On Gersten's conjecture

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Conventions

A	:	Commutative noetherian ring with 1.
$\mathscr{B}$	:	Abelian category.
$\mathscr{C}$	:	Category.
$\mathscr{E}$	:	Exact category.
f	:	Element of A or R.
$\mathscr{G}$, $\mathscr{H}$	:	Exact subcategories of $\mathscr{B}$.
I, J	:	Ideals of A.
$\mathscr{K}$	:	Subcategories of $\mathscr{B}$.
$\mathscr{M}$	:	Category of finitly generatd modules.
n, p	:	Non-negative integers.
R	:	Commutative regular local ring.
S	:	Finite set.
T, U, V	:	subsets of S.
$\mathscr{X}$	:	$R/\mathfrak{f}_{S\smallsetminus\{s\}}R$
$\mathscr{Y}$	:	$\operatorname{Kos}_{\mathscr{X}}^{\{f_s\}}$
$\mathscr{Z}$	:	$\mathbf{Ch}_{[0,1]}(\mathscr{M}_{\mathscr{X}}(1))$.

References

[Moc13] S. Mochizuki, *Higher K-theory of Koszul cubes*, Homology, Homotopy and Applications Vol. **15** (2013), p. 9-51.

[Moc15] S. Mochizuki, *On Gersten's conjecture*, prepreint, avilable at arXiv:1503.07966v2 (2015).

1 What is Gersten's conjecture?

For any commutative noetherian ring A with 1 and any natural number $0 \leq p \leq \dim A$, let $\mathscr{M}_A^p$ denote the category of finitely generated A-modules M whose support has codimension $\geq p$ in $\operatorname{Spec} A$.

Gersten's conjecture.
For any commutative regular local ring R and natural number $1 \leq p \leq \dim R$, the canonical inclusion $\mathscr{M}_R^p \hookrightarrow \mathscr{M}_R^{p-1}$ induces the zero map on K-theory

$$K(\mathscr{M}_R^p) \to K(\mathscr{M}_R^{p-1})$$

where $K(\mathscr{M}_R^i)$ denotes the K-theory of the abelian category $\mathscr{M}_R^i$.

It is well-known that Gersten's conjecture has several consequences. We illustrate a part of examples.

Corollary.
(1) For any commutative regular local ring R, The chow group $\mathrm{CH}_k(\operatorname{Spec} R)$ is trivial for any $k < \dim R$.
(2) For any regular noetherian separated scheme X, we have the canonical isomorphism

$$\mathrm{CH}^p(X) \xrightarrow{\sim} \mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{K}_p)$$

where $\mathscr{K}_p$ is the Zariski sheafication of the K-presheaf $U \mapsto K_p(U)$.

2 Idea of the proof

In my viewpoint, difficulty of solving Gersten's conjecture consists of ring theoretic aspect and homotopy theoretic aspect. We will explain ideas about how to overcome each difficulty.

Ring theoretic aspect

Gersten's conjecture for the Grothendieck groups K_0 is equivalent to the following *Generator conjecture*:

Geneartor conjecture.
For any commutative regular local ring R and any natural number $0 \leq p \leq \dim R$, the Grothendieck group $K_0(\mathscr{M}_R^p)$ is generated by cyclic modules $R/(f_1, \cdots, f_p)$ where the sequence $f_1, \cdots, f_p$ forms an R-regular sequence.

We illustrate how to prove that the generator conjecture implies Gersten's conjecture for K_0.
Let a sequence $f_1, \cdots, f_p$ be an R-regular sequence. Then there exists the short exact sequence

$$0 \to R/(f_1, \cdots, f_{p-1}) \overset{f_p}{\to} R/(f_1, \cdots, f_{p-1}) \to R/(f_1, \cdots, f_p) \to 0$$

in $\mathscr{M}_R^{p-1}$. Thus the class $[R/(f_1, \cdots, f_p)]$ in $K_0(\mathscr{M}_R^p)$ goes to

$$[R/(f_1, \cdots, f_p)] = [R/(f_1, \cdots, f_{p-1})] - [R/(f_1, \cdots, f_{p-1})] = 0$$

in $K_0(\mathscr{M}_R^{p-1})$.

Gambit 1.
We will establish and prove a higher analogue of the generator conjecture.

Homotopy theoretic aspect

Roghly saying, we will try to compare the following two functors on K-theory. We denote the category of bounded complexes on $\mathscr{M}_R^{p-1}$ by $\mathbf{Ch}_b(\mathscr{M}_R^{p-1})$.

$$\mathscr{M}_R^p \to \mathbf{Ch}_b(\mathscr{M}_R^{p-1}),$$

$$R/(f_1,\cdots,f_p) \mapsto \begin{cases} \left|\begin{matrix} R/(f_1,\cdots,f_p) \\ \downarrow f_p \\ R/(f_1,\cdots,f_p) \end{matrix}\right| \underset{\text{qis}}{\sim} R/(f_1,\cdots,f_p) \\ \left|\begin{matrix} R/(f_1,\cdots,f_p) \\ \downarrow \mathrm{id} \\ R/(f_1,\cdots,f_p) \end{matrix}\right| \underset{\text{qis}}{\sim} 0 \end{cases}$$

The functors above shall be homotopy equivalence on K-theory by the additivity theorem.

