ have diameter $\leq (6n{+}1)\delta$, and $(K^{(1)}, L)$ is $(9n{+}1)\delta$-connected with respect to $p^{(1)}$.

Let $k \leq r+1$ and suppose inductively that we have constructed constants $\kappa^{(i)}$ $(i = 1, \ldots, k)$ and homotopy equivalences $f^{(i)} : (K^{(i)}, L) \to (K^{(i-1)}, L)$ *rel* L such that

(1) $(K^{(i)}, L)$ has only cells of of dimension i, $i+1$, $\ldots$, $\max\{n+1, i+1\}$,
(2) $(K^{(i)}, L)$ has only cells of diameter $\leq \kappa^{(i)}\delta$ with respect to $p^{(i)} = p \circ f^{(1)} \circ f^{(2)} \circ \cdots \circ f^{(i)}$,
(3) $(K^{(i)}, L)$ is $(\kappa^{(i)}\delta, r)$-connected with respect to $p^{(i)}$, and
(4) $f^{(i)}$ is a $\kappa^{(i)}\delta$-homotopy equivalence with respect to $p^{(i)}$ and $p^{(i-1)}$.

We eliminate k-cells of $(K^{(k)}, L)$ (and at the same time introduce new $(k+2)$-cells) by a method called "cell-trading" ([2], p.25). Let $\{e_j^k\}$ be the k-cells and let $\theta_j : (D^k, S^{k-1}) \to (K^{(k)}, L)$ be the characteristic map for e_j^k, and set $\delta' = \kappa^{(k)}\delta$, then by assumption there is a δ'-homotopy $h_j : D^k \times [0,1] \to K^{(k)}$ *rel* L from θ_j to a map into L. Use a cellular approximation to replace h_j by a $(2n - 2k + 1)\delta'$-homotopy $h_j : D^k \times [0,1] \to K_{k+1}^{(k)}$. Consider a $(k{+}2)$-ball D^{k+2}. Its boundary is the union of two $(k+1)$-balls $B_1 \cup B_2$. Fix an identification of B_1 with $D^k \times [0,1]$, and hence an identification of $\partial B_1 = \partial B_2$ with $\partial(D^k \times [0,1])$. Attach new $(k+1)$-cells $\{e_j^{k+1}\}$ to $K^{(k)}$ using the map

$$h_j | \partial(D^k \times [0,1]) \ : \ \partial B_2 \longrightarrow K_k^{(k)}$$

to define a new complex $\bar{K}^{(k)}$, and then attach new $(k+2)$-cells $\{e_j^{k+2}\}$ to $\bar{K}^{(k)}$ using the map

$$h_j \cup 1_{B_2} \ : \ \partial D^{k+2} = B_1 \cup B_2 \longrightarrow \bar{K}_{k+1}^{(k)} \ ,$$

to obtain a complex $\hat{K}^{(k)}$. Since $\hat{K}^{(k)}$ is obtained from $K^{(k)}$ by simultaneous elementary expansions, the collapsing map $\hat{K}^{(k)} \to K^{(k)}$ is a 0-homotopy equivalence, and $(\hat{K}^{(k)}, L)$ is (δ', r)-connected. The diameter of added cells of $\hat{K}^{(k)}$ is $\leq (4n{-}4k{+}3)\delta'$ Note that the subcomplex $(K_k^{(k)} \cup \bigcup_j e_j^{k+1}, L)$ deforms into L fixing L by a $(4n - 4k + 3)\delta'$-homotopy, and that the homotopy extends to a δ''-homotopy of $\hat{K}^{(k)}$, where $\delta'' = (1 + 2(4n - 4k + 3)(n + 1 - k))\delta'$. Now define $K^{(k+1)}$ to be the space obtained from shrinking each of the added closed $(k+1)$-cells of $\hat{K}^{(k)}$ to a point, then $K^{(k+1)}$ has no cells of dimension $\leq k$, and there is a δ''-homotopy equivalence $f^{(k+1)} : K^{(k+1)} \to K^{(k)}$. The pair $(K^{(k+1)}, L)$ is $(\delta' {+} 3\delta'', r)$-connected, so

$$\kappa^{(k+1)} = (4 + 6(4n - 4k + 3)(n + 1 - k))\kappa^{(k)}$$

works. This completes the inductive step and the proof is finished. $\square$

The proof above is an adaptation of Quinn's proof of the geometric connectivity theorem for handlebodies [4] to the CW complex case. The following is a special case of it:

Theorem 1.6 (Quinn). *Let $n \geq 5$ be an integer. There exists a $\kappa > 1$ depending on n so that if*

(1) *X is a metric space,*
(2) *r is an integer such that $0 \leq r \leq n-4$, and*
(3) *$p : (N, \partial_0 N) \to X$ is a (δ, r)-connected n-dimensional compact manifold with a handlebody structure of diameter $\leq \delta$,*

then $(N, \partial_0 N)$ has a handlebody structure of diameter $\leq \kappa\delta$ with no handles of dimension $\leq r$.

