Some problems on hypergeometric integrals associated with hypersphere arrangements, (in memory of Tetsusuke Ohkawa)

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1 Introduction

Let $\alpha_j = (\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jn})\,(1 \leq j \leq m)$ be m pieces of n dimensional real vectors, and $\alpha_{j0} \in \mathbf{R}$. We take quadratic polynomials $f_j(x)$ of $x = (x_1, x_2, \ldots, x_n) \in \mathbf{C}^n$

$$f_j(x) = Q(x) + 2(\alpha_j, x) + \alpha_{j0},$$

where we put $Q(x) = (x, x) = \sum_{\nu=1}^n x_\nu^2,\ (\alpha_j, x) = \sum_{\nu=1}^n \alpha_{j\nu} x_\nu$.

For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{C}^m$, we consider an analytic integral

$$\mathcal{J}(\varphi) = \mathcal{J}_\lambda(\varphi) = \int_{\mathfrak{z}} \Phi(x)\varphi(x)\varpi, \tag{1}$$
$$\varpi = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

associated with the multiplicative meromorphic function

$$\Phi(x) = f_1(x)^{\lambda_1} f_2(x)^{\lambda_2} \cdots f_m(x)^{\lambda_m} \quad (\lambda_j \in \mathbf{C}).$$

Here $\varphi(x)$ is a rational function on $\mathbf{C}^n$ which is holomorphic in the complement $X = \mathbf{C}^n - S$ $(S = \bigcup_{j=1}^m S_j)$, and S_j denotes the $n-1$ dimensional complex hypersphere

$$S_j : f_j(x) = 0.$$

The RHS of (1) will also be denoted by the bilinear form $\langle \varphi, \mathfrak{z} \rangle$. We put λ_∞ to be $\sum_{j=1}^m \lambda_j$. From now on, we assume

$$\lambda_j \neq 0, 1, 2, 3, \ldots \quad (1 \leq j \leq m),$$
$$\lambda_\infty \neq 0, \pm 1, \pm 2, \pm 3, \ldots.$$

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For the formulation of the integral (1), we use the twisted de Rham cohomology on X denoted by $H^*_\nabla(X, \Omega^\cdot(*S))$, namely the cohomology related to the twisted exterior differentiation ∇ :

$$\nabla\psi = d\psi + d\log\Phi \wedge \psi$$

over the complex of rational differential forms on the complex domain $X = \mathbf{C}^n - S$:

$$\Omega^\cdot(*S) = \oplus_{\nu=0}^n \Omega^\nu(*S).$$

In particular $\Omega^0(*S)$ is the set of all rational functions which are holomorphic on X. Let $\mathcal{L}$ denote the local system on X defined by $\Phi(x)$ and $\mathcal{L}^*$ be its dual. We can take $\mathfrak{z}$ as an element (twisted cycle) of the n dimensional homology $H_n(X, \mathcal{L}^*)$ with coefficients in $\mathcal{L}^*$. In fact $H^n(X, \Omega^\cdot(*S))$ and $H_n(X, \mathcal{L}^*)$ are dual to each other due to A.Grothendieck and P.Deligne theorem (see [9],[13] for details).

Since the integral (1) admits the group of isometric transformations on the Euclidean space $\mathbf{R}^n$ or its complexification $\mathbf{C}^n$, differential equations or contiguity relations can be represented by invariants related to $\Phi(x)$ under this group .

Denote by $\Re S_j$ the real $n-1$ dimensional hypersphere being the real part of S_j, by O_j its center $-\alpha_j = (-\alpha_{j1}, -\alpha_{j2}, \ldots, -\alpha_{jn})$. Then the radius of $\Re S_j$ and the distance between O_j and O_k are given by

$$r_j^2 = Q(\alpha_j) - \alpha_{j0},\ \rho_{jk}^2 = Q(\alpha_j - \alpha_k)$$

respectively. These are the basic invariants with respect to the group of isometry.

Definition 1 Let $B = (b_{jk})_{1\le j,k\le m+2}$ be the Cayley-Menger matrix of degree $m+2$ associated with the hypersphere arrangement S , i.e., the symmetric matrix of degree $m+2$ such that

$$b_{jj} = 0\ (1 \le j \le m+2),\ b_{1j} = 1\,(2 \le j \le m+2),$$
$$b_{2j} = r_{j-2}^2\,(3 \le j \le m+2),\ b_{j,k} = \rho_{j-2,k-2}^2\ (3 \le j < k \le m+2).$$

Cayley-Menger determinants with the components r_j^2 and $\rho_{j,k}^2$ are defined as follows ([10],[14],[16]). For the basic property, see [11],[15]:

$$B\begin{pmatrix} 0 & i_1 & \ldots & i_p \\ 0 & j_1 & \ldots & j_p \end{pmatrix} = \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & \rho_{i_1,j_1}^2 & \rho_{i_1,j_2}^2 & \cdots & \rho_{i_1,j_p}^2 \\ 1 & \rho_{i_2,j_1}^2 & \rho_{i_2,j_2} & \cdots & \rho_{i_2,j_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{i_p,j_1}^2 & \rho_{i_p,j_2}^2 & \cdots & \rho_{i_p,j_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} \star & i_1 & \ldots & i_p \\ 0 & j_1, & \ldots & j_p \end{pmatrix} = \begin{vmatrix} 1 & r_{j_1}^2 & r_{j_2}^2 & \ldots & r_{j_p}^2 \\ 1 & \rho_{i_1,j_1}^2 & \rho_{i_1,j_2}^2 & \cdots & \rho_{i_1,j_p}^2 \\ 1 & \rho_{i_2,j_1}^2 & \rho_{i_2,j_2} & \cdots & \rho_{i_2,j_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{i_p,j_1}^2 & \rho_{i_p,j_2}^2 & \cdots & \rho_{i_p,j_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} \star & i_1 & \dots & i_p \\ \star & j_1 & \dots & j_p \end{pmatrix} = \begin{vmatrix} 0 & r_{j_1}^2 & r_{j_2}^2 & \dots & r_{j_p}^2 \\ r_{i_1}^2 & \rho_{i_1,j_1}^2 & \rho_{i_1,j_2}^2 & \dots & \rho_{i_1,j_p}^2 \\ r_{i_2}^2 & \rho_{i_2,i_1}^2 & \rho_{i_2,j_2}^2 & \dots & \rho_{i_2,j_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{i_p}^2 & \rho_{i_p,i_1}^2 & \rho_{i_p,j_2}^2 & \dots & \rho_{i_p,j_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} 0 & \star & i_1 & \dots & i_p \\ 0 & \star & j_1 & \dots & j_p \end{pmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & r_{j_1}^2 & r_{j_2}^2 & \dots & r_{j_p}^2 \\ 1 & r_{i_1}^2 & \rho_{i_1,j_1}^2 & \rho_{i_1,j_2}^2 & \dots & \rho_{i_1,j_p}^2 \\ 1 & r_{i_2}^2 & \rho_{i_2,i_1}^2 & \rho_{i_2,j_2}^2 & \dots & \rho_{i_2,j_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{i_p}^2 & \rho_{i_p,i_1}^2 & \rho_{i_p,j_2}^2 & \dots & \rho_{i_p,j_p}^2 \end{vmatrix}.$$

When $i_\nu = j_\nu$ $(1 \le \nu \le p)$, we simply write the principal minors $B(0i_1 \dots i_p)$, $B(i_1 \dots i_p)$, $B(0 \star i_1 \dots i_p)$ instead of $B\begin{pmatrix} 0 & i_1 & \dots & i_p \\ 0 & i_1 & \dots & i_p \end{pmatrix}$, $B\begin{pmatrix} i_1 & \dots & i_p \\ i_1 & \dots & i_p \end{pmatrix}$, $B\begin{pmatrix} 0 & \star & i_1 & \dots & i_p \\ 0 & \star & i_1 & \dots & i_p \end{pmatrix}$ etc. For example,

$$\begin{aligned} & B(0ij) = 2\rho_{ij}^2,\ B(0 \star j) = 2r_j^2, \\ & \rho_{ik}^2 + \rho_{jk}^2 - \rho_{ij}^2 = B\begin{pmatrix} 0 & i & k \\ 0 & j & k \end{pmatrix}, r_i^2 + r_j^2 - \rho_{ij}^2 = B\begin{pmatrix} 0 & \star & i \\ 0 & \star & j \end{pmatrix}, \\ & \rho_{jk}^2 + r_k^2 - r_j^2 = B\begin{pmatrix} 0 & \star & k \\ 0 & j & k \end{pmatrix}, \\ & B(0jkl) = \rho_{kl}^4 + \rho_{jl}^4 + \rho_{jk}^4 - 2\rho_{jk}^2\rho_{jl}^2 - 2\rho_{jk}^2\rho_{kl}^2 - 2\rho_{jl}^2\rho_{kl}^2. \end{aligned}$$

As is seen below, the arrangement of hyperspheres $\{S_j\}_{1\le j\le m}$ is equivalent to an arrangement of hyperplane sections of the fundamental unit hypersphere $\mathbf{C}S_0^n$ through the stereographic projection.

Let

$$\begin{aligned} & \iota : x_j = \frac{\xi_j}{\xi_{n+1}+1} \quad (1 \le j \le n), \\ & \iota^{-1} : \xi_j = \frac{2x_j}{Q(x)+1}\ (1 \le j \le n),\ \xi_{n+1} = \frac{1-Q(x)}{1+Q(x)}, \\ & (\xi = (\xi_1, \xi_2, \dots, \xi_{n+1}) \in \mathbf{C}S_0^n), \end{aligned}$$

be the stereographic projection onto $\mathbf{C}^n$ from the fundamental unit hypersphere:

$$\mathbf{C}S_0^n : \quad \xi_1^2 + \xi_2^2 + \dots + \xi_{n+1}^2 = 1.$$

We have the conformal isomorphism

$$\iota : \mathbf{C}S_0^n - \{\xi_{n+1} + 1 = 0\} \cong \mathbf{C}^n - \{Q(x) + 1 = 0\}.$$

A quadratic polynomial $f_j(x)$ is linearized to the function $\tilde{f}_j(\xi)$:

$$\begin{aligned}\tilde{f}_j(\xi) &= \frac{(\xi_{n+1}+1)}{2r_j}\, f_j\Big(\frac{\xi_1}{\xi_{n+1}+1}, \frac{\xi_2}{\xi_{n+1}+1}, \dots, \frac{\xi_n}{\xi_{n+1}+1}\Big) \\ &= u_{j0} + \sum_{\nu=1}^{n+1} u_{j\nu}\xi_\nu \ (\,\text{linear function}\,), \\ u_{j0} &= \frac{1+\alpha_{j0}}{2r_j},\ u_{j\nu} = \frac{2\alpha_{j\nu}}{2r_j}\ (1 \le j \le n),\ u_{j\,n+1} = \frac{\alpha_{j0}-1}{2r_j}.\end{aligned}$$

Then $\tilde{S}_j : \{\tilde{f}_j = 0\} \cap \mathbf{C}S_0^n\, (1 \le j \le m)$ define hyperplane sections of $\mathbf{C}S_0^n$.