Problem.
The functors above are not 1-**funtorial**!!

We need to a good notion of K-theory for higher category theory or need to discuss more subtle argument for such exotic functors.

Gambit 2.
We give a modified definition of algebraic K-theory in a particular situation and by utilizing this definition, we will treat such exotic functors inside the classical Waldhausen K-theory.

3 Strategy of the proof

Let A be a commutative noetherian ring with 1 and let I be an ideal of A with codimension $Y = V(I) \geq p$ in $\operatorname{Spec} A$. Let $\mathscr{M}_A^I$ be a full subcategory of $\mathscr{M}_A^p$ consisting of those modules M supported on $Y = V(I)$ and $\mathscr{M}_{A,\mathrm{red}}^I$ be a full subcategory of $\mathscr{M}_A^I$ consisting of those modules M such that IM are trivial. We call a module in $\mathscr{M}_{A,\mathrm{red}}^I$ a *reduced module with respect to I*. Let $\mathscr{P}_A$ be the category of finitely generated projective A-modules. Let J be an ideal generated by R-regular sequence $f_1, \cdots, f_p$ such that f_i is a prime element for any $1 \leq i \leq p$. First notice the following isomorphisms:

$$K(\mathscr{M}_R^p) \underset{\mathbf{I}}{\overset{\sim}{\to}} \operatorname*{colim}_{\substack{\operatorname{codim}_{\operatorname{Spec} R} V(I) = p \\ \operatorname{Spec} R/I \underset{\text{regular}}{\hookrightarrow} \operatorname{Spec} R}} K(\mathscr{M}_R^I),$$

$$K(\mathscr{P}_{R/J}) \underset{\mathbf{II}}{\overset{\sim}{\to}} K(\mathscr{M}_{R,\mathrm{red}}^J) \underset{\mathbf{III}}{\overset{\sim}{\to}} K(\mathscr{M}_R^J).$$

Since R is Cohen-Macaulay, the ordered set of all ideals of R that contains an R-regular sequence of length p with usual inclusion is directed. Thus $\mathscr{M}_R^p$ is the filtered limit $\operatorname*{colim}_I \mathscr{M}_R^I$ where I runs through any ideal generated by any R-regular sequence of length p. Thus the isomorphism **I** follows from cocontinuity of K-theory. The isomorphism **II** follows from the resolution theorem and regularity of R. Finally the isomorphism **III** follows from the dévissage theorem. Hence the problem reduces to show the following. Let $\mathfrak{f}_S = \{f_s\}_{s \in S}$ be an R-regular sequence such that f_s is a prime element of R for any $s \in S$. We call such a sequence $\mathfrak{f}_S$ a *prime regular sequence*:

The inclusion functor $\mathscr{P}_{R/\mathfrak{f}_S R} \hookrightarrow \mathscr{M}_R^{\#S-1}$ induces the zero map

$$K(\mathscr{P}_{R/\mathfrak{f}_S R}) \to K(\mathscr{M}_R^{\#S-1})$$

on K-theory.

We construct the exact category $\mathrm{Kos}_R^{\mathfrak{f}_S}$ of *Koszul cubes* associated with $\mathfrak{f}_S$ and prove that

(i) There exists the commutative diagram below,
(ii) The map **I** is a split epimorphism and
(iii) The map **II** is the zero map.

$$\begin{array}{ccc} K(\mathrm{Kos}_R^{\mathfrak{f}_S}) & & \\ \downarrow \mathbf{I} & \searrow^{\mathbf{II}} & \\ K(\mathscr{P}_{R/\mathfrak{f}_S R}) & \longrightarrow & K(\mathscr{M}_R^{\# S-1}). \end{array}$$

Assertion (ii) corresponds with Gambit 1 and assertion (iii) corresponds with Gambit 2 in the previous section.

4 Cubes

Definitions.