2. Geometric Chain Complexes

In this section, we first review the theory of geometric chain complexes associated with transverse relative CW complexes developed by Quinn [6], and then study a Hurewicz-type criterion for (δ, n)-connectivity. We use the $\mathbb{Z}$-coefficients.

First let K be any space. Consider a map $S : |S| \to K$ from a set $|S|$. We identify S with its graph in $|S| \times K$. The two components of an element $s \in S \subset |S| \times K$ will be denoted $|s| \in |S|$ and $[s](= S(|s|)) \in K$, respectively. The free $\mathbb{Z}$-module on S is called the *geometric $\mathbb{Z}$-module* on K generated by S, and is denoted $\mathbb{Z}[S]$. It is *finitely generated* if $|S|$ is a finite set. If $\mathbb{Z}[S]$ and $\mathbb{Z}[T]$ are geometric modules on K, a *geometric morphism* $f : \mathbb{Z}[S] \to \mathbb{Z}[T]$ is a $\mathbb{Z}$-coefficient linear combination of '*paths*' (s, ρ, t) from $s \in S$ to $t \in T$, where $\rho : [0, \tau] \to K$ is a Moore path such that its initial point $\rho(0)$ is $[s]$ and its terminal point $\rho(\tau)$ is $[t]$. The composite $g \circ f$ of consecutive geometric morphisms

$$f = \sum_{\lambda \in \Lambda} m_\lambda (s_\lambda, \rho_\lambda, t_\lambda) : \mathbb{Z}[S] \to \mathbb{Z}[T] \ , \quad g = \sum_{\gamma \in \Gamma} n_\gamma (t'_\gamma, \sigma_\gamma, u_\gamma) : \mathbb{Z}[T] \to \mathbb{Z}[U]$$

is defined to be

$$\sum_{\lambda \in \Lambda, \gamma \in \Gamma, t_\lambda = t'_\gamma} n_\gamma m_\lambda (s_\lambda, \sigma_\gamma \rho_\lambda, u_\gamma).$$

We are using the Moore composition of paths here. Two paths $(s, \rho : [0, \tau], t)$ and $(s', \rho' : [0, \tau'], t')$ are *homotopic* if $s = s'$, $t = t'$, and there exists a homotopy from ρ to ρ' through Moore paths. A *homotopy* ($\simeq$) of a geometric morphism is a finite sequence of the following two operations:

- homotopies of the paths,
- combining two terms $m(s, \rho, t) + n(s, \rho, t)$ into $(m+n)(s, \rho, t)$, and its inverse.

Definition. A (*geometric $\mathbb{Z}$-module*) *chain complex on K* is a sequence of maps of geometric modules on K

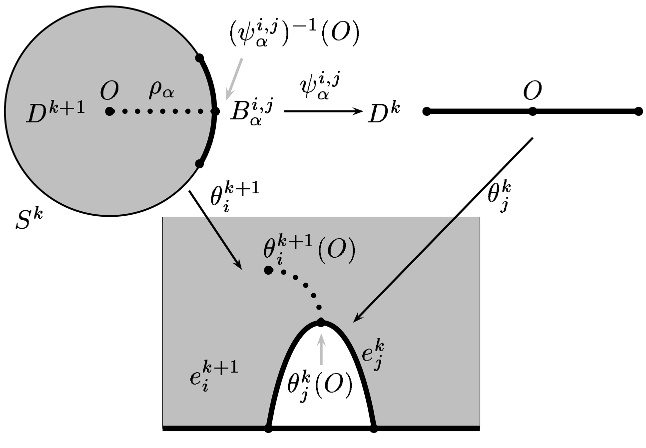
$$(C, d) \ : \ \cdots \to C_{r+1} \xrightarrow{d_{r+1}} C_r \xrightarrow{d_r} C_{r-1} \to \cdots$$

such that $d_r \circ d_{r+1} \simeq 0$.