We now normalize $\tilde{f}_j$ by

$$-u_{j0}^2 + \sum_{\nu=1}^{n+1} u_{j\nu}^2 = 1$$

such that they are invariant under the standard Lorentz transformations. It is convenient to put

$$\tilde{f}_{m+1}(\xi) = \xi_{n+1} + 1.$$

The Lorentz inner product $a_{jk} = (\tilde{f}_j, \tilde{f}_k)$ between $\tilde{f}_j, \tilde{f}_k$ can be defined as

$$\begin{aligned}a_{jk} &= -u_{j0}u_{k0} + \sum_{\nu=1}^{n+1} u_{j\nu}u_{k\nu} \\ &= \frac{r_j^2 + r_k^2 - \rho_{jk}^2}{2r_jr_k} \quad (1 \le j,k \le m+1)\end{aligned}$$

such that $a_{jj} = 1\,(1 \le j \le m), a_{m+1\,m+1} = 0$. We put further

$$a_{j0} = u_{j0} \quad (1 \le \jmath \le m+1),\ a_{00} = -1$$

and then obtain the $(m+2)\times(m+2)$ symmetric configuration matrix $A = (a_{jk})_{0 \le j,k \le m+1}$.

Denote the minor determinant of $i_1, \dots, i_p$th row, $j_1, \dots, j_p$th column by

$$A\begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_p \end{pmatrix}.$$

As is well-known ([20]), all the singularities appearing in the integral (1) are contained in the zero set of at least one of the principal minor determinants:

$$A(i_1 \dots i_p) = A\begin{pmatrix} i_1 & \dots & i_p \\ i_1 & \dots & i_p \end{pmatrix} = 0.$$

Lemma 2 *The following identities hold*:

$$A(i_1 \dots i_p) = (-1)^{p-1} \frac{B(0 \star i_1 \cdots i_p)}{\prod_{\nu=1}^{p} B(0 \star i_\nu)} \quad (p \geq 1).$$

We can define two kinds of bases of $H^n_\nabla(X, \Omega^\cdot(*S))$.

Assume $m = n+1$. We call "admissible" an arbitrary subset of indices J of size $p = |J|$ in the interval $[1, n+1]$. Denote by $\mathcal{B}$ the set of all admissible sets of indices.

We can define two kinds of bases of $H^n_\nabla(X, \Omega^\cdot(*S))$ wich is of dimension $2^{n+1} - 1$ as follows.

The one (of first kind) is given as follows : For an admissible set J one can take

$$F_J = \frac{\varpi}{\prod_{j \in J} f_j} \quad (J \in \mathcal{B}, 1 \leq |J| \leq n+1).$$

The other (of second kind) is given as follows:

$$W_0(J)\varpi = -\sum_{\nu=1}^{p} B\begin{pmatrix} 0 & \star & \partial_\nu J \\ 0 & j_\nu & \partial_\nu J \end{pmatrix} F_{\partial_\nu J} + B(0 \star J)\, F_J \quad (1 \leq p \leq n+1)$$
$$(J = \{j_1, \dots, j_p\} \in \mathcal{B}, 1 \leq p \leq n+1).$$

The following Lemma immediately follows from the definition:

Lemma 3 *For an admissible* J*,* F_J *can be described as a linear combination of* $W_0(K)\varpi$ $(K \subset J)$*,* i.e.*, there exists a triangular matrix* $(\beta_{K,J})$ *such that*

$$B(0 \star J)\, F_J = \sum_{K \subset J} \beta_{K,J}\, W_0(K)\varpi, \tag{2}$$

where $\beta_{K,J}$ *are uniquely determined by the relations*

$$\beta_{J,J} = 1, \quad \beta_{J,J\cup\{l\}} = \frac{B\begin{pmatrix} 0 & \star & J \\ 0 & l & J \end{pmatrix}}{B(0 \star J)} \quad (l \notin J),$$
$$\beta_{K,J} = \sum_{l \in J-K} \beta_{K,K\cup\{l\}} \beta_{K\cup\{l\},J}.$$

An arbitrary element $\varphi\varpi \in \Omega^n(*S)$ can be uniquely represented in $H^n_\nabla(X, \Omega^\cdot(*S))$ by a linear combination of either of the above bases.

First of all we give an explicit expression of ϖ in terms of the basis of second kind.

$\mathcal{J}_\lambda(\varphi)$ can be regarded as an analytic function of the parameters $\alpha_{j\nu}$. Denote by d_B the total differentiation with respect to the parameters, then

$$d_B \mathcal{J}_\lambda(\varphi) = \int_{\mathfrak{z}} \Phi(x) \nabla_B(\varphi\varpi), \tag{3}$$

where ∇_B denotes the covariant differentiation (Gauss-Manin connection) operating on $H^n_\nabla(X, \Omega^{\cdot}(*S))$:

$$\nabla_B(\varphi\varpi) = (d_B\varphi + d_B \log \Phi\, \varphi)\varpi.$$

More explicitly

$$\nabla_B \varpi = \sum_{j=1}^{n+1} 2\lambda_j \left(\sum_{\nu=1}^{n+1-j} d\alpha_{j,\nu} x_j + \frac{1}{2} d\alpha_{j0} \right) F_j.$$

The RHS can be described by a linear combination of the basis (of first kind or of second kind) in $H^n_\nabla(X, \Omega^{\cdot}(*S))$.

The second aim of this talk is to give an explicit expression of $\nabla_B\varpi$, using the basis $\{W_0(J)\varpi\}$ and invariant special 1-forms θ_J defined as follows.

Definition 4 We introduce the following invariant differential 1-forms θ_J $(J \in \mathcal{B})$:

$$\theta_j = -\frac{1}{2} d \log r_j^2, \tag{4}$$

$$\theta_{jk} = \frac{1}{2} d \log \rho_{jk}^2, \tag{5}$$

$$\begin{aligned}
\theta_{jkl} = & -\frac{B\begin{pmatrix} \star & j & k & l \\ 0 & j & k & l \end{pmatrix}}{B(0kl)B(0jl)B(0jk)} d \log B(0jkl) \\
& - \frac{r_j^2}{B(0jk)B(0jl)} dB\begin{pmatrix} 0 & k & j \\ 0 & l & j \end{pmatrix} - \frac{r_k^2}{B(0jk)B(0kl)} dB\begin{pmatrix} 0 & j & k \\ 0 & l & k \end{pmatrix} \\
& - \frac{r_l^2}{B(0jl)B(0kl)} dB\begin{pmatrix} 0 & j & l \\ 0 & k & l \end{pmatrix} \\
= & -\frac{1}{2} \frac{B\begin{pmatrix} 0 & \star & k & l \\ 0 & j & k & l \end{pmatrix}}{B(0jkl)} d \log \rho_{kl}^2 - \frac{1}{2} \frac{B\begin{pmatrix} 0 & \star & j & l \\ 0 & k & j & l \end{pmatrix}}{B(0jkl)} d \log \rho_{jl}^2 \\
& - \frac{1}{2} \frac{B\begin{pmatrix} 0 & \star & j & k \\ 0 & l & j & k \end{pmatrix}}{B(0jkl)} d \log \rho_{jk}^2,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\theta_{j_1j_2j_3j_4} = & \frac{1}{2} \sum_{i<j,\, k<l} d \log \rho_{ij}^2 \left\{ \frac{B\begin{pmatrix} 0 & \star & i & j \\ 0 & k & i & j \end{pmatrix} B\begin{pmatrix} 0 & \star & i & j & k \\ 0 & l & i & j & k \end{pmatrix}}{B(0ijk)B(0j_1j_2j_3j_4)} \right. \\
& \left. + \frac{B\begin{pmatrix} 0 & \star & i & j \\ 0 & l & i & j \end{pmatrix} B\begin{pmatrix} 0 & \star & i & j & l \\ 0 & k & i & j & l \end{pmatrix}}{B(0ijl)B(0j_1j_2j_3j_4)} \right\},
\end{aligned} \tag{7}$$

where $\{i,j,k,l\}$ move over the set of all permutations of $J = \{j_1 j_2 j_3 j_4\}$ such that $i < j$, $k < l$.

More generally for $J = \{j_1, \ldots, j_p\} \in \mathcal{B}\,(2 \le p \le n+1)$,

$$\theta_J := \frac{(-1)^p}{2} \sum_{\{L\}=\{J\}; l_1<l_2} d\log \rho^2_{l_1 l_2}$$
$$\frac{B\begin{pmatrix} 0 & \star & l_1 & l_2 \\ 0 & l_3 & l_1 & l_2 \end{pmatrix} B\begin{pmatrix} 0 & \star & l_1 & l_2 & l_3 \\ 0 & l_3 & l_1 & l_2 & l_3 \end{pmatrix} \cdots B\begin{pmatrix} 0 & \star & l_1 & l_2 & l_3 \cdots l_{p-1} \\ 0 & l_3 & l_1 & l_2 & l_3 \cdots l_{p-1} \end{pmatrix}}{\prod_{\nu=3}^{p} B(0 l_1 l_2 l_3 \cdots l_p)} \tag{8}$$

where $L = \{l_1, l_2, l_3, \ldots, l_p\}$ move over the set of sequences such that L coincides with J as a set in $[1, n+1]$ and satisfies $l_1 < l_2 < l_3 < \cdots < l_p$.

We can state the the following two conjectures:

Conjecture I.
ϖ is represented cohomologically in terms of the basis of second kind

$$(2\lambda_\infty + n)\varpi \sim \sum_{p=1}^{n+1} \sum_{I \subset \mathcal{B}, |I|=p} (-1)^p \frac{\prod_{j \in I} \lambda_j}{\prod_{\nu=1}^{p-1}(\lambda_\infty + n - \nu)} W_0(I)\varpi$$

in $H^n_\nabla(X, \Omega^{\cdot}(*S))$.

Conjecture II.

$$\nabla_B \varpi \sim \sum_{p=1}^{n+1} V_p \varpi,$$
$$V_p = \sum_{J \in \mathcal{B}, |J|=p} \frac{\prod_{j \in J} \lambda_j}{\prod_{\nu=1}^{p-1}(\lambda_\infty + n - \nu)} \theta_J W_0(J).$$

Remark The forms θ_J all seem "weakly logarithmic (logarithmic in the sense of K.Saito)". We shall show that it is true for $|J| \le 4$.

Remark To prove the above Conjectures it is necessary to give explicit expressions for the forms $x_j F_j$ in $H^n_\nabla(X, \Omega^{\cdot}(*S))$ in terms of the basis $\{W_0(J)\varpi\}$. The latter follows from contiguity relation in $H^n_\nabla(Z, \Omega^{\cdot}(*S))$. In fact we take the standard basis ε_j of $\mathbf{Z}^{n+1}$ in the space of exponents λ such that $\lambda = \sum_{j=1}^{n+1} \lambda_j \varepsilon_j$. The contiguity relation with respect to the shifts $\lambda \to \lambda + \epsilon_j$ gives the equalities in the above Conjectures.

As is well-known, the classical Schläfli formula can be stated as follows.

Let $\tilde{D}$ be a geodesic n-simplex in the fundamental unit hypersphere $\Re S_0$ defined by

$$\tilde{D} : \tilde{f}_1 \leq 0, \ldots, \tilde{f}_{n+1} \leq 0.$$

where we assume that all the hyperplanes : $\tilde{f}_j = 0$ contain the origin.