An S-*cube* in a category $\mathscr{C}$ is a contravariant functor

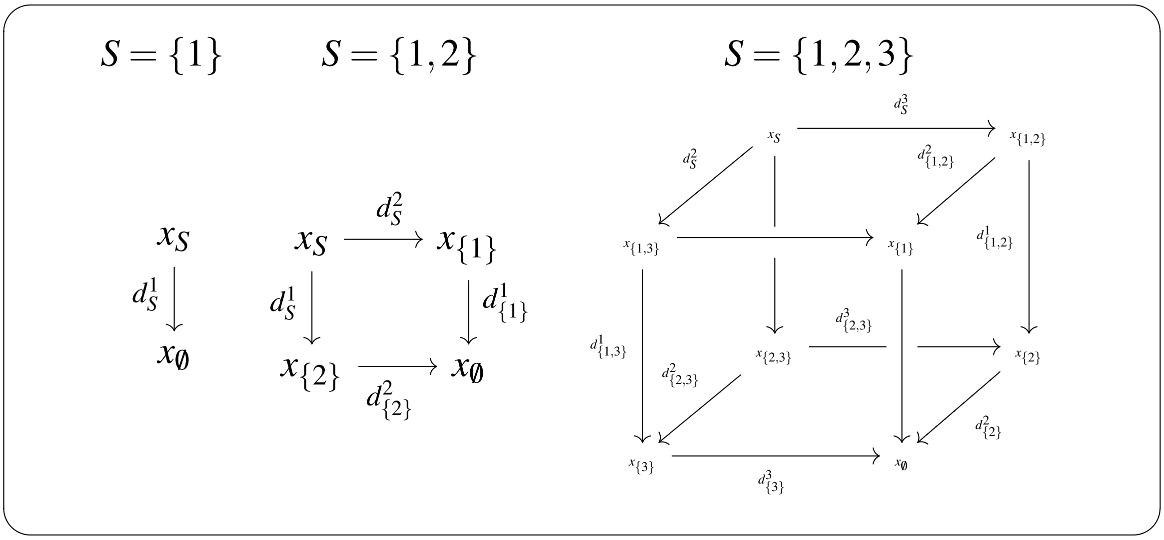
$$\mathscr{P}(S)^{\mathrm{op}}(\overset{\sim}{\to} [1]^{S^{\mathrm{op}}}) \to \mathscr{C}\,.$$

For any $U \in \mathscr{P}(S)$ and $k \in U$, we call

• $x_U (:= x(U))$ a *vertex of* x *at* U and

• $d_U^{k,x} (= d_U^k) := x(U \setminus \{k\} \hookrightarrow U)$ a k-*boundary map at* U.

We denote the category of S-cubes by $\mathbf{Cub}^S(\mathscr{C})$.



Example 1

For a family of morphisms $\mathfrak{x} := \{x_s \overset{d_s}{\to} x\}_{s\in S}$ in $\mathscr{B}$, we set

$$\operatorname{Fib}\mathfrak{x}_U := \begin{cases} x & \text{if } U = \emptyset \\ x_s & \text{if } U = \{s\} \\ x_{t_1} \times_x x_{t_2} \times_x \cdots \times_x x_{t_r} & \text{if } U = \{t_1, \cdots, t_r\} \end{cases}$$

Definition.

An S-cube x in $\mathscr{B}$ is *fibered* if the canonical map

$$x \to \operatorname{Fib}\{x_{\{s\}} \overset{d^s_{\{s\}}}{\to} x_\emptyset\}_{s\in S}$$

is an isomorphism.

Example 2

For a family of elements $\mathfrak{f}_S = \{f_s\}_{s\in S}$ in A, we set

$$\operatorname{Typ}_A(\mathfrak{f}_S)_U := A \text{ and } d^s_U = f_s$$

for any $U \in \mathscr{P}(S)$ and $s \in U$.

(Notice that $\operatorname{Tot}\operatorname{Typ}_A(\mathfrak{f}_S)$ is the Koszul complex associated with $\mathfrak{f}_S$.)

More generally for any non-negative integer r and a family of non-negative integers $\mathfrak{n}_S = \{n_s\}_{s\in S}$ less than r indexed by S, we define $\operatorname{Typ}_A(\mathfrak{f}_S; r, \mathfrak{n}_S)$ to be an S-cube of finitely generated free A-modules by setting for each element s in S and subsets $U \subset S$ and $V \subset S \smallsetminus \{s\}$, $\operatorname{Typ}_A(\mathfrak{f}_S; r, \mathfrak{n}_S)_U := A^{\oplus r}$ and $d^{\operatorname{Typ}_A(\mathfrak{f}_S;r,\mathfrak{n}_S),s}_{V\sqcup\{s\}} := \begin{pmatrix} f_s E_{n_s} & 0 \\ 0 & E_{r-n_s} \end{pmatrix}$ where E_m is the $m \times m$ unit matrix. We call $\operatorname{Typ}_A(\mathfrak{f}_S; r, \mathfrak{n}_S)$ the *typical cube of type* $(r, \mathfrak{n}_S)$ *associated with* $\mathfrak{f}_S$.

Faces and homoligy of cubes

Definitions.

$k \in S$, x: S-cube

$$B^k,\ F^k : \mathscr{P}(S \smallsetminus \{k\}) \to \mathscr{P}(S)$$

$$F^k : U \mapsto U$$

$$B^k : U \mapsto U \sqcup \{k\}$$

We call

- xB^k the *backside k-face of x*,
- xF^k the *frontside k-face of x* and
- $\mathrm{H}_0^k(x) := \mathrm{Coker}(xB^k \to xF^k)$ the *k-direction 0-th homology of x*.

$S = \{1, 2\}$

$$\begin{array}{ccccc} x_S & \longrightarrow & x_{\{1\}} & & \mathrm{H}_0^2(x)_{\{1\}} \\ \downarrow & & \downarrow & \Rightarrow & \downarrow \\ x_{\{2\}} & \longrightarrow & x_{\emptyset} & & \mathrm{H}_0^2(x)_{\emptyset} \end{array}$$

$$\Downarrow$$

$$\mathrm{H}_0^1(x)_{\{2\}} \longrightarrow \mathrm{H}_0^2(x)_{\emptyset}$$

By taking faces and homologies, the notion of S-cubes conforms itself to inductive argument on the cardinality of S.

