A Generalization of Ohkawa's Theorem

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Summary of the Article

Abstract

A theorem due to Ohkawa states that the collection of Bousfield equivalence classes of spectra is a set. We extend this result to arbitrary combinatorial model categories.

A model category (in the sense of Quillen) is **combinatorial** if it is **locally presentable** and **cofibrantly generated.**

Locally Presentable Categories

Locally presentable categories were first considered by Gabriel and Ulmer in 1971.

For a cardinal λ, a small category $\mathcal{K}$ is **λ-filtered** if the following two conditions are satisfied:

- Given any set of objects $\{k_i \mid i \in I\}$ with $|I| < \lambda$, there is an object k and a morphism $k_i \to k$ for each $i \in I$.
- Given any set of parallel arrows $\{\alpha_j \colon k \to k' \mid j \in J\}$ where $|J| < \lambda$, there is a morphism $\gamma \colon k' \to k''$ such that $\gamma \circ \alpha_j$ is the same morphism for all $j \in J$.

Locally Presentable Categories

Let λ be a regular cardinal.

An object X of a cocomplete category $\mathcal{C}$ is **λ-presentable** if the functor $\mathcal{C}(X, -)$ preserves λ-filtered colimits; that is,

$$\mathcal{C}(X, \mathrm{colim}_{k \in \mathcal{K}} Y_k) \cong \mathrm{colim}_{k \in \mathcal{K}} \mathcal{C}(X, Y_k)$$

for every λ-filtered category $\mathcal{K}$ and every diagram $Y \colon \mathcal{K} \to \mathcal{C}$.

Examples:

- A set X if λ-presentable if and only if $|X| < \lambda$.
- A group G is λ-presentable if it admits a presentation $G = \langle X \mid R \rangle$ where $|X| < \lambda$ and $|R| < \lambda$.

Locally Presentable Categories

A cocomplete category $\mathcal{C}$ is **locally λ-presentable** if the isomorphism classes of λ-presentable objects form a set and every object of $\mathcal{C}$ is a λ-filtered colimit of λ-presentable objects.

A category is **locally presentable** if it is locally λ-presentable for some regular cardinal λ.

Examples:

- The category of sets is $\aleph_0$-presentable.
- Every functor category from a small category to a locally λ-presentable category is locally λ-presentable.
- The category of simplicial sets is $\aleph_0$-presentable.
- The category of symmetric spectra over simplicial sets is $\aleph_0$-presentable.

Cofibrantly Generated Model Categories

A **model category** is a category $\mathcal{M}$ equipped with collections of **weak equivalences, fibrations,** and **cofibrations** that satisfy Quillen's axioms. We assume functorial factorizations.

A model category $\mathcal{M}$ is **cofibrantly generated** if there are two sets of maps I (called **generating cofibrations**) and J (called **generating trivial cofibrations**) such that

- the domains of maps in I are small for I-cellular maps;
- the domains of maps in J are small for J-cellular maps;
- the fibrations in $\mathcal{M}$ are the maps with the right lifting property with respect to J;
- the trivial fibrations in $\mathcal{M}$ are the maps with the right lifting property with respect to I.

Combinatorial Model Categories

A model category is **combinatorial** if it is cofibrantly generated and the underlying category is locally presentable.

Dugger proved in 2001 that a model category is combinatorial if and only if it is Quillen equivalent to a left Bousfield localization of a category of diagrams of simplicial sets equipped with the projective model structure.

Examples:

- Simplicial sets
- Symmetric spectra over simplicial sets
- Motivic spaces and motivic spectra
- Module spectra over a ring spectrum
- Chain complexes of modules over a ring

Combinatorial Model Categories

If a model category $\mathcal{M}$ is combinatorial, then there are arbitrarily large regular cardinals λ with the following properties:

- The category $\mathcal{M}$ is locally λ-presentable.
- There are sets of generating cofibrations and generating trivial cofibrations in $\mathcal{M}$ whose domains and codomains are λ-presentable.
- There are fibrant and cofibrant replacement functors on $\mathcal{M}$ that preserve λ-filtered colimits.
- The terminal object of $\mathcal{M}$ is λ-presentable.