Let us recall the construction of the geometric cellular chain complex $C_*(K, L)$ of a relative CW complex (K, L) from [6]. Let us assume that (K, L) is transverse in the following sense. As in the previous section, K_k will denote the k-skeleton of (K, L).

Definition. A map $f : (M^k, \partial M) \to (K_k, K_{k-1})$ from a smooth k-dimensional manifold (possibly with boundary) is said to be *transverse to the k-cells* if for each open k-cell e^k of (K, L), $f^{-1}(e^k)$ is the union of the interiors of a finitely many disjoint closed k-balls B^k_α in M such that there exists a homeomorphism $\psi_\alpha : B^k_i \to D^k$ to the k-ball D^k with $\theta \circ \psi_\alpha = f | B^k_\alpha$ for each α, where $\theta : D^k \to K$ denotes the characteristic map for the k-cell e^k. A relative CW complex (K, L) is *transverse* if the attaching maps $\theta : S^k \to K_k$ of the $(k+1)$-cells are all transverse to the k-cells for every k. Any continuous map $f : (M^k, \partial M) \to (K_k, K_{k-1})$ is homotopic rel ∂ to one that is transverse to the k-cells, and any finite relative CW complex is simple homotopy equivalent to a transverse relative CW complex.

Suppose (K, L) is transverse. The geometric celluar chain group $C_k(K, L)$ is defined to be the free abelian group generated by the set $S_k = \{e^k_j\}$ of the k-cells of (K, L) together with the map $S_k \to K$ sending each k-cell e^k_j to the image $\theta^k_j(O)$ of the origin O of the k-ball D^k by the characteristic map $\theta^k_j : D^k \to K$ for e^k_j. Let e^{k+1}_i be a $(k+1)$-cell of (K, L) and e^k_j be a k-cell of (K, L). Let $\theta^{k+1}_i : D^{k+1} \to K$ and $\theta^k_j : D^k \to K$ be the characteristic maps for e^{k+1}_i and e^k_j. By assumption, the restriction $\theta = \theta^{k+1}_i | S^k : (S^k, \emptyset) \to (K_k, K_{k-1})$ is transverse to the k-cells, and the inverse image $\theta^{-1}(e^k)$ is the union of the interiors of a finitely many disjoint closed k-balls $\{B^{i,j}_\alpha\}$ and there are homeomorphisms $\psi^{i,j}_\alpha : B^{i,j}_\alpha \to D^k$ such that $\theta^k_j \circ \psi^{i,j}_\alpha = \theta^{k+1}_i | B^{i,j}_\alpha$. Let ρ_α be a line segment (viewed as a path of constant speed 1) connecting the center O of D^{k+1} and the inverse image of the center O of D^k by $\psi^{i,j}_\alpha$. Now we need to fix our convention on orientaions of manifolds. We give the standard orientations $[D^{k+1}]$ and $[D^k]$ to D^{k+1} and D^k, and give the orientation $[S^k]$ to S^k so that [the unit outward normal vector] $\times [S^k] = [D^{k+1}]$. We give the orientation induced from that of S^k to $B^{i,j}_\alpha$. The *sign* of a homeomorphism $g : N_1 \to N_2$ of oriented manifolds is defined to be $+1$ or -1 according as g is orientsation preserving or not, and is denoted $\mathrm{sign}(g)$. Now let $d^{i,j}$ be the geometric morphism from $C_{k+1}(K, L)$ to $C_k(K, L)$ defined by the sum $\sum_\alpha \mathrm{sign}(\psi^{i,j}_\alpha)\theta^{k+1}_i \circ \rho_\alpha$. The boundary map $d_{k+1} : C_{k+1}(K, L) \to C_k(K, L)$ is defined by the sum $\sum_{i,j} d^{i,j}$. A transversality argument with respect to $(k-1)$-cells applied to $\theta^{k+1}_i | : S^k - \mathrm{Int} \bigcup_{j,\alpha} B^{i,j}_\alpha \to K_{k-1}$ shows that $d \circ d$ is homotopic to 0.



A geometric morphism $f : S \to T$ between geometric modules on a space K is said to *have radius* $\leq \epsilon$ with respect to a control map $p : K \to X$ if, for each path $\rho : [0, \tau] \to K$ with non-zero coefficient in f, the image of $p \circ \rho$ in X is contained in the intersection of the closed ϵ-neighborhoods of $p \circ \rho(0)$ and $p \circ \rho(\tau)$. This definition extends to geometric module chain complexes, chain maps, chain homotopies, chain contractions, and so on in an obvious manner. See [7], for example. If all the cells of (K, L) have diameter $\leq \delta$ with respect to $p : X \to X$, then $C_*(K, L)$ is a chain complex of radius $\leq \delta$.