5 Admissibility

Definition.
We say that an S-cube x in $\mathscr{B}$ is *admissible* if
(1) its boundary morphism(s) is (are) monomorphism(s) and
(2) if for every k in S, $\mathrm{H}_0^k(x)$ is admissible.

• We can prove that any admissible cube is fibered.

• We can prove that an S-cube x in $\mathscr{B}$ is admissible if and only if
(1) all faces of the S-cube x are admissible and
(2) $\mathrm{H}_k(\mathrm{Tot}\, x) = 0$ for any $k > 0$.

Definition.
For an admissible S-cube x in $\mathscr{B}$ and a subset T of S, we set

$$\mathrm{H}_0^T(x) := \begin{cases} x & \text{if } T = \emptyset \\ \mathrm{H}_0^{t_1}(\mathrm{H}_0^{t_2}(\mathrm{H}_0^{t_3}(\cdots(\mathrm{H}_0^{t_r}(x))\cdots))) & \text{if } T = \{t_1, \cdots, t_r\}. \end{cases}$$

We can prove that the definition above does not depend upon an order of $t_1, \cdots, t_r$ up to isomorphisms.

Example 1.

For a commutative diagram of monomorphisms in $\mathscr{B}$

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet, \end{array}$$

the diagram above is admissible if and only if it is Cartesian.

Example 2.

For any family of elements $\mathfrak{f}_S = \{f_s\}_{s \in S}$ in A, $\mathrm{Typ}(\mathfrak{f}_S)$ is admissible if and only if $\mathfrak{f}_S$ is a regular sequence in any order.

Admissibility

= a higher analogue of the notion about Cartesian squares

= a categorical variant of the notion about regular sequences.

6 Multi semi-direct products of categories

Definition.
Let $\mathfrak{K} = \{\mathscr{K}_T\}_{T \in \mathscr{P}(S)}$ be a family of full subcategories of an abelian category $\mathscr{B}$. We define $\ltimes \mathfrak{K} = \underset{T \in \mathscr{P}(S)}{\ltimes} \mathscr{K}_T$ the *multi semi-direct products of the family* $\mathfrak{K}$ as follows. $\ltimes \mathfrak{K}$ is the full subcategory of $\mathbf{Cub}^S(\mathscr{B})$ consisting of those S-cubes x such that x is admissible and each vertex of $\mathrm{H}_0^T(x)$ is in $\mathscr{K}_T$ for any $T \in \mathscr{P}(S)$.
If S is a singleton (namely $\#S = 1$), then we write $\mathscr{K}_S \ltimes \mathscr{K}_\emptyset$ for $\ltimes \mathfrak{K}$.

We often use the following formula to demonstrate propositions about S-cubes by induction on the cardinality of S. For any element $s \in S$, we have the equalitiy

$$\underset{T \in \mathscr{P}(S)}{\ltimes} \mathscr{K}_T = \left(\underset{T \in \mathscr{P}(S \smallsetminus \{s\})}{\ltimes} \mathscr{K}_{T \sqcup \{s\}} \right) \ltimes \left(\underset{T \in \mathscr{P}(S \smallsetminus \{s\})}{\ltimes} \mathscr{K}_T \right).$$

7 Koszul cubes

Let A be a commutative noetherian ring with 1 and $\mathfrak{f}_S = \{f_s\}_{s\in S}$ be a family of A-regular sequence in any order indexed by a finite set S. For any subset $T \subset S$, we write $\mathfrak{f}_T$ for the family $\{f_t\}_{t\in T}$.

Definition.
We set

$$\mathrm{Kos}_A^{\mathfrak{f}_S} := \ltimes_{T\in\mathscr{P}(S)} \mathscr{P}_{A/\mathfrak{f}_T A}$$

and call it the category of *Koszul cubes*[1] *associated with* $\mathfrak{f}_S$.

[1] In the paper [Moc15], we write $\mathrm{Kos}_{A,\mathrm{simp}}^{\mathfrak{f}_S}$ for $\mathrm{Kos}_A^{\mathfrak{f}_S}$ and call it the category of *simple* Koszul cubes. But in this note, for simplicity, we use this terminology.

We can prove that $\mathrm{Kos}_A^{\mathfrak{f}_S}$ is a semisimple exact category such that the inclusion functor $\mathrm{Kos}_A^{\mathfrak{f}_S} \hookrightarrow \mathbf{Cub}^S(\mathscr{M}_A)$ is exact and reflects exactness.