Such a choice of items will be called a **$\boldsymbol{\lambda}$-combinatorial structure** on $\mathcal{M}$.

Main Result

Let $\mathcal{M}$ be a model category with a λ-combinatorial model structure for a regular cardinal λ. Let I be the given set of generating cofibrations, and let R be the given fibrant replacement functor. Let 0 denote the terminal object of $\mathcal{M}$.

An object X of $\mathcal{M}$ is **contractible** if the unique morphism $X \to 0$ is a weak equivalence.

For a functor $H\colon \mathcal{M} \to \mathcal{M}$, an object X of $\mathcal{M}$ is called ***H*-acyclic** if HX is contractible.

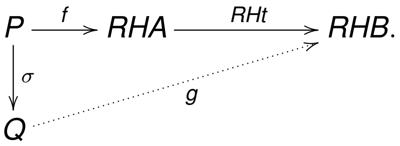
Let $\mathcal{A}(H)$ denote the collection of all H-acyclic objects in $\mathcal{M}$.

Main Result

Let $\mathcal{M}$ be a model category with a λ-combinatorial model structure. Choose a set $\mathcal{S}$ of representatives of all isomorphism classes of λ-presentable objects in $\mathcal{M}$.

Suppose given a functor $H\colon \mathcal{M} \to \mathcal{M}$.

For each triple (σ, A, f) where $\sigma\colon P \to Q$ is in the set I of generating cofibrations of $\mathcal{M}$ and $f\colon P \to RHA$ is a morphism in which $A \in \mathcal{S}$, let $\boldsymbol{T_H(\sigma, A, f)}$ denote the set of all morphisms $t\colon A \to B$ where $B \in \mathcal{S}$ and for which there exists $g\colon Q \to RHB$ such that $RHt \circ f = g \circ \sigma$.



Main Result

Denote

$$\boldsymbol{T(H)} = \{T_H(\sigma, A, f) \mid \text{all such triples } (\sigma, A, f)\}.$$

Theorem Let $\mathcal{M}$ be a model category with a λ-combinatorial structure for a regular cardinal λ. Let H_1 and H_2 be functors $\mathcal{M} \to \mathcal{M}$ that preserve λ-filtered colimits. If $T(H_2) \subseteq T(H_1)$ and the terminal object of $\mathcal{M}$ is H_2-acyclic, then $\mathcal{A}(H_1) \subseteq \mathcal{A}(H_2)$.

Proof It is essentially the same argument as in Ohkawa's article about elementary equivalence classes of spectra.

Main Result

Corollary If a model category $\mathcal{M}$ admits a λ-combinatorial structure for a regular cardinal λ, then there is only a set of distinct acyclic classes $\mathcal{A}(H)$ where H runs over all functors $\mathcal{M} \to \mathcal{M}$ that preserve λ-filtered colimits and such that the terminal object is H-acyclic.

Proof If $\{H_i \mid i \in I\}$ is any collection of such functors, then each $T(H_i)$ is a set of subsets of the union of $\mathcal{M}(A, B)$ for all $A, B \in \mathcal{S}$.

Therefore, the cardinality of the set of acyclic classes $\mathcal{A}(H)$ is bounded by $2^{2^{\kappa}}$ where κ is the cardinality of the set of morphisms between representatives of all isomorphism classes of λ-presentable objects of $\mathcal{M}$.

Monoidal Model Categories

Let $\mathcal{M}$ be a **monoidal** model category. The **Bousfield class** of an object E is defined as

$$\langle E \rangle = \{X \in \mathcal{M} \mid E \otimes X = 0 \text{ in } \mathrm{Ho}(\mathcal{M})\}.$$

Theorem If $\mathcal{M}$ is a pointed combinatorial monoidal model category, then the Bousfield classes in $\mathcal{M}$ form a set.