Denote by $\tilde{V}$ the volume of $\tilde{D}$ with respect to the standard volume form

$$\tilde{V} = \int_{\tilde{D}} \tilde{\varpi},$$
$$\tilde{\varpi} = \sum_{\nu=1}^{n+1} (-1)^{\nu-1} \xi_\nu d\xi_1 \wedge \cdots \wedge d\xi_{\nu-1} \wedge d\xi_{\nu+1} \wedge \cdots \wedge d\xi_{n+1}.$$

Then $\tilde{V}$ can be regarded as a (many valued) analytic function of the symmetric matrix A. The variational differential formula for $\tilde{V}$ can be represented as

$$d_A \tilde{V} = \sum_{j<k} V_{jk}^* \, d\langle j, k\rangle$$

where $\langle j, k\rangle$ denotes the dihedral angle between the two hyperplanes sections $\Re\tilde{S}_j$, $\Re\tilde{S}_k$ and V_{jk}^* denotes the lower dimensional volume of the intersection $\Re\tilde{S}_j \cap \Re\tilde{S}_k$ with respect to the induced volume form $\tilde{\varpi}_{jk}$:

$$d\langle j, k\rangle = \frac{1}{2i} d\log\left(\frac{-a_{jk} + i\sqrt{1 - a_{jk}^2}}{-a_{jk} - i\sqrt{1 - a_{jk}^2}}\right),$$
$$\tilde{V}_{jk}^* = \int_{\Re\tilde{S}_j \cap \Re\tilde{S}_k} [\frac{\tilde{\varpi}}{d\tilde{f}_j \wedge d\tilde{f}_k}]_{S_j \cap S_k}.$$

We shall give an analogue of L.Schläfli formula for a n-simplex with spherical faces as a consequence of Conjecture II.

In the sequel we prove the above conjectural formulae in case $n = 3$.

2 Basic Facts

We denote by $\varepsilon_j \, (1 \leq j \leq 4)$ the standard basis of $\mathbf{C}^4$, the space of exponents λ : $\lambda = \sum_{j=1}^4 \lambda_j \varepsilon_j$. For $J \subset \{1, 2, 3, 4\}$, denote by $|J|$ the size of J. Denote by $\mathfrak{S}_4$ the symmetric group of degree 4 generated by σ_{ij} the transposition between the indices $i, j \, (i \neq j)$.

A subset of indices $J \subset \{1,2,3,4\}$ such that $|J| \geq 1$ will be called "admissible". Denote by $\mathcal{B}$ the family of all admissible sets of indices.

We assume that none of the principal Cayley-Menger determinants vanish. Namely

$$(\mathcal{H}1) \quad B(0J) \neq 0, \ B(0\star J) \neq 0$$

for all $J \subset \{1,2,3,4\}$ and $|J| \geq 1$.

Through ι, the arrangement of (complex) spheres in $\mathbf{C}^3$ is identified by an arrangement of hyperplane sections in $\mathbf{C}S_0^3$. Since the Euler numbers of X and of the complement $\mathbf{C}S_0^3 - \bigcup_{j=1}^4 \tilde{S}_j$ (denoted by Eu) are related as

$$Eu(X) = Eu\{\mathbf{C}S_0^3 - \bigcup_{j=1}^{4} \tilde{S}_j\} - 1 = -15,$$

we have the equality(see [9] Corollary 2.2)

$$\dim H^3_\nabla(X, \Omega^\cdot(*S)) = 15.$$

The purpose of this article is stated as follows:

We can define two kinds of bases of $H^3_\nabla(X, \Omega^\cdot(*S))$ (see Proposition 9). One is given as follows :

$$F_j = \frac{\varpi}{f_j} \ (1 \leq j \leq 4), \ F_{jk} = \frac{\varpi}{f_j f_k} \ (1 \leq j < k \leq 4),$$
$$F_{jkl} = \frac{\varpi}{f_j f_k f_l} \ (1 \leq j < k < l \leq 4), \ F_{1234} = \frac{\varpi}{f_1 f_2 f_3 f_4}.$$

The other is given as follows:

$$W_0(j)\varpi \ (1 \leq j \leq 4), \ W_0(jk)\varpi \ (1 \leq j < k \leq 4),$$
$$W_0(jkl)\varpi \ (1 \leq j < k < l \leq 4), \ W_0(1234)\varpi,$$

where $W_0(J)\varpi$ are defined as

$$W_0(j)\varpi = B(0\star j)\,\frac{\varpi}{f_j},$$

$$W_0(jk)\varpi = -B\begin{pmatrix} 0 & \star & k \\ 0 & j & k \end{pmatrix}\frac{\varpi}{f_k} - B\begin{pmatrix} 0 & \star & j \\ 0 & k & j \end{pmatrix}\frac{\varpi}{f_j} + B(0\star jk)\,\frac{\varpi}{f_j f_k},$$

$$W_0(jkl)\varpi = -B\begin{pmatrix} 0 & \star & k & l \\ 0 & j & k & l \end{pmatrix}\frac{\varpi}{f_k f_l} - B\begin{pmatrix} 0 & \star & j & l \\ 0 & k & j & l \end{pmatrix}\frac{\varpi}{f_j f_l}$$
$$-B\begin{pmatrix} 0 & \star & j & k \\ 0 & l & j & k \end{pmatrix}\frac{\varpi}{f_j f_k} + B(0\star jkl)\,\frac{\varpi}{f_j f_k f_l},$$

$$W_0(1234)\varpi = -B\begin{pmatrix} 0 & \star & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}\frac{\varpi}{f_2 f_3 f_4} - B\begin{pmatrix} 0 & \star & 1 & 3 & 4 \\ 0 & 2 & 1 & 3 & 4 \end{pmatrix}\frac{\varpi}{f_1 f_3 f_4}$$
$$-B\begin{pmatrix} 0 & \star & 1 & 2 & 4 \\ 0 & 3 & 1 & 2 & 4 \end{pmatrix}\frac{\varpi}{f_1 f_2 f_4} - B\begin{pmatrix} 0 & \star & 1 & 2 & 3 \\ 0 & 4 & 1 & 2 & 3 \end{pmatrix}\frac{\varpi}{f_1 f_2 f_3}$$
$$+B(0\star 1234)\,\frac{\varpi}{f_1 f_2 f_3 f_4}.$$

F_J and $W_0(J)\varpi$ both can be extended to an arbitrary admissible J such that they are symmetric with respect to any permutation which preserves the set J. We shall call the former basis F_J "of the first kind", and the latter basis $W_0(J)\varpi$ "of the second kind".

An arbitrary element $\varphi\varpi \in \Omega^3(*S)$ can be uniquely represented in $H^3_\nabla(X, \Omega^\cdot(*S))$ by a linear combination of either of the above bases.

The following three lemmas are elementary and easily checked.

Lemma 5

$$A(j\,k) = \frac{\Delta^2(r_j, r_k, \rho_{jk})}{4r_j^2 r_k^2},$$

where

$$\begin{aligned}\Delta^2(r_j, r_k, \rho_{jk}) &= -r_j^4 - r_k^4 - \rho_{jk}^4 + 2r_j^2 r_k^2 + 2r_j^2\rho_{jk}^2 + 2r_k^2\rho_{jk}^2 \\ &= (r_j + r_k + \rho_{jk})(-r_j + r_k + \rho_{jk})(r_j - r_k + \rho_{jk})(r_j + r_k - \rho_{jk}) \\ &= -B(0 \star jk).\end{aligned}$$

$A(j\,k) = B(0 \star jk) = 0$ *holds if and only if* S_j *and* S_k *have contact with each other.*

Lemma 6

$$\begin{aligned}&4r_1^2 r_2^2 r_3^2\, A(123) = \frac{1}{2}\, B(0 \star 123) = -\{r_1^4\rho_{23}^2 + \cdots\} \\ &+\{r_2^2 r_3^2(\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2) + \cdots\} + \{r_1^2\rho_{23}^2(\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2) + \cdots\} - \rho_{23}^2\rho_{13}^2\rho_{12}^2.\end{aligned}$$

$A(1\,2\,3) = 0$ *if and only if* S_1, S_2, S_3 *have a common tangent line.*

$$\begin{aligned}a_{j\ m+1} &= -\frac{1}{r_j}, \quad A(j\ m+1) = -\frac{1}{r_j^2} \quad (1 \le j \le m), \\ A(j\ k\ m+1) &= -\frac{\rho_{jk}^2}{r_j^2 r_k^2} \quad (1 \le j < k \le m),\end{aligned}$$

so that $A(j\ m+1) = 0$ *and* $A(j\ k\ m+1) = 0$ *are equivalent to* $r_j = \infty$ *and* $\rho_{j,k} = 0$ *respectively.*

Lemma 7 S_1, S_2, S_3, S_4 *have a common point if and only if*

$$A(1234) = B(0 \star 1234) = 0.$$

O_k *the centers of* S_k $(1 \le k \le 4)$ *lie in the same plane if and only if*

$$B(01234) = 0.$$

For the proof see [6].

Now we take the following hypothesis which is crucial in the sequel:

$$(\mathcal{H}2) \qquad \begin{array}{l} A(j\,k) > 0 (1 \leq j,k \leq 4),\ A(j\,5) < 0 (1 \leq j \leq 4),\ A(i\,j\,k) > 0, \\ A(j\,k\,5) < 0,\ A(1234) > 0,\ A(i\,j\,k\,5) > 0,\ A(01234) < 0 \end{array}$$

for different $i, j, k\, (1 \leq i, j, k \leq 4)$.

Namely each triple S_i, S_j, S_k has common two different points, $K_1 \cap K_2 \cap K_3 \cap K_4 \neq \emptyset$, where K_j denotes a closed disc bounded by $\Re S_j$.

Remark that then $B(0jk) > 0, B(0jkl) < 0, B(01234) > 0$ and $B(0 \star j) > 0$, $B(0 \star jk) < 0$, $B(0 \star jkl) > 0$.

We see that S_j $(1 \leq j \leq 4)$ are in general position.

We may assume that f_j are normalized by an isometry in the following form:

$$\begin{aligned} f_1(x) &= Q(x) + 2\alpha_{11}x_1 + 2\alpha_{12}x_2 + 2\alpha_{13}x_3 + \alpha_{10}, \\ f_2(x) &= Q(x) + 2\alpha_{21}x_1 + 2\alpha_{22}x_2 + \alpha_{20}, \\ f_3(x) &= Q(x) + 2\alpha_{31}x_1 + \alpha_{30}. \\ f_4(x) &= Q(x) + \alpha_{40}. \end{aligned} \tag{9}$$

for $\alpha_{j\nu} \in \mathbf{R}$. We may take $\alpha_{31} > 0, \alpha_{22} > 0, \alpha_{13} > 0$.