Proposition.
Assume that A *and* $\mathfrak{f}_S$ *satisfy the following two conditions:*
(1) *For any subset* T *of* S*, finitely generated projective* $A/\mathfrak{f}_T A$*-modules are free.*
(2) $\mathfrak{f}_S$ *is contained in the Jacobson radical of* A*.*
Then for any object x *of* $\mathrm{Kos}_A^{\mathfrak{f}_S}$*, there exists a non-negative integer* r *and a family* $\mathfrak{n}_S := \{n_s\}_{s\in S}$ *of non-negative integers less than* r *such that* x *is isomorphic to* $\mathrm{Typ}_A(\mathfrak{f}_S;\mathfrak{n}_S,r)$*.*

8 Adroit systems

Let R be a commutative regular local ring and let $\mathfrak{f}_S = \{f_s\}_{s\in S}$ be an R-regular sequence such that f_s is a prime element for any $s \in S$. We will show that the exact functor

$$\mathrm{Kos}_R^{\mathfrak{f}_S} \to \prod_{T\in\mathscr{P}(S)} \mathscr{P}_{R/\mathfrak{f}_T R}, \quad x \mapsto (\mathrm{H}_0^T(x)_{\emptyset})_{T\in\mathscr{P}(S)}$$

induces a homotopy equivalence

$$K(\mathrm{Kos}_R^{\mathfrak{f}_S}) \overset{\sim}{\to} \bigoplus_{T\in\mathscr{P}(S)} K(\mathscr{P}_{R/\mathfrak{f}_T R})$$

on K-theory. Then in particular we will prove assertion (ii) in §3.

We will analyze the problem by abstracting more general situation.

$\boxed{1}$ For a pair of exact subcategories[1] $\mathscr{G} \hookrightarrow \mathscr{H}$ in $\mathscr{B}$ such that $\mathscr{G} \ltimes \mathscr{H}$ is also an exact subcategory in $\mathbf{Cub}^{[1]}\mathscr{B}$. Then the exact functor $\mathscr{G} \ltimes \mathscr{H} \to \mathscr{G} \times \mathscr{H}$ induces a homotopy equivalece $K(\mathscr{G} \ltimes \mathscr{H}) \to K(\mathscr{G}) \times K(\mathscr{H})$ on K-theory by the additivity theorem.

$\boxed{2}$ If we have a triple of exact subcategories $\mathscr{H}_1 \hookrightarrow \mathscr{H}_2 \hookleftarrow \mathscr{G}$ in $\mathscr{B}$ satisfying the following two conditions:

(1) $\mathscr{G} \ltimes \mathscr{H}_1$ and $\mathscr{G} \ltimes \mathscr{H}_2$ are exact subcategories in $\mathbf{Cub}^{[1]}\mathscr{B}$.

(2) The inclusion functors $\mathscr{H}_1 \hookrightarrow \mathscr{H}_2$ and $\mathscr{G} \ltimes \mathscr{H}_1 \hookrightarrow \mathscr{G} \ltimes \mathscr{H}_2$ induce equivalences of triangulated categories:

$$\mathscr{D}^b(\mathscr{H}_1) \overset{\sim}{\to} \mathscr{D}^b(\mathscr{H}_2) \quad \text{and} \quad \mathscr{D}^b(\mathscr{G} \ltimes \mathscr{H}_1) \overset{\sim}{\to} \mathscr{D}^b(\mathscr{G} \ltimes \mathscr{H}_2).$$

Then the exact functor $\mathscr{G} \ltimes \mathscr{H}_1 \to \mathscr{G} \times \mathscr{H}_1$ induces a homotopy equivalence $K(\mathscr{G} \ltimes \mathscr{H}_1) \to K(\mathscr{G}) \times K(\mathscr{H}_1)$ on K-theory.

Proof. Let us consider the following commutative diagram

$$\begin{array}{ccc} K(\mathscr{G} \ltimes \mathscr{H}_1) & \longrightarrow & K(\mathscr{G}) \times K(\mathscr{H}_1) \\ \downarrow & & \downarrow \\ K(\mathscr{G} \ltimes \mathscr{H}_2) & \longrightarrow & K(\mathscr{G}) \times K(\mathscr{H}_2). \end{array}$$

The bottom horizontal line and the vertical lines are homotopy equivalences by $\boxed{1}$ and condition (2). Thus we obtain the result. □

[1] We say that a full subcategory $\mathscr{G}$ of an exact category $\mathscr{H}$ is an *exact subcategory* if the inclusion functor $\mathscr{G} \hookrightarrow \mathscr{H}$ is exact and reflects exactness.

We axiomatize the conditions of the triple $(\mathscr{G}, \mathscr{H}_1, \mathscr{H}_2)$ which imply conditions (1) and (2).

Definition.
An *adroit system* in an abelian category $\mathscr{B}$ is a system $(\mathscr{H}_1, \mathscr{H}_2, \mathscr{G})$ consisting of exact subcategories $\mathscr{H}_1 \hookrightarrow \mathscr{H}_2 \hookleftarrow \mathscr{G}$ in $\mathscr{B}$ and they satisfy the following axioms **(Adr 1)**, **(Adr 2)**, **(Adr 3)** and **(Adr 4)**:

(Adr 1) $\mathscr{G} \ltimes \mathscr{H}_1$ and $\mathscr{G} \ltimes \mathscr{H}_2$ are exact subcategories of $\mathbf{Ch}_b(\mathscr{B})$.
(Adr 2) $\mathscr{H}_1$ is closed under extensions in $\mathscr{H}_2$.
(Adr 3) Let $x \rightarrowtail y \twoheadrightarrow z$ be an admissible short exact sequence in $\mathscr{B}$. Assume that y is isomorphic to an object in $\mathscr{H}_1$ and z is isomorphic to an object in $\mathscr{H}_1$ or $\mathscr{G}$. Then x is isomorphic to an object in $\mathscr{H}_1$.
(Adr 4) For any object z in $\mathscr{H}_2$, there exists an object y in $\mathscr{H}_1$ and an admissible epimorphism $y \twoheadrightarrow z$.