The following proposition illustrates a typical use of geometric morphisms in the theory of controlled topology.

Proposition 2.1. *Let n be an integer ≥ 2, M be a connected and simply-connected oriented n-dimensional manifold with a possibly empty boundary ∂M, and (K, L) be an n-dimensional relative CW complex whose n-cells have diameter $\leq \delta$ with respect to a control map $p : K \to X$. Choose a basepoint x_0 of M. If a map $f : (M, \partial M) \to (K, K_{n-1})$ is transverse to all the n-cells of (K, L) and the image $f(M)$ has diameter $\leq \delta$ and meeets only finitely many n-cells of (K, L), then f has an associated geometric morphism $\sigma(f) : \mathbb{Z}[\{f(x_0)\}] \to C_n(K, L)$ of radius $\leq \delta$ well-defined up to homotopy of radius $\leq \delta$ such that if $\sigma(f)$ is δ-homotopic to 0 then f is 3δ-homotopic rel ∂ to a map into K_{n-1}.*

Proof. The geometric morphism $\sigma(f)$ is defined as follows. Let e^n_j be the open n-cells of (K, L), and let $\theta_j : D^n \to K$ be the characteristic map for e^n_j. Since f is transverse to the n-cells of (K, L), there are finitely many closed n-balls $B^n_{j,\alpha}$ and homeomorphisms $\psi_{j,\alpha} : B^n_{j,\alpha} \to D^n$ such that $\theta_j \circ \psi_{j,\alpha} = f|B^n_{j,\alpha}$ and $f^{-1}(e^n_j)$ is the union $\bigcup_\alpha B^n_{j,\alpha}$. For each $B^n_{j,\alpha}$, connect x_0 and $\psi^{-1}_{j,\alpha}(O)$ by a path $\rho_{j,\alpha}$ in M. Now $\sigma(f)$ is defined to be the sum $\sum_{j,\alpha} \mathrm{sign}(\psi_{j,\alpha}) f \circ \rho_{j,\alpha}$. The only ambiguity is the choice of the paths, and these paths are unique up to homotopy, since M is simply-connected.

Now suppose $\sigma(f)$ is modified by a homotopy of $f \circ \rho_{j,\alpha}$.

□

Proposition 2.2 (Controlled Hurewicz Theorem)**.** *Let $n \geq 1$ be an integer. There exists a $\kappa > 1$ depending on n so that if*

(1) *X is a metric space,*
(2) *r is an integer such that $0 \leq r \leq n$,*
(3) *(K, L) is a $(\delta, 1.5)$-connected relative n-dimensional CW complex with cells of diameter $\leq \delta$ with respect to a control map $p : K \to X$, and*
(4) *there are geometric morphisms $s_k : C_k(K, L) \to C_{k+1}(K, L)$ of radius $\leq \delta$ $(k = 0, \ldots, r)$ such that $d_{k+1}s_k + s_{k-1}d_k \simeq_{2\delta} 1$,*

then (K, L) is $(\kappa\delta, r)$-connected.

Proof. □

References

1. T. A. Chapman, *Controlled Simple Homotopy Theory and Applications*, Lecture Notes in Math.,**1009**, Springer-Verlag, New York-Heidelberg-Berlin (1983).
2. M. M. Cohen, *A Course in Simple-Homotopy Theory*, Graduate Texts in Mathematics, **10**, Springer-Verlag, New York-Heidelberg-Berlin (1973).
3. S. C.Ferry, *Epsilon-delta Surgery over $\mathbb{Z}$*, preprint.
4. F. Quinn, *Ends of Maps I*, Annals. Math., **110**, 275–331 (1979).
5. ______, *Lecture Note on Algebraic Topology*, Va. Tech (1979/1980).
6. ______, *Geometric Algebra*, Proc. 1983 Rutgers Topology Conference, Springer Lecture Notes, **1126**, 182–198 (1985).
7. A. Ranicki and M. Yamasaki, *Controlled K-theory*, Topology Appl. **61**, 1 – 59 (1995).
8. G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, **61**, Springer-Verlag, New York-Heidelberg-Berlin (1978).