Then

$$\begin{aligned} & r_1^2 = \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 - \alpha_{10}, \\ & r_2^2 = \alpha_{21}^2 + \alpha_{22}^2 - \alpha_{20}, \\ & r_3^2 = \alpha_{31}^2 - \alpha_{30},\ r_4^2 = -\alpha_{40}, \\ & \rho_{34}^2 = \alpha_{31}^2,\ \rho_{24}^2 = \alpha_{21}^2 + \alpha_{22}^2,\ \rho_{14}^2 = \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 \\ & \rho_{13}^2 = (\alpha_{11} - \alpha_{31})^2 + \alpha_{12}^2 + \alpha_{13}^2, \\ & \rho_{23}^2 = (\alpha_{21} - \alpha_{31})^2 + \alpha_{22}^2, \\ & \rho_{12}^2 = (\alpha_{11} - \alpha_{21})^2 + (\alpha_{12} - \alpha_{22})^2 + \alpha_{13}^2. \end{aligned} \tag{10}$$

Lemma 8 *Under the condition* (9) *we have*

$$
\begin{aligned}
\alpha_{31}^2 &= \frac{1}{2}B(034),\ \alpha_{31} = \rho_{34}, \\
\alpha_{31}^2\alpha_{22}^2 &= -\frac{1}{4}B(0234),\ \alpha_{22} = \sqrt{\frac{-B(0234)}{2B(034)}}, \\
\alpha_{31}^2\alpha_{22}^2\alpha_{13}^2 &= \frac{1}{8}B(01234),\ \alpha_{13} = \sqrt{\frac{B(01234)}{-2B(0234)}}, \\
\alpha_{21}\alpha_{31} &= \frac{1}{2}B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix} = \frac{1}{2}(\rho_{24}^2 + \rho_{34}^2 - \rho_{23}^2), \\
\alpha_{11}\alpha_{31} &= \frac{1}{2}B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix} = \frac{1}{2}(\rho_{14}^2 + \rho_{34}^2 - \rho_{13}^2), \\
\alpha_{12}\alpha_{22} &= -\frac{B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{4\rho_{34}^2},
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{10} - \alpha_{40} = \rho_{14}^2 + r_4^2 - r_1^2 = B\begin{pmatrix} 0 & 1 & 4 \\ 0 & \star & 4 \end{pmatrix},\ \alpha_{10} + \alpha_{40} = -B\begin{pmatrix} 0 & \star & 1 \\ 0 & \star & 4 \end{pmatrix}, \\
\alpha_{20} - \alpha_{40} = \rho_{24}^2 + r_4^2 - r_2^2 = B\begin{pmatrix} 0 & 2 & 4 \\ 0 & \star & 4 \end{pmatrix},\ \alpha_{20} + \alpha_{40} = -B\begin{pmatrix} 0 & \star & 2 \\ 0 & \star & 4 \end{pmatrix}, \\
\alpha_{30} - \alpha_{40} = \rho_{34}^2 + r_4^2 - r_3^2 = B\begin{pmatrix} 0 & 3 & 4 \\ 0 & \star & 4 \end{pmatrix},\ \alpha_{30} + \alpha_{40} = -B\begin{pmatrix} 0 & \star & 3 \\ 0 & \star & 4 \end{pmatrix}.
\end{aligned}
$$

From (9) and (10), we have

$$
\begin{aligned}
A(034) &= -\frac{(1+r_4^2)^2\, B(034)}{8r_3^2r_4^2}, \\
A(0234) &= \frac{(1+r_4^2)^2\, B(0234)}{16r_2^2r_3^2r_4^2}, \\
A(01234) &= -\frac{(1+r_4^2)^2\, B(01234)}{32r_1^2r_2^2r_3^2r_4^2}.
\end{aligned}
$$

Hence $A(034) = 0, A(0234) = 0$ or $A(01234) = 0$ holds if and only if the length of the segment ΔO_3O_4, the area of the triangle $\Delta O_2O_3O_4$, or the volume of the tetrahedron $\Delta O_1O_2O_3O_4$ vanishes respectively.

The following two Propositions follow from the preceding results in [6],[7],[8],[9]. See also [16].

Proposition 9 $\dim H_\nabla^3(X, \Omega^{\cdot}(*S))$ *is equal to* 15*, the absolute value of the* Euler *number of X. We can choose the following basis of* $H^3_\nabla(X, \Omega^{\cdot}(*S))$:

$$H^3_\nabla(X, \Omega^{\cdot}(*S)) \cong \langle \frac{\varpi}{f_j} \ (1 \le j \le 4), \ \frac{\varpi}{f_j f_k} \ (1 \le j < k \le 4),$$
$$\frac{\varpi}{f_i f_j f_k} \ (1 \le i, j, k \le 4), \ \frac{\varpi}{f_1 f_2 f_3 f_4} \rangle.$$

As for the dual basis, the following Proposition holds:

Proposition 10 *As a basis of* $H_3(X, \mathcal{L}^*)$, *we can choose the homology classes of twisted 3-cycles identified with all relatively compact connected components of* $\Re X - S$. *Namely*

(i) $\mathfrak{z}_{1234} = K_1 \cap K_2 \cap K_3 \cap K_4$,
(ii) $\mathfrak{z}_{234} = K_2 \cap K_3 \cap K_4 - K_1$, $\mathfrak{z}_{134} = K_1 \cap K_3 \cap K_4 - K_2$,
$\mathfrak{z}_{124} = K_1 \cap K_2 \cap K_4 - K_3$, $\mathfrak{z}_{123} = K_1 \cap K_2 \cap K_3 - K_4$,
(iii) $\mathfrak{z}_{ij} = K_i \cap K_j - K_k \cup K_l \ (1 \le i < j \le 4)$,
(iv) $\mathfrak{z}_i = K_i - K_j \cup K_k \cup K_l \ (1 \le i \le 4))$,

$\{i, j, k, l\}$ *denotes a permutation of* $\{1, 2, 3, 4\}$, *where none of them are not empty.*

For the proof and its background see [6],[7].

Remark 1 In case where each of $S_j \cap S_k \cap S_l$ has two different points, Proposition 10 may be generalized by taking all relatively compact components of $\Re X - S$ even if one of the regions stated in (i) - (iv) is empty. For example if $K_1 \cap K_2 \cap K_3 - K_4$ is empty, $K_1 \cap K_2 - K_3 \cup K_4$ has two connected components etc.

Remark 2 Put $\lambda_\infty = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. In Proposition 9, $\mathfrak{z}_\infty = \mathbf{R}^3 - K_1 \cup K_2 \cup K_3 \cup K_4$, the complement of $K_1 \cup K_2 \cup K_3 \cup K_4$ can also be regarded as a twisted cycle. This cycle can be represented by a linear combination of $\mathfrak{z}_J \ (J \in \mathcal{B})$. We give it without proof:

$$\mathfrak{z}_\infty \sim -\frac{1}{\cos \pi\lambda_\infty} \mathfrak{z}_{1234} - \sum_{j=1}^{4} \frac{\cos \pi\lambda_j}{\cos \pi\lambda_\infty} \mathfrak{z}_{\hat{j}} - \sum_{j<k} \frac{\cos \pi(\lambda_j + \lambda_k)}{\cos \pi\lambda_\infty} \mathfrak{z}_{\widehat{jk}}$$
$$- \sum_{j<k<l} \frac{\cos \pi(\lambda_j + \lambda_k + \lambda_l)}{\cos \pi\lambda_\infty} \mathfrak{z}_{\widehat{jkl}} \tag{11}$$

Here $\widehat{J}$ denotes the complement $\{1, 2, 3, 4\} - J$. The above formula can be proved by applying Cauchy Integral Theorem on the upper half planes of a family of complex lines going through a point in $\bigcap_{j=1}^4 K_j$.

[**Notation**]

We define the logarithmic forms

$$\begin{aligned}
&e_j = \frac{df_j}{f_j}, e_{jk} = \frac{df_j}{f_j} \wedge \frac{df_k}{f_k}, e_{jkl} = \frac{df_j}{f_j} \wedge \frac{df_k}{f_k} \wedge \frac{df_l}{f_l}, \\
&e_{ijkl} = \frac{df_i}{f_i} \wedge \frac{df_j}{f_j} \wedge \frac{df_k}{f_k} \wedge \frac{df_l}{f_l},
\end{aligned}$$

and the function W such that

$$e_{234} - e_{134} + e_{124} - e_{123} = 8W\varpi \quad (\sqrt{B(01234)}\, W \in \Omega^0(*S)).$$

Note that $\sqrt{B(01234)}$ and W should be alternating with respect to $\mathfrak{S}_4$.

The following identity can be obtained by an elementary calculation:

Lemma 11

$$\begin{aligned}
W &= \frac{c_1}{f_2 f_3 f_4} + \frac{c_2}{f_1 f_3 f_4} + \frac{c_3}{f_1 f_2 f_4} + \frac{c_4}{f_1 f_2 f_3} + \frac{c_0}{f_1 f_2 f_3 f_4} \\
&= \frac{W_0(1234)}{\sqrt{8\, B(01234)}}, \qquad (12)
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= -\frac{B\begin{pmatrix} 0 & \star & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}}{\sqrt{8\, B(01234)}}, c_2 = \frac{B\begin{pmatrix} 0 & \star & 1 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}}{\sqrt{8\, B(01234)}}, \\
c_3 &= -\frac{B\begin{pmatrix} 0 & \star & 1 & 2 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}}{\sqrt{8\, B(01234)}}, c_4 = \frac{B\begin{pmatrix} 0 & \star & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}}{\sqrt{8\, B(01234)}}, \\
c_0 &= \frac{B(0 \star 1234)}{\sqrt{8\, B(01234)}}.
\end{aligned}$$

The RHS *of* (12) *is uniquely expressed.*

Proposition 12 *In* $H^3_\nabla(X, \Omega^\cdot(*S))$, *we have the identities*

$$e_{234} \sim \frac{\lambda_1}{\lambda_\infty}(e_{234} - e_{134} + e_{124} - e_{123}) = \frac{8\lambda_1}{\lambda_\infty} W\varpi, \qquad (13)$$

$$e_{134} \sim -\frac{\lambda_2}{\lambda_\infty}(e_{234} - e_{134} + e_{124} - e_{123}) = -\frac{8\lambda_2}{\lambda_\infty} W\varpi, \qquad (14)$$

$$e_{124} \sim \frac{\lambda_3}{\lambda_\infty}(e_{234} - e_{134} + e_{124} - e_{123}) = \frac{8\lambda_3}{\lambda_\infty} W\varpi, \qquad (15)$$

$$e_{123} \sim \frac{\lambda_1}{\lambda_\infty}(e_{234} - e_{134} + e_{124} - e_{123}) = -\frac{8\lambda_4}{\lambda_\infty} W\varpi. \qquad (16)$$

Proof. Indeed by Stokes formulae,

$$\nabla(e_{34}) = \lambda_1 e_{134} + \lambda_2 e_{234} \sim 0,$$
$$\nabla(e_{24}) = \lambda_1 e_{124} - \lambda_3 e_{234} \sim 0,$$
$$\nabla(e_{23}) = \lambda_1 e_{123} + \lambda_4 e_{234} \sim 0.$$

From this and Lemma 11, Proposition 12 follows. □

3 Hierarchy of Contiguity Relations

In this section we represent the contiguity relations involved in the basis F_J by the shifts $\lambda \to \lambda + \varepsilon_j$.

Definition 13 For $\varphi\,\varpi \in \Omega^3(*S)$, the multiplication by $f_j^{\pm 1}$ is denoted by $T_{f_j^{\pm 1}}\varphi\varpi$, the multiplication by $(f_j f_k)^{\pm 1}$ by $T^{\pm 1}_{f_j f_k}\varphi\varpi$ $(j \neq k)$. These correspond to the shift operators shifting the exponent λ to $\lambda \pm \varepsilon_j$, and the exponent λ to $\lambda \pm (\varepsilon_j + \varepsilon_k)$ respectively:

$$T_{\pm\varepsilon_j}\varphi\varpi = T^{\pm 1}_{f_j}\varphi\varpi = f_j^{\pm 1}\,\varphi\varpi;\ T_{\pm(\varepsilon_j+\varepsilon_k)}\varphi\varpi = T^{\pm 1}_{f_j f_k}\varphi\varpi = (f_j f_k)^{\pm 1}\varphi\varpi.$$

These multiplications operate on $H^3_\nabla(X, \Omega^{\cdot}(*S))$:

$$H^3_\nabla(X, \Omega^{\cdot}(*S)) \xrightarrow{T^{\pm 1}_{f_j}} H^3_\nabla(X, \Omega^{\cdot}(*S)).$$

Generally $T_{-\varepsilon_J} = \prod_{j\in J} T_{-\varepsilon_j}$ are well-defined for $\varepsilon_J = \sum_{j\in J}\varepsilon_j$ $(J \subset \{1, 2, 3, 4\})$.