Corollary.
Let $(\mathscr{H}_1, \mathscr{H}_2, \mathscr{G})$ be an adorit system in an abelian category $\mathscr{B}$. Then the canonical exact functor $\mathscr{G} \ltimes \mathscr{H}_1 \to \mathscr{G} \times \mathscr{H}_1$ induces a homotopy equivalence

$$K(\mathscr{G} \ltimes \mathscr{H}_1) \to K(\mathscr{G}) \times K(\mathscr{H}_1)$$

on K-theory.

We will utilize this corollary inductively. To discuss inductive argument, we need to prepare notations.

For a commutative noetherian ring with 1 and for an ideal I of A, and for a negative integer r, let $\mathscr{M}_A^I(r)$ be the full subcategory of $\mathscr{M}_A^I$ consisting of those A-modules M with $\operatorname{Projdim}_A M \leq r$. Let $S = U \sqcup V$ be a disjoint decomposition of S and let the letter p be a natural number such that $p \geq \# U$. We define $\mathscr{P}_A(\mathfrak{f}_U;\mathfrak{f}_V)(p)$ to be a full subcategory of $\mathbf{Cub}\ V\,\mathscr{M}_A$ by setting

$$\mathscr{P}_A(\mathfrak{f}_U;\mathfrak{f}_V)(p) := \underset{T\in\mathscr{P}(S)}{\ltimes} \mathscr{M}_{A/\mathfrak{f}_{T\sqcup U}A}(p-\#U)$$

In particular we have equalities

$$\mathrm{Kos}_A^{\mathfrak{f}_S} = \mathscr{P}_A(\mathfrak{f}_{\emptyset};\mathfrak{f}_S)(0)$$

and for any element v of V,

$$\begin{aligned}\mathscr{P}_A(\mathfrak{f}_U;\mathfrak{f}_V)(p) &= \left(\underset{T\in\mathscr{P}(V\smallsetminus\{v\})}{\ltimes} \mathscr{M}_{A/\mathfrak{f}_{T\sqcup U\sqcup\{v\}}A}(p-\#U)\right) \ltimes \left(\underset{T\in\mathscr{P}(V\smallsetminus\{v\})}{\ltimes} \mathscr{M}_{A/\mathfrak{f}_{T\sqcup U}A}(p-\#U)\right)\\ &= \mathscr{P}_A(\mathfrak{f}_{U\sqcup\{v\}};\mathfrak{f}_{V\smallsetminus\{v\}})(p+1) \ltimes \mathscr{P}_A(\mathfrak{f}_U;\mathfrak{f}_{V\smallsetminus\{v\}})(p).\end{aligned}$$

Lemma.
For any element v of V, the triple

$$(\mathscr{P}_R(\mathfrak{f}_U;\mathfrak{f}_{V\smallsetminus\{v\}})(p), \mathscr{P}_R(\mathfrak{f}_U;\mathfrak{f}_{V\smallsetminus\{v\}})(p+1), \mathscr{P}_R(\mathfrak{f}_{U\sqcup\{v\}};\mathfrak{f}_{V\smallsetminus\{v\}})(p+1))$$

is an adroit system in $\mathbf{Cub}^V\,\mathscr{M}_R$.

To check axiom **(Adr 4)**, we shall use the following lemma.

Lemma (Koszul resolution).
For any object x in $\mathscr{P}_R(\mathfrak{f}_U;\mathfrak{f}_V)(p+1)$, there exists a non-negative integer r and a family of non-negative integers $\mathfrak{n}_V = \{n_v\}_{v\in V}$ less than r and an admissible epimorphism $\mathrm{Typ}_{R/\mathfrak{f}_U R}(\mathfrak{f}_V;\mathfrak{n}_V,r) \twoheadrightarrow x$.

Corollary.
The canonical exact functor $\mathrm{Kos}_R^{\mathfrak{f}_S} \to \prod_{T\in\mathscr{P}(S)} \mathscr{P}_{R/\mathfrak{f}_T R}$ induces a homotopy equivalence

$$K(\mathrm{Kos}_R^{\mathfrak{f}_S}) \to \prod_{T\in\mathscr{P}(S)} K(\mathscr{P}_{R/\mathfrak{f}_T R})$$

on K-theory.

9 Zero map theorem

Let s be an element of S. We will prove that the composition

$$\mathrm{Kos}_R^{\mathfrak{f}_S} \overset{\mathrm{H}_0^S}{\to} \mathscr{P}_{R/\mathfrak{f}_S R} \hookrightarrow \mathscr{M}_R^{\mathfrak{f}_{S \smallsetminus \{s\}}}(\# S)$$

induces the zero map $K(\mathrm{Kos}_R^{\mathfrak{f}_S}) \to K(\mathscr{M}_R^{\mathfrak{f}_{S \smallsetminus \{s\}}}(\# S))$ on K-theory.