Proof Let λ be a regular cardinal such that $\mathcal{M}$ has a λ-combinatorial structure and let Q be a cofibrant replacement functor that preserves λ-filtered colimits. For each E, let

$$H_E X = QE \otimes QX.$$

Then H_E preserves λ-filtered colimits for all E, since $QE \otimes (-)$ has a right adjoint. Our claim follows since $\langle E \rangle = \mathcal{A}(H_E)$.

Special Cases

- For every commutative ring R there is only a set of distinct Bousfield classes in the derived category $\mathcal{D}(R)$.
- For every commutative ring spectrum E there is only a set of distinct Bousfield classes in the homotopy category of E-module spectra.
- For each Noetherian scheme S of finite Krull dimension there is only a set of distinct Bousfield classes in the stable motivic homotopy category $\mathrm{SH}(S)$ with base scheme S.
- For every field k of zero characteristic there is only a set of distinct Bousfield classes in the derived category $\mathrm{DM}(k)$ of motives over k, i.e., modules over the $H\mathbb{Z}$-spectrum.

Nonadditive Homology Theories

Question

Does Ohkawa's Theorem still hold if we omit representability?

In other words, do the kernels of non necessarily additive homology theories form a set?

Answer

Obviously not.

Homology with Zero Coefficients

The **James–Whitehead** homology theory is defined as

$$\mathrm{JW}_n(X) = \prod_{i=0}^{\infty} H_i(X) \,/\, \bigoplus_{i=0}^{\infty} H_i(X),$$

for all n, where H_* denotes reduced singular homology.

This theory is not additive, since $\mathrm{JW}_n(X) = 0$ for all n if X is a finite CW complex, while

$$\mathrm{JW}_n(\vee_{k=0}^{\infty} S^k) = \prod_{i=0}^{\infty} \mathbb{Z} \,/\, \bigoplus_{i=0}^{\infty} \mathbb{Z}$$

is nonzero.

Note that H_* and $H_* \oplus \mathrm{JW}_*$ are in the same Bousfield class.

Nonadditive Bousfield Classes

For a collection of elements $a = \{a_i \mid i \in I\}$ in an abelian group A, define its **support** and its **content** as

$$\text{supp } a = \{i \in I \mid a_i \neq 0\} \subseteq I;$$
$$\text{cont } a = \cup_{i \in I} \{a_i\} \subseteq A.$$

Example: $\text{supp}(1,1,1,\dots) = \mathbb{N}$ while $\text{cont}(1,1,1,\dots) = \{1\}$.

For each cardinal α, denote $A^\alpha = \prod_{i<\alpha} A$ and

$$A^{<\alpha} = \{(a_i) \in A^\alpha \ : \ |\text{cont}(a_i)| < \alpha\}.$$

The functor on abelian groups

$$F_\alpha A = A^\alpha / A^{<\alpha}$$

is exact and satisfies $F_\alpha A = 0$ if $|A| < \alpha$ while $F_\alpha(\oplus_{i<\alpha} \mathbb{Z}) \neq 0$.

Nonadditive Bousfield Classes

Theorem There is a proper class of distinct kernels of nonadditive homology theories.

This is joint work with Pau Casassas and Fernando Muro (unpublished).

Proof For each cardinal α we construct a homology theory h_* such that the CW complexes with less than α cells are h_*-acyclic while there exists a CW complex with α cells which is not h_*-acyclic. Namely, let

$$h_n(X) = F_\alpha H_n(X)$$

where H_* denotes singular homology and $F_\alpha A = A^\alpha / A^{<\alpha}$.

Note that $h_1(\vee_{i<\alpha} S^1) \neq 0$. Thus, such homology theories are in distinct Bousfield classes for distinct cardinals α.

Nonadditive Bousfield Classes

Theorem For each regular cardinal λ there is only a set of Bousfield classes of (non necessarily additive) homology theories that commute with λ-filtered colimits.

Proof This is shown in the same way as Ohkawa's Theorem.