We are interested in representing the shift operators in terms of the basis stated in Proposition 9.

As the first step, the following is obvious and there is nothing to prove.

$$T_{f_1}\frac{\varpi}{f_1 f_2 f_3 f_4} = \frac{\varpi}{f_2 f_3 f_4}$$

The similar formula holds for the other T_{f_j} $(2 \leq j \leq 4)$.

As the second step,

Lemma 14 *The following identity holds:*

$$T_{f_1}\left(\frac{\varpi}{f_2 f_3 f_4}\right) = \frac{B\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(0234)}\frac{\varpi}{f_2 f_3} - \frac{B\begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(0234)}\frac{\varpi}{f_2 f_4}$$
$$+\frac{B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(0234)}\frac{\varpi}{f_3 f_4} - \frac{B\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & \star & 2 & 3 & 4 \end{pmatrix}}{B(0234)}\frac{\varpi}{f_2 f_3 f_4}$$
$$+\frac{1}{\sqrt{8}}\frac{\sqrt{B(01234)}}{B(0234)}e_{234},$$

where e_{234} *can be expressed by* (13)*, so that the last term in the* RHS *is cohomologous to*

$$\frac{\lambda_1}{\lambda_\infty}\frac{W_0(1234)}{B(0234)}\varpi.$$

Hence $T_{f_1}(\frac{\varpi}{f_2f_3f_4})$ *can be represented cohomologously as a linear combination of* F_{ij}, F_{ijk} *and* F_{1234}.

The similar formula can be obtained by symmetry for $T_{f_2}, T_{f_3}, T_{f_4}$ *too.*

From Lemma 11, Lemma 14 and Proposition 12,

Corollary 15

$$\begin{aligned} &T_{f_1}(e_{234} - e_{134} + e_{124} - e_{123}) = 8\,T_{f_1}(W\varpi) \\ &= h_{23}\frac{\varpi}{f_2f_3} + h_{24}\frac{\varpi}{f_2f_4} + h_{34}\frac{\varpi}{f_3f_4} + h_{234}\frac{\varpi}{f_2f_3f_4} + h'_{234}e_{234} \qquad (17)\\ &\sim \frac{\sqrt{8B(01234)}}{B(0234)}W_0(234)\varpi - \frac{\lambda_1}{\lambda_\infty}\frac{\sqrt{8}B\begin{pmatrix} 0 & \star & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}}{B(0234)\sqrt{B(01234)}}W_0(1234)\varpi, \\ &. \qquad (18) \end{aligned}$$

where

$$\begin{aligned} h_{23} &= -\frac{\sqrt{8\,B(01234)}\,B\begin{pmatrix} 0 & \star & 2 & 3 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(0234)}, \\ h_{234} &= \frac{\sqrt{8\,B(01234)}\,B(0\star 234)}{B(0234)}, \\ h'_{234} &= -\frac{B\begin{pmatrix} 0 & \star & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}}{B(0234)}. \end{aligned}$$

In the RHS *of* (17) *the signs of* $h_{23}, h_{24}, h_{34}, h_{234}, h'_{234}$ *shoud be determined such that they are alternating with respect to the permutations of the indices* $\{1, 2, 3, 4\}$.

The formulae of $T_{f_i}(W\varpi)$ $(2 \le i \le 4)$ *can be likewise obtained.*

As the third step, first remark the following identity:

$$\frac{2\alpha_{31}x_1}{f_3f_4}\varpi = \frac{f_3 - f_4}{f_3f_4}\varpi - \frac{\alpha_{30} - \alpha_{40}}{f_3f_4}\varpi, \qquad (19)$$

$$4\alpha_{31}x_2\frac{\varpi}{f_3f_4} = dx_3 \wedge e_{34}. \qquad (20)$$

The latter can be rewritten as follows:

Lemma 16

$$dx_3 \wedge e_{34} = \frac{f_1}{2\alpha_{13}} e_{134} - \frac{\alpha_{12} f_2}{2\alpha_{22}\alpha_{13}} e_{234}.$$

Proof. Indeed from the identity

$$\begin{aligned}&2\alpha_{31}\alpha_{22}\alpha_{13}x_3 = \alpha_{31}\alpha_{22}(f_1 - f_4) - \alpha_{31}\alpha_{12}(f_2 - f_4) - (\alpha_{11}\alpha_{12} - \alpha_{21}\alpha_{12})(f_3 - f_4)\\ &+\{-(\alpha_{10} - \alpha_{40})\alpha_{31}\alpha_{22} + (\alpha_{20} - \alpha_{40})\alpha_{31}\alpha_{12} + (\alpha_{30} - \alpha_{40})(\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})\},\end{aligned}$$

we have

$$2\alpha_{31}\alpha_{22}\alpha_{13}dx_3 = \alpha_{31}\alpha_{22}(df_1 - df_4) - \alpha_{31}\alpha_{12}(df_2 - df_4) - (\alpha_{11}\alpha_{12} - \alpha_{21}\alpha_{12})(df_3 - df_4).$$

Since $df_3 \wedge e_{34} = df_4 \wedge e_{34} = 0$, we have further

$$\begin{aligned}&2\alpha_{31}\alpha_{22}\alpha_{13}dx_3 \wedge e_{34} = \alpha_{31}\alpha_{22}df_1 \wedge e_{34} - \alpha_{31}\alpha_{12}df_2 \wedge e_{34}\\ &= \alpha_{31}\alpha_{22}f_1e_{134} - \alpha_{31}\alpha_{12}f_2e_{234}.\end{aligned}$$

□

From (19),(20) and Lemma 16,

$$\begin{aligned}&\alpha_{31}T_{f_2}\left(\frac{\varpi}{f_3f_4}\right) = \alpha_{31}T_{f_2-f_4}\left(\frac{\varpi}{f_3f_4}\right) + \alpha_{31}\frac{\varpi}{f_3}\\ &= \alpha_{31}\frac{2\alpha_{21}x_1 + 2\alpha_{22}x_2 + \alpha_{20} - \alpha_{40}}{f_3f_4}\varpi + \alpha_{31}\frac{\varpi}{f_3}\\ &= (\alpha_{31} - \alpha_{21})\frac{\varpi}{f_3} + \alpha_{21}\frac{\varpi}{f_4} + \frac{\alpha_{31}(\alpha_{20} - \alpha_{40}) - \alpha_{21}(\alpha_{30} - \alpha_{40})}{f_3f_4}\varpi\\ &\quad + \frac{1}{4}\{\frac{\alpha_{22}}{\alpha_{13}}f_1e_{134} - \frac{\alpha_{12}}{\alpha_{13}}f_2e_{234}\}.\end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned}&f_1e_{134} = T_{f_1}e_{134} \sim -T_{f_1}\left(8\frac{\lambda_2}{\lambda_\infty}W\varpi\right) \sim -8\frac{\lambda_2}{\lambda_\infty + 1}T_{f_1}(W\varpi),\\ &f_2e_{234} = T_{f_2}e_{234} \sim T_{f_2}\left(8\frac{\lambda_1}{\lambda_\infty}W\varpi\right) \sim 8\frac{\lambda_1}{\lambda_\infty + 1}T_{f_2}(W\varpi).\end{aligned}$$

Since

$$\begin{aligned}\frac{\alpha_{22}}{\alpha_{31}\alpha_{13}} &= -\frac{\sqrt{2}B(0234)}{B(034)\sqrt{B(01234)}},\\ \frac{\alpha_{12}}{\alpha_{31}\alpha_{13}} &= -\frac{\sqrt{2}B\begin{pmatrix} 0 & 1 & 3 & 4\\ 0 & 2 & 3 & 4\end{pmatrix}}{B(034)\sqrt{B(01234)}},\end{aligned}$$

then Lemma 8 and (21) leads us to

Lemma 17

$$T_{f_2}(\frac{\varpi}{f_3f_4}) = -\frac{B\begin{pmatrix} 0 & 3 & 4 \\ 0 & 2 & 3 \end{pmatrix}}{B(034)}\frac{\varpi}{f_3} + \frac{B\begin{pmatrix} 0 & 3 & 4 \\ 0 & 2 & 4 \end{pmatrix}}{B(034)}\frac{\varpi}{f_4} - \frac{B\begin{pmatrix} 0 & \star & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(034)}\frac{\varpi}{f_3f_4}$$
$$+\frac{\lambda_1}{\lambda_\infty+1}\frac{\sqrt{8}\,B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(034)\sqrt{B(01234)}}T_{f_2}(W\varpi)$$
$$+\frac{\lambda_2}{\lambda_\infty+1}\frac{\sqrt{8}\,B(0234)}{B(034)\sqrt{B(01234)}}T_{f_1}(W\varpi), \tag{22}$$

where $T_{f_j}(W\varpi)$ *can be obtained from* Cororally 15.

$T_{f_i}(\frac{\varpi}{f_jf_k})$ *for different* i,j,k *can be likewise derived by symmetry.*

The following Proposition is an immediate consequence of (22) and Corollary 15:

Proposition 18

$$T_{f_2}(\frac{\varpi}{f_3f_4}) \sim V_0 + \frac{\lambda_1}{\lambda_\infty+1}V_1 + \frac{\lambda_2}{\lambda_\infty+1}V_2 + \frac{\lambda_1\lambda_2}{(\lambda_\infty+1)\lambda_\infty}V_{12}, \tag{23}$$

where

$$V_0 = \frac{1}{B(034)}\left\{-B\begin{pmatrix} 0 & 3 & 4 \\ 0 & 2 & 3 \end{pmatrix}\frac{\varpi}{f_3} + B\begin{pmatrix} 0 & 3 & 4 \\ 0 & 2 & 4 \end{pmatrix}\frac{\varpi}{f_4}\right.$$
$$\left. -B\begin{pmatrix} 0 & \star & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}\frac{\varpi}{f_3f_4}\right\},$$
$$V_1 = \frac{B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(034)B(0134)}W_0(134)\varpi,\ V_2 = \frac{1}{B(034)}W_0(234)\varpi,$$
$$V_{12} = -\frac{B\begin{pmatrix} 0 & \star & 3 & 4 \\ 0 & 1 & 3 & 4 \end{pmatrix}}{B(034)\,B(0134)}W_0(1234)\varpi.$$

The RHS *of* (23) *is symmetric with respect to* $3,4$.

$T_{f_1}(\frac{\varpi}{f_3f_4})$ *can be expressed from* (23) *by the transposition* σ_{12}. *In general for different* j,k,l, $T_{f_j}(\frac{\varpi}{f_kf_l})$ *can be derived from* (23) *by a suitable permutation.*

As a final step, we want to give the expression of $T_{f_3-f_4}\frac{\varpi}{f_4}$.