For simplicity, we set $\mathscr{X} = R/\mathfrak{f}_{S \smallsetminus \{s\}} R$ and $\mathscr{Y} = \mathrm{Kos}_{\mathscr{X}}^{\{f_s\}}$. Let $\mathbf{Ch}_b(\mathscr{M}_{\mathscr{X}}(1))$ be the category of bounded complexes on $\mathscr{M}_{\mathscr{X}}(1)$ and let $\eta\colon \mathscr{Y} \to \mathbf{Ch}_b(\mathscr{M}_{\mathscr{X}}(1))$ and $\eta'\colon \mathscr{M}_{\mathscr{X}}(1) \hookrightarrow \mathbf{Ch}_b(\mathscr{M}_{\mathscr{X}}(1))$ be the canonical inclusion functors. By considering the commutative diagrams below

$$\begin{array}{ccccccc} \mathrm{Kos}_R^{\mathfrak{f}_S} & \xrightarrow{\mathrm{H}_0^{S \smallsetminus \{s\}}} & \mathscr{Y} & & & & \\ & \searrow^{\mathrm{H}_0^S} & \downarrow{\scriptstyle \mathrm{H}_0^{\{s\}}} & & & & \\ & & \mathscr{P}_{R/\mathfrak{f}_S R} & \longrightarrow & \mathscr{M}_{\mathscr{X}}(1) & \longrightarrow & \mathscr{M}_R^{\mathfrak{f}_{S \smallsetminus \{s\}}}(1), \end{array}$$

$$\begin{array}{ccc} K(\mathscr{Y}) & \xrightarrow{K(\eta)} & K(\mathbf{Ch}_b(\mathscr{M}_{\mathscr{X}}(1)); \mathrm{qis}) \\ {\scriptstyle K(\mathrm{H}_0^{\{s\}})}\downarrow & & \uparrow{\scriptstyle K(\eta')} \\ K(\mathscr{P}_{R/\mathfrak{f}_S R}) & \longrightarrow & K(\mathscr{M}_{\mathscr{X}}(1)), \end{array}$$

we reduced the problem to show that the inclusion functor $\eta\colon \mathscr{Y} \to \mathbf{Ch}_b(\mathscr{M}_{\mathscr{X}}(1))$ induces the zero map

$$K(\mathscr{Y}) \to K(\mathbf{Ch}_b(\mathscr{M}_{\mathscr{X}}(1)); \mathrm{qis})$$

on K-theory.

Let $\mathscr{Z}$ be the full subcategory of $\mathbf{Ch}_b \mathscr{M}_{\mathscr{X}}(1)$ consiting of those complexes x such that $x_k = 0$ unless $k = 0$ or $k = 1$. There is a natural exact functor

$$\gamma\colon\ \mathscr{Z} \to \mathscr{M}_{\mathscr{X}}(1) \times \mathscr{M}_{\mathscr{X}}(1), \quad x \mapsto (x_{\{s\}}, x_{\emptyset})$$

which induces a homotopy equivalence

$$K(\mathscr{Z}) \to K(\mathscr{M}_{\mathscr{X}}(1)) \times K(\mathscr{M}_{\mathscr{X}}(1))$$

on K-theory by the additivity theorem. Let $\iota\colon\ \mathscr{Z} \to \mathbf{Ch}_b(\mathscr{M}_{\mathscr{X}}(1))$ be the inclusion functor.

Hope.
We wish to define two exact 'functors' $\mu_1,\ \mu_2\colon\ \mathscr{Y} \to \mathscr{Z}$ such that
(1) We have the equality $\gamma\mu_1 = \gamma\mu_2$.
(2) There are natural transformations $\iota\mu_1 \to \eta$ and $0 \to \iota\mu_2$ such that all components are quasi-isomorphisms.

Then we have the equalities

$$K(\eta) = K(\iota\mu_1) = K(\iota)K(\gamma)^{-1}K(\gamma\mu_1) = K(\iota)K(\gamma)^{-1}K(\gamma\mu_2) = K(\iota\mu_2) = 0.$$

To define the 'functors' μ_i ($i = 1, 2$), we analyze morphisms in $\mathscr{Y}$.

An object x is isomorphic to $\mathrm{Typ}_{\mathscr{X}}(\{f_s\};\{n_s\},r)$ for some pair of non-negative integers $n_s \leq r$. We set $x_{\text{non-deg}} := \mathrm{Typ}_{\mathscr{X}}(\{f_s\};\{n_s\},n_s)$ and $x_{\text{deg}} := \mathrm{Typ}_{\mathscr{X}}(\{1\};\{n_s\},r-n_s)$ and call $x_{\text{non-deg}}$ the *non-degenerated part of* x and call x_{deg} the *degenerated part of* x. We have an isomorphism of $\{s\}$-cubes $x \overset{\sim}{\to} x_{\text{non-deg}} \oplus x_{\text{deg}}$.