First remark the following Lemma:

Lemma 19

$$T_{f_3-f_4}(\frac{\varpi}{f_4}) \sim B\begin{pmatrix} 0 & \star & 4 \\ 0 & 3 & 4 \end{pmatrix}\frac{\varpi}{f_4} - \frac{\lambda_1}{\lambda_\infty+2}\frac{\sqrt{8}B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{\sqrt{B(01234)}}T_{f_2f_3}(W\varpi)$$
$$-\frac{\lambda_2}{\lambda_\infty+2}\frac{\sqrt{8}B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{\sqrt{B(01234)}}T_{f_1f_3}(W\varpi) - \frac{\lambda_3}{\lambda_\infty+2}\frac{\sqrt{8}B(034)}{\sqrt{B(01234)}}T_{f_1f_2}(W\varpi)$$
$$. \tag{24}$$

Proof. Indeed the following identity holds:

$$\begin{aligned} &d(f_1-f_4)\wedge d(f_2-f_4)\wedge df_4 = df_1\wedge df_2\wedge df_4 \\ &\quad = 8\{-\alpha_{22}\alpha_{13}x_1 - \alpha_{21}\alpha_{13}x_2 + (\alpha_{11}\alpha_{22}-\alpha_{21}\alpha_{12})x_3\}\varpi, \end{aligned} \tag{25}$$
$$\begin{aligned} &d(f_1-f_4)\wedge d(f_3-f_4)\wedge df_4 = df_1\wedge df_3\wedge df_4 \\ &\quad = -8\{\alpha_{31}\alpha_{13}x_2 + \alpha_{31}\alpha_{12}x_3\}\varpi, \end{aligned} \tag{26}$$
$$d(f_2-f_4)\wedge d(f_3-f_4)\wedge df_4 = df_2\wedge df_3\wedge df_4 = -8\alpha_{22}\alpha_{31}x_3\varpi. \tag{27}$$

On the other hand, for different j,k such that $j,k\neq 4$

$$df_j\wedge df_k\wedge e_4 = f_jf_ke_{jk4} = T_{f_jf_k}(e_{jk4}).$$

From (25)-(27), $x_1\varpi$ can be represented as a linear combination of ϖ and $f_1f_2e_{124}$, $f_1f_3e_{134}$, $f_2f_3e_{234}$. Hence the following equality is valid:

$$\begin{aligned} T_{f_3-f_4}\left(\frac{\varpi}{f_4}\right) &= \frac{f_3-f_4}{f_4}\varpi = \frac{2\alpha_{31}x_1+\alpha_{30}-\alpha_{40}}{f_4}\varpi \\ &= \frac{\alpha_{30}-\alpha_{40}}{f_4}\varpi + \frac{1}{4\alpha_{22}\alpha_{13}}(-\alpha_{31}f_1f_2e_{124}+\alpha_{21}f_1f_3e_{134}-\alpha_{11}f_2f_3e_{234}). \end{aligned} \tag{28}$$

Since any cohomologous relation is preserved under the shift operations, Proposition 12 implies

$$\begin{aligned} f_1f_2e_{124} &\sim 8T_{f_1f_2}(\frac{\lambda_3}{\lambda_\infty}W\varpi) \sim 8\frac{\lambda_3}{\lambda_\infty+2}T_{f_1f_2}(W\varpi), \\ f_1f_3e_{134} &\sim -8T_{f_1f_3}(\frac{\lambda_2}{\lambda_\infty}W\varpi) \sim -8\frac{\lambda_2}{\lambda_\infty+2}T_{f_1f_3}(W\varpi), \\ f_2f_3e_{234} &\sim 8T_{f_2f_3}(\frac{\lambda_1}{\lambda_\infty}W\varpi) \sim 8\frac{\lambda_1}{\lambda_\infty+2}T_{f_2f_3}(W\varpi). \end{aligned}$$

From Lemma 8, it follows that

$$\alpha_{30}-\alpha_{40} = B\begin{pmatrix} 0 & \star & 4 \\ 0 & 3 & 4 \end{pmatrix},\ \frac{\alpha_{11}}{4\alpha_{22}\alpha_{13}} = \frac{B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{\sqrt{8\,B(01234)}},$$
$$\frac{\alpha_{21}}{4\alpha_{22}\alpha_{13}} = \frac{B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{\sqrt{8\,B(01234)}},\ \frac{\alpha_{31}}{4\alpha_{22}\alpha_{13}} = \frac{B(034)}{\sqrt{8\,B(01234)}}.$$

□

Seeing that the formulae for $T_{f_j f_k}(W\varpi)$ $(j \neq k)$ can be derived from Corollary 15 and Proposition 18, we have

Lemma 20

$$T_{f_1 f_2}(W\varpi) = c_1 T_{f_1}(\frac{\varpi}{f_3 f_4}) + c_2 T_{f_2}(\frac{\varpi}{f_3 f_4}) + c_3 \frac{\varpi}{f_4} + c_4 \frac{\varpi}{f_3} + c_0 \frac{\varpi}{f_3 f_4}.$$

Generally $T_{f_j f_k}(W\varpi)$ can be obtained by symmetry.

From Lemma 19-20,

$$\begin{aligned}
&T_{f_3 - f_4}\left(\frac{\varpi}{f_4}\right) \sim B\begin{pmatrix} 0 & \star & 4 \\ 0 & 3 & 4 \end{pmatrix} \frac{\varpi}{f_4} \\
&- \frac{\lambda_1}{\lambda_\infty + 2} \frac{\sqrt{8} B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{\sqrt{B(01234)}} \left\{ c_1 \frac{\varpi}{f_4} + c_4 \frac{\varpi}{f_1} + c_0 \frac{\varpi}{f_1 f_4} + c_2 T_{f_2}\left(\frac{\varpi}{f_1 f_4}\right) + c_3 T_{f_3}\left(\frac{\varpi}{f_1 f_4}\right) \right\} \\
&- \frac{\lambda_2}{\lambda_\infty + 2} \frac{\sqrt{8} B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{\sqrt{B(01234)}} \left\{ c_2 \frac{\varpi}{f_4} + c_4 \frac{\varpi}{f_2} + c_0 \frac{\varpi}{f_2 f_4} + c_1 T_{f_1}\left(\frac{\varpi}{f_2 f_4}\right) + c_3 T_{f_3}\left(\frac{\varpi}{f_2 f_4}\right) \right\} \\
&- \frac{\lambda_3}{\lambda_\infty + 2} \frac{B(034)}{\sqrt{B(01234)}} \left\{ c_3 \frac{\varpi}{f_4} + c_4 \frac{\varpi}{f_3} + c_0 \frac{\varpi}{f_3 f_4} + c_1 T_{f_1}\left(\frac{\varpi}{f_3 f_4}\right) + c_2 T_{f_2}\left(\frac{\varpi}{f_3 f_4}\right) \right\}.
\end{aligned} \tag{29}$$

Finally (29), Lemma 17 and Proposition 18 lead us to the following Proposition:

Proposition 21 *We have the contiguity relation*

$$T_{f_3 - f_4}(\frac{\varpi}{f_4}) \sim I + II + III + IV, \tag{30}$$

where

$$\begin{aligned}
I &= U_0, \\
II &= \frac{\lambda_1}{\lambda_\infty + 2} U_1 + \frac{\lambda_2}{\lambda_\infty + 2} U_2 + \frac{\lambda_3}{\lambda_\infty + 2} U_3, \\
III &= \frac{\lambda_1 \lambda_2}{(\lambda_\infty + 2)(\lambda_\infty + 1)} U_{12} + \frac{\lambda_1 \lambda_3}{(\lambda_\infty + 2)(\lambda_\infty + 1)} U_{13} + \frac{\lambda_2 \lambda_3}{(\lambda_\infty + 2)(\lambda_\infty + 1)} U_{23}, \\
IV &= \frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_\infty + 2)(\lambda_\infty + 1)\lambda_\infty} U_{123}.
\end{aligned}$$

$U_0, U_j, U_{jk}, U_{123}$ *are defined as*

$$
\begin{aligned}
&U_0 = B\begin{pmatrix} 0 & \star & 4 \\ 0 & 3 & 4 \end{pmatrix}\frac{\varpi}{f_4}, \\
&U_1 = -\frac{B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{B(014)}\,W_0(14)\varpi,\ U_2 = -\frac{B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{B(024)}\,W_0(24)\varpi, \\
&U_3 = -W_0(34)\varpi, \\
&U_{12} = U'_{12}\,W_0(124)\varpi, U_{13} = U'_{13}\,W_0(134)\varpi, U_{23} = U'_{23}\,W_0(234)\varpi, \\
&U_{123} = U'_{123}\,W_0(1234)\varpi,
\end{aligned}
$$

where

$$
\begin{aligned}
&B(0124)\,U'_{12} \\
&= \frac{-B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix}B\begin{pmatrix} 0 & \star & 1 & 4 \\ 0 & 1 & 2 & 4 \end{pmatrix}}{B(014)} + \frac{B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}B\begin{pmatrix} 0 & \star & 2 & 4 \\ 0 & 1 & 2 & 4 \end{pmatrix}}{B(024)}, \\
&U'_{13} = \frac{B\begin{pmatrix} 0 & \star & 4 \\ 0 & 1 & 4 \end{pmatrix}}{B(014)}, \\
&U'_{23} = \frac{B\begin{pmatrix} 0 & \star & 4 \\ 0 & 2 & 4 \end{pmatrix}}{B(024)}, \\
&U'_{123} = -\frac{B\begin{pmatrix} 0 & \star & 1 & 2 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}}{B(01234)}\left\{\frac{-B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix}B\begin{pmatrix} 0 & \star & 1 & 4 \\ 0 & 1 & 2 & 4 \end{pmatrix}}{B(0124)B(014)}\right. \\
&\left. + \frac{B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}B\begin{pmatrix} 0 & \star & 2 & 4 \\ 0 & 1 & 2 & 4 \end{pmatrix}}{B(0124)B(024)}\right\} - \frac{B\begin{pmatrix} 0 & \star & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}B\begin{pmatrix} 0 & \star & 4 \\ 0 & 2 & 4 \end{pmatrix}}{B(01234)\,B(024)} \\
&+ \frac{B\begin{pmatrix} 0 & \star & 1 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}B\begin{pmatrix} 0 & \star & 4 \\ 0 & 1 & 4 \end{pmatrix}}{B(01234)\,B(014)}.
\end{aligned}
$$

$U'_{12}, U'_{13}, U'_{23}, U'_{123}$ *are linearly related as follows*:

$$
\begin{aligned}
B(01234)\,U'_{123} = -B\begin{pmatrix} 0 & \star & 1 & 2 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}U'_{12} - B\begin{pmatrix} 0 & \star & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}U'_{23} \\
+B\begin{pmatrix} 0 & \star & 1 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}U'_{13}.
\end{aligned}
$$

$T_{f_j - f_k}(\frac{\varpi}{f_k})\,(j \neq k)$ *can be likewise obtained by symmetry.*

Proof. Indeed, I and II are obtained immediately from (29). On the other hand, (28) and Lemma 17 shows that $III+IV$ can be expressed by a linear combination of $T_{f_k}(W\varpi)\,(1\le k\le 3)$:

$$III+IV=\frac{\sqrt{8}\lambda_2\lambda_3}{(\lambda_\infty+2)(\lambda_\infty+1)}\frac{B\begin{pmatrix}0&\star&4\\0&2&4\end{pmatrix}B(0234)}{\sqrt{B(01234)}\,B(024)}T_{f_1}(W\varpi)$$
$$+\frac{\sqrt{8}\lambda_1\lambda_3}{(\lambda_\infty+2)(\lambda_\infty+1)}\frac{B\begin{pmatrix}0&\star&4\\0&1&4\end{pmatrix}B(0134)}{\sqrt{B(01234)}\,B(014)}T_{f_2}(W\varpi)$$
$$+\frac{\sqrt{8}\lambda_1\lambda_2}{(\lambda_\infty+2)(\lambda_\infty+1)}\left\{\frac{-B\begin{pmatrix}0&1&4\\0&3&4\end{pmatrix}B\begin{pmatrix}0&\star&1&4\\0&1&2&4\end{pmatrix}}{\sqrt{B(01234)}B(014)}\right.$$
$$\left.+\frac{B\begin{pmatrix}0&2&4\\0&3&4\end{pmatrix}B\begin{pmatrix}0&\star&2&4\\0&1&2&4\end{pmatrix}}{\sqrt{B(01234)}\,B(024)}\right\}T_{f_3}(W\varpi).$$

We may then apply Corollary 14 to the RHS. □

U'_{123} is simplified through Jacobi identity into the following expression:

Lemma 22

$$U'_{123}$$
$$=\frac{B\begin{pmatrix}0&\star&4\\0&1&4\end{pmatrix}B\begin{pmatrix}0&\star&1&4\\0&1&2&4\end{pmatrix}}{B(014)\,B(0124)}-\frac{B\begin{pmatrix}0&\star&4\\0&2&4\end{pmatrix}B\begin{pmatrix}0&\star&2&4\\0&1&2&4\end{pmatrix}}{B(024)\,B(0124)}$$
$$=-\frac{B\begin{pmatrix}0&\star&4\\0&1&4\end{pmatrix}B\begin{pmatrix}0&\star&4\\0&2&4\end{pmatrix}}{B(014)\,B(024)}-\frac{B\begin{pmatrix}0&\star&1&4\\0&1&2&4\end{pmatrix}B\begin{pmatrix}0&\star&2&4\\0&1&2&4\end{pmatrix}}{B(014)\,B(024)\,B(0124)}.$$

4 The expression of ϖ

We want to express cohomologically the standard 3-form $\varpi = dx_1\wedge dx_2\wedge dx_3$ as a linear combination of the basis of second kind $W_0(j)\varpi$, $W_0(jk)\varpi$, $W_0(jkl)\varpi$, $W_0(1234)\varpi$.

Lemma 23 *The following cohomological relation is valid in* $H^3_\nabla(X, \Omega^\cdot(*S))$:

$$(2\lambda_\infty + 3)\varpi \sim \lambda_1 T_{(f_1-f_4)}(\frac{\varpi}{f_1}) + \lambda_2 T_{(f_2-f_4)}(\frac{\varpi}{f_2}) + \lambda_3 T_{(f_3-f_4)}(\frac{\varpi}{f_3})$$
$$+\lambda_1(\alpha_{10}+\alpha_{40})\frac{\varpi}{f_1} + \lambda_2(\alpha_{20}+\alpha_{40})\frac{\varpi}{f_2} + \lambda_3(\alpha_{30}+\alpha_{40})\frac{\varpi}{f_3} + 2\lambda_4\alpha_{40}\frac{\varpi}{f_4} \quad (31)$$

Proof. Indeed

$$\begin{aligned}
0 &\sim \nabla(x_1dx_2\wedge dx_3 - x_2dx_1\wedge dx_3 + x_3dx_1\wedge dx_2)\\
&= (2\lambda_\infty+3)\varpi - 2\lambda_1\left(\frac{\alpha_{11}x_1+\alpha_{12}x_2+\alpha_{13}x_3+\alpha_{10}}{f_1}\right)\varpi\\
&-2\lambda_2\left(\frac{\alpha_{21}x_1+\alpha_{22}x_2+\alpha_{20}}{f_2}\right)\varpi - 2\lambda_3\left(\frac{\alpha_{31}x_1+\alpha_{30}}{f_3}\right)\varpi - 2\lambda_4\frac{\alpha_{40}}{f_4}\varpi\\
&= \text{LHS of (26)} - \text{RHS of (26)}. \qquad (32)
\end{aligned}$$

Hence (31) is valid. □

We arrive at the following result by applying Proposition 21 to (31):

Theorem 24 *We have*

$$(2\lambda_\infty+3)\varpi \sim I^* + II^* + III^* + IV^*,$$

where

$$\begin{aligned}
I^* &= -\sum_{j=1}^4 \lambda_j W_0(j)\varpi = -2\sum_{j=1}^4 \lambda_j r_j^2 \frac{\varpi}{f_j},\\
II^* &= \sum_{1\le j<k\le 4} \frac{\lambda_j\lambda_k}{\lambda_\infty+2} W_0(jk)\varpi,\\
III^* &= -\sum_{1\le j<k<l\le 4} \frac{\lambda_j\lambda_k\lambda_l}{(\lambda_\infty+2)(\lambda_\infty+1)} W_0(jkl)\varpi,\\
IV^* &= \frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\lambda_\infty+2)(\lambda_\infty+1)\lambda_\infty} W_0(1234)\varpi.
\end{aligned}$$

5 Variational formula for ϖ

The derivation of the variation formula for ϖ is based on the relation (3). First notice the following Lemma:

Lemma 25

$$\begin{aligned}\nabla_B \varpi &= \sum_{j=1}^{4} \sum_{\nu=1}^{4-j} d\alpha_{j\nu} \nabla_{B,\frac{\partial}{\partial \alpha_{j\nu}}} \varpi \\ &= 2\lambda_1 (d\alpha_{11} x_1 + d\alpha_{12} x_2 + d\alpha_{13} x_3 + \frac{1}{2}\, d\alpha_{10})\, \frac{\varpi}{f_1} \\ &+ 2\lambda_2 (d\alpha_{21} x_1 + d\alpha_{22} x_2 + \frac{1}{2}\, d\alpha_{20})\, \frac{\varpi}{f_2} \\ &+ 2\lambda_3 (d\alpha_{31} x_1 + \frac{1}{2}\, d\alpha_{30})\, \frac{\varpi}{f_3} + \lambda_4 d\alpha_{40}\, \frac{\varpi}{f_4}. \end{aligned} \tag{33}$$

We want to express (33) explicitly as a linear combination of the basis stated in Proposition 9.

From (9) we have

$$\begin{aligned} x_1 &= \frac{1}{2\alpha_{31}}\{f_3 - f_4 - (\alpha_{30} - \alpha_{40})\}, \\ x_2 &= \frac{1}{2\alpha_{31}\alpha_{22}}\{\alpha_{31}(f_2 - f_4) - \alpha_{21}(f_3 - f_4) - \alpha_{31}(\alpha_{20} - \alpha_{40}) + \alpha_{21}(\alpha_{30} - \alpha_{40})\}, \\ x_3 &= \frac{1}{2\alpha_{31}\alpha_{22}\alpha_{13}}\{\alpha_{31}\alpha_{22}(f_1 - f_4) - \alpha_{31}\alpha_{12}(f_2 - f_4) - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})(f_3 - f_4) \\ &\quad -\alpha_{31}\alpha_{22}(\alpha_{10} - \alpha_{40}) + \alpha_{31}\alpha_{12}(\alpha_{20} - \alpha_{40}) + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})(\alpha_{30} - \alpha_{40})\}, \end{aligned}$$

and from Lemma 7 we have

$$\begin{aligned} &\alpha_{31}\alpha_{22} = \sqrt{\frac{-B(0234)}{4}},\ \alpha_{31}\alpha_{12} = -\frac{1}{2}\frac{B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{\sqrt{-B(0234)}}, \\ &\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \frac{B\begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{2\sqrt{-B(0234)}}, \\ &-\alpha_{31}\alpha_{22}(\alpha_{10} - \alpha_{40}) + \alpha_{31}\alpha_{12}(\alpha_{20} - \alpha_{40}) + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})(\alpha_{30} - \alpha_{40}) \\ &\quad = -\frac{B\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & \star & 2 & 3 & 4 \end{pmatrix}}{2\sqrt{-B(0234)}}. \end{aligned}$$

By substituting these equalities into (33), we have

$$\nabla_B \varpi = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 + \lambda_4 V_4, \tag{34}$$

where

$$
\begin{aligned}
V_1 &= \theta^1_{34}\,\frac{f_3-f_4-(\alpha_{30}-\alpha_{40})}{f_1}\varpi+\theta^1_{24}\,\frac{f_2-f_4-(\alpha_{20}-\alpha_{40})}{f_1}\varpi \\
&\quad +\theta^1_{14}\,\frac{f_1-f_4-(\alpha_{10}-\alpha_{40})}{f_1}\varpi+d\alpha_{10}\,\frac{\varpi}{f_1}, \\
V_2 &= \theta^2_{34}\,\frac{f_3-f_4-(\alpha_{30}-\alpha_{40})}{f_2}\varpi+\theta^2_{24}\,\frac{f_2-f_4-(\alpha_{20}-\alpha_{40})}{f_2}\varpi+d\alpha_{20}\,\frac{\varpi}{f_2}, \\
V_3 &= \theta^3_{34}\,\frac{f_3-f_4-(\alpha_{30}-\alpha_{40})}{f_3}\varpi+d\alpha_{30}\,\frac{\varpi}{f_3}, \\
V_4 &= d\alpha_{40}\,\frac{\varpi}{f_4}.
\end{aligned}
$$

$\theta^1_{34}, \theta^1_{24}, \theta^1_{14}, \theta^2_{34}, \theta^2_{24}, \theta^3_{34}$ are defined respectively in the following manner:

$$
\begin{aligned}
\theta^1_{34} &= \frac{\alpha_{22}\alpha_{13}d\alpha_{11}-\alpha_{21}\alpha_{13}d\alpha_{12}-(\alpha_{11}\alpha_{22}-\alpha_{21}\alpha_{12})d\alpha_{13}}{\alpha_{31}\alpha_{22}\alpha_{13}}, \\
\theta^1_{24} &= \frac{\alpha_{13}d\alpha_{12}-\alpha_{12}d\alpha_{13}}{\alpha_{22}\alpha_{13}},\ \theta^1_{14}=\frac{d\alpha_{13}}{\alpha_{13}}, \\
\theta^2_{34} &= \frac{\alpha_{22}d\alpha_{21}-\alpha_{21}d\alpha_{22}}{\alpha_{31}\alpha_{22}},\ \theta^2_{24}=\frac{d\alpha_{22}}{\alpha_{22}}, \\
\theta^3_{34} &= \frac{d\alpha_{31}}{\alpha_{31}}. \qquad (35)
\end{aligned}
$$

The above six differential forms θ^k_{ij} can be characterized by the lower triangular matrix 1-form ω defined below.