Let y be an object in $\mathscr{Y}$ and $\varphi\colon x \to y$ a morphism in $\mathscr{Y}$. Then we can denote φ by

$$\begin{bmatrix} (x_{\text{non-deg}} \oplus x_{\text{deg}})_{\{s\}} \\ \downarrow \\ (x_{\text{non-deg}} \oplus x_{\text{deg}})_{\emptyset} \end{bmatrix} \begin{matrix} \overset{\varphi_{\{s\}}}{\to} \\ \\ \underset{\varphi_{\emptyset}}{\to} \end{matrix} \begin{bmatrix} (y_{\text{non-deg}} \oplus y_{\text{deg}})_{\{s\}} \\ \downarrow \\ (y_{\text{non-deg}} \oplus y_{\text{deg}})_{\emptyset} \end{bmatrix}$$

with $\varphi_{\{s\}} = \begin{pmatrix} \varphi_{n\to n} & \varphi_{n\to d} \\ f_s\varphi_{d\to n} & \varphi_{d\to d} \end{pmatrix}$ and $\varphi_{\emptyset} = \begin{pmatrix} \varphi_{n\to n} & f_s\varphi_{n\to d} \\ \varphi_{d\to n} & \varphi_{d\to d} \end{pmatrix}$ where the letter n means nondegenerate and the letter d means degenerate and $\varphi_{n\to n}$ is a morphism of S-cubes $x_{\text{non-deg}} \to y_{\text{non-deg}}$ from the non-degenerated part of x to the non-degenerated part of y and $\varphi_{n\to d}$ is a morphism of S-cubes $x_{\text{non-deg}} \to y_{\text{deg}}$ from the non-degenerated part of x to the degenerated part of y and so on. In this case we write $\begin{pmatrix} \varphi_{n\to n} & \varphi_{n\to d} \\ \varphi_{d\to n} & \varphi_{d\to d} \end{pmatrix}$ for φ.

Candidates of $\mu_i\colon \mathscr{Y} \to \mathscr{Z}$ ($i = 1, 2$) are following.

$$\mu_1 : x \mapsto x_{\text{non-deg}},$$

$$\mu_2 : x \mapsto \begin{bmatrix} (x_{\text{non-deg}})_{\{s\}} \\ \downarrow \ \mathrm{id} \\ (x_{\text{non-deg}})_{\emptyset} \end{bmatrix}.$$

But they are not 1-functorial. We need to make revisions in the previous idea.

Modified algebraic K-theory

We introduce a modified version of algebraic K-theory of $\mathscr{Y}$. We say that a morphism $\varphi\colon x \to y$ in $\mathscr{Y}$ is *upper triangle* if $\varphi_{d\to n}$ is the zero morphism and say that φ is *lower triangle* if $\varphi_{n\to d}$ is the zero morphism. We denote the class of all upper triangle isomorphisms in $\mathscr{Y}$ by $i^{\triangle}$. We define $S^{\nabla}_{\cdot}\mathscr{Y}$ to be a simplicial full subcategory of $S_{\cdot}\mathscr{Y}$ the Segal-Waldhausen S-construction of $\mathscr{Y}$, consisting of those objects x such that $x(i \leq j) \to x(i' \leq j')$ is a lower triangle for each $i \leq i', j' \leq j$.

Lemma.
(1) The inclusion functor $i^{\triangle}S^{\nabla}_{\cdot}\mathscr{Y} \to iS_{\cdot}\mathscr{Y}$ is a homotopy equivalence.
(2) The simplicial functors $\mu'_1,\ \mu'_2\colon i^{\triangle}S^{\nabla}_{\cdot}\mathscr{Y} \to iS_{\cdot}\mathscr{X}$ which sending x to $x_{\text{non-deg}}$ and $\begin{bmatrix} (x_{\text{non-deg}})_{\{s\}} \\ \downarrow \text{ id} \\ (x_{\text{non-deg}})_{\emptyset} \end{bmatrix}$ respectively are well-defined.

The main reason for (1) is the fact that $\mathscr{Y}$ is a semisimple exact category. To prove well defined-ness of μ'_i, we utilize the following lemma.

Lemma.
Let

$$\operatorname{Typ}_{\mathscr{X}}(\{f_s\})^{\oplus t} \overset{\alpha}{\to} \operatorname{Typ}_{\mathscr{X}}(\{f_s\})^{\oplus m} \overset{\beta}{\to} \operatorname{Typ}_{\mathscr{X}}(\{f_s\})^{\oplus n} \qquad (1)$$

be a sequence of fundamental typical cubes such that $\beta\alpha = 0$. If the induced sequence of $\mathscr{X}/f_s\mathscr{X}$-modules

$$\mathrm{H}_0^s(\operatorname{Typ}_{\mathscr{X}}(\{f_s\})^{\oplus t}) \overset{\mathrm{H}_0^s(\alpha)}{\to} \mathrm{H}_0^s(\operatorname{Typ}_{\mathscr{X}}(\{f_s\})^{\oplus m}) \overset{\mathrm{H}_0^s(\beta)}{\to} \mathrm{H}_0^s(\operatorname{Typ}_{\mathscr{X}}(\{f_s\})^{\oplus n})$$

is exact, then the original sequence (1) is exact.

By replacing μ_i with μ'_i, we can justify the arguments in the previous pages. Hence we complete the proof.