Definition 26 Let g_α be the lower triangular matrix of size 5×5 defined by

$$
g_\alpha = \begin{pmatrix} 1 & & & & \\ \alpha_{40} & 1 & & & \\ \alpha_{30} & 1 & \alpha_{31} & & \\ \alpha_{20} & 1 & \alpha_{21} & \alpha_{22} & \\ \alpha_{10} & 1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \end{pmatrix}.
$$

We denote by $\omega=(\omega_{jk})_{1\le j,k\le 5}$ the differential 1-form of g_α which is invariant under the right multiplication:

$$
\omega = dg_\alpha\cdot g_\alpha^{-1} = \begin{pmatrix} \omega_{11} & & & & \\ \omega_{21} & \omega_{22} & & & \\ \omega_{31} & \omega_{32} & \omega_{33} & & \\ \omega_{41} & \omega_{42} & \omega_{43} & \omega_{44} & \\ \omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & \omega_{55} \end{pmatrix}.
$$

Then

$$\omega_{jk} = 0 \quad (j<k),$$

$$\omega_{11} = \omega_{22} = 0,\ \omega_{21} = d\alpha_{40},$$

$$\omega_{31} = d\alpha_{30} - (\alpha_{30}-\alpha_{40})d\log\alpha_{31},\ \omega_{32} = -\theta^3_{34},\ \omega_{33} = \theta^3_{34} = d\log\alpha_{31},$$

$$\omega_{41} = d\alpha_{20} - \frac{\alpha_{30}-\alpha_{40}}{\alpha_{31}}d\alpha_{21} - \frac{\alpha_{31}(\alpha_{20}-\alpha_{40}) - \alpha_{21}(\alpha_{30}-\alpha_{40})}{\alpha_{31}}d\log\alpha_{22},$$

$$\omega_{42} = -\theta^2_{34} - \theta^2_{24},\ \omega_{43} = \theta^2_{34},\ \omega_{44} = \theta^2_{24} = d\log\alpha_{22},$$

$$\omega_{51} = d\alpha_{10} - \frac{\alpha_{30}-\alpha_{40}}{\alpha_{31}}d\alpha_{11} - \frac{\alpha_{31}(\alpha_{20}-\alpha_{40}) - \alpha_{21}(\alpha_{30}-\alpha_{40})}{\alpha_{31}\alpha_{22}}d\alpha_{12}$$

$$-\frac{\alpha_{31}\alpha_{22}(\alpha_{10}-\alpha_{40}) - \alpha_{31}\alpha_{12}(\alpha_{20}-\alpha_{40}) - (\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})(\alpha_{30}-\alpha_{40})}{\alpha_{31}\alpha_{22}}d\log\alpha_{13},$$

$$\omega_{52} = -\theta^1_{34} - \theta^1_{24} - \theta^1_{14},\ \omega_{53} = \theta^1_{34},\ \omega_{54} = \theta^1_{24},\ \omega_{55} = \theta^1_{14} = d\log\alpha_{13}.$$

Lemma 8 and (35) imply the following Proposition:

Proposition 27 θ^k_{ij} *are written down in terms of Cayley-Menger determinants*:

$$\theta^3_{34} = \frac{1}{2}d\log(\rho^2_{34}),$$

$$\theta^2_{34} = \frac{dB\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{2\rho^2_{34}} - \frac{B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{4\rho^2_{34}}d\log(-B(0234)),$$

$$\theta^2_{24} = \frac{1}{2}\,d\log\left(\frac{-B(0234)}{2B(034)}\right),$$

$$\theta^1_{14} = \frac{1}{2}\,d\log\left(\frac{B(01234)}{-2B(0234)}\right),$$

$$\theta^1_{24} = \frac{1}{B(0234)}\left\{dB\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix} - \frac{1}{2}B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}d\log(\rho^2_{34}) - \frac{B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{2}d\log B(01234)\right\},$$

$$\theta^1_{34} = \frac{1}{2\rho^2_{34}}\,dB\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix} + \frac{B\begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{2B(0234)}d\log(\rho^2_{34})$$
$$-\frac{B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{2\rho^2_{34}\,B(0234)}dB\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix} + \frac{B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{4\rho^2_{34}}d\log(-B(0234))$$
$$+\frac{B\begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{2B(0234)}d\log\ B(01234).$$

On the other hand, $\omega_{21}, \omega_{31}, \omega_{41}, \omega_{51}$ *are expressed respectively as*

$$\omega_{21} = -d(r_4^2) = -\frac{1}{2} dB(0 \star 4),$$

$$\omega_{31} = d(-r_3^2 + \rho_{34}^2) - \frac{1}{2}\, B\begin{pmatrix} 0 & \star & 4 \\ 0 & 3 & 4 \end{pmatrix} d\log(\rho_{34}^2),$$

$$\omega_{41} = d(-r_2^2 + \rho_{24}^2) - \frac{B\begin{pmatrix} 0 & \star & 4 \\ 0 & 3 & 4 \end{pmatrix}}{2\rho_{34}^2} \left\{ dB\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix} - \frac{1}{2}\, B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix} d\log(\rho_{34}^2) \right\}$$
$$+ \frac{B\begin{pmatrix} 0 & \star & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{4\rho_{34}^2} d\log\left(\frac{-B(0234)}{2B(034)}\right),$$

$$\omega_{51} = d(-r_1^2 + \rho_{14}^2) - \frac{B\begin{pmatrix} 0 & \star & 4 \\ 0 & 3 & 4 \end{pmatrix}}{2\rho_{34}^2} \left\{ dB\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix} - \frac{1}{2} B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix} d\log(\rho_{34}^2) \right\}$$
$$+ \frac{B\begin{pmatrix} 0 & \star & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(0234)} \left\{ \frac{1}{2} d\left(\frac{B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{\rho_{34}^2} \right) - \frac{B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{4\rho_{34}^2} d\log\left(\frac{-B(0234)}{2B(034)}\right) \right\}$$
$$- \frac{1}{2} \frac{B\begin{pmatrix} 0 & \star & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}}{B(0234)} d\log\left(\frac{B(01234)}{-2B(0234)}\right).$$

We have further

Lemma 28

$$\frac{d\alpha_{31}}{\alpha_{31}} = \frac{dB(034)}{2B(034)} = \frac{d\rho_{34}}{\rho_{34}},$$

$$\frac{d\alpha_{11}}{\alpha_{31}} = \frac{d(\alpha_{11}\alpha_{31}) - \alpha_{11}d\alpha_{31}}{\alpha_{31}^2} = \frac{dB\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{2\rho_{34}^2} - \frac{B\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 4 \end{pmatrix} d\rho_{34}}{2\rho_{34}^3},$$

$$\frac{d\alpha_{22}}{\alpha_{22}} = \frac{d(\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{22}} - \frac{d\alpha_{31}}{\alpha_{31}} = \frac{dB(0234)}{2B(0234)} - \frac{d\rho_{34}}{\rho_{34}},$$

$$\frac{\alpha_{21}d\alpha_{22}}{\alpha_{31}\alpha_{22}} = \frac{\rho_{24}^2 + \rho_{34}^2 - \rho_{23}^2}{2\rho_{34}^2}\left\{\frac{dB(0234)}{2B(0234)} - \frac{d\rho_{34}}{\rho_{34}}\right\},$$

$$\frac{d\alpha_{21}}{\alpha_{31}} = \frac{d(\alpha_{21}\alpha_{31}) - \alpha_{21}d\alpha_{31}}{\alpha_{31}^2}$$

$$= \frac{dB\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{2\rho_{34}^2} - \frac{B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}}{2\rho_{34}^3} d\rho_{34},$$

$$\frac{(\alpha_{21} - \alpha_{31})d\alpha_{22}}{\alpha_{31}\alpha_{22}} = -\frac{B\begin{pmatrix} 0 & 2 & 3 \\ 0 & 4 & 3 \end{pmatrix}}{2\rho_{34}^2}\left\{\frac{dB(0234)}{2B(0234)} - \frac{d\rho_{34}}{\rho_{34}}\right\}.$$

Summing up Lemma 14, Proposition 18, Proposition 21, and the equality (34), we can conclude

Theorem 29 *We have*

$$\nabla_B \varpi \sim I^{**} + II^{**} + III^{**} + IV^{**}, \tag{36}$$

where

$$I^{**} = \sum_{j=1}^{4} \lambda_j \theta_j W_0(j)\varpi,$$

$$II^{**} = \sum_{j<k} \frac{\lambda_j \lambda_k}{\lambda_\infty + 2} \theta_{jk} W_0(jk)\varpi$$

$$III^{**} = \sum_{j<k<l} \frac{\lambda_j \lambda_k \lambda_l}{(\lambda_\infty + 2)(\lambda_\infty + 1)} \theta_{jkl} W_0(jkl)\varpi$$

$$IV^{**} = \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{(\lambda_\infty + 2)(\lambda_\infty + 1)\lambda_\infty} \theta_{1234} W_0(1234)\varpi,$$

and

$$\theta_j = -\frac{1}{2} d\log(r_j^2) = -\frac{1}{2} d\log B(0\star j), \tag{37}$$

$$\theta_{jk} = \frac{1}{2} d\log(\rho_{jk}^2) = \frac{1}{2} d\log B(0jk), \tag{38}$$

$$\begin{aligned}
\theta_{jkl} &= -\frac{B\begin{pmatrix} \star & j & k & l \\ 0 & j & k & l \end{pmatrix}}{B(0kl)B(0jl)B(0jk)} d\log B(0jkl) \\
&\quad -\frac{r_j^2}{B(0jk)B(0jl)} dB\begin{pmatrix} 0 & k & j \\ 0 & l & j \end{pmatrix} - \frac{r_k^2}{B(0jk)B(0kl)} dB\begin{pmatrix} 0 & j & k \\ 0 & l & k \end{pmatrix} \\
&\quad -\frac{r_l^2}{B(0jl)B(0kl)} dB\begin{pmatrix} 0 & j & l \\ 0 & k & l \end{pmatrix} \\
&= -\frac{1}{2} \frac{B\begin{pmatrix} 0 & \star & k & l \\ 0 & j & k & l \end{pmatrix}}{B(0jkl)} d\log\rho_{kl}^2 - \frac{1}{2} \frac{B\begin{pmatrix} 0 & \star & j & l \\ 0 & k & j & l \end{pmatrix}}{B(0jkl)} d\log\rho_{jl}^2 \\
&\quad -\frac{1}{2} \frac{B\begin{pmatrix} 0 & \star & j & k \\ 0 & l & j & k \end{pmatrix}}{B(0jkl)} d\log\rho_{jk}^2,
\end{aligned} \tag{39}$$

$$\begin{aligned}
\theta_{1234} &= \frac{1}{2} \sum_{i<j,\, k<l} d\log\rho_{ij}^2 \left\{ \frac{B\begin{pmatrix} 0 & \star & i & j \\ 0 & k & i & j \end{pmatrix} B\begin{pmatrix} 0 & \star & i & j & k \\ 0 & l & i & j & k \end{pmatrix}}{B(0ijk)B(01234)} \right. \\
&\quad \left. + \frac{B\begin{pmatrix} 0 & \star & i & j \\ 0 & l & i & j \end{pmatrix} B\begin{pmatrix} 0 & \star & i & j & l \\ 0 & k & i & j & l \end{pmatrix}}{B(0ijl)B(01234)} \right\},
\end{aligned} \tag{40}$$

where $\{i,j,k,l\}$ *move over the set of all permutations of* $\{1,2,3,4\}$ *such that* $i < j$, $k < l$.

Proof. Proposition 27 and the equality (34) show that there exist uniquely 1-forms $\theta_j, \theta_{jk}, \theta_{jkl}, \theta_{1234}$ such that (36) holds. Since the RHS of (36) is symmetric with respect to $\mathfrak{S}_4$, we have only to compute explicitly $\theta_4, \theta_{34}, \theta_{234}, \theta_{1234}$.

In view of (34), the coefficient of the monomial λ_4 in the RHS of (36) equals

$$\frac{\varpi}{f_4} d\alpha_{40} = -\frac{\varpi}{f_4} dr_4^2 = -\frac{1}{2} W_0(4)\varpi\, d\log r_4^2.$$

Hence we have (37).

Next the coefficient of the term $\frac{\lambda_3\lambda_4}{\lambda_\infty+2}$ in the RHS of (38) coincides with the one in $\lambda_3 \frac{f_3-f_4}{f_3} \theta_{34}^3$. From Proposition 27, the latter equals

$$\frac{1}{2} W_0(34)\varpi d\log\rho_{34}^2.$$

Hence we obtain (38).

Thirdly the coefficient of $\frac{\lambda_2\lambda_3\lambda_4}{(\lambda_\infty+2)(\lambda_\infty+1)}$ in the RHS of (39) coincides with the one in

$$\lambda_2\left(\frac{f_3-f_4}{f_2}\varpi\,\theta^2_{34}+\frac{f_2-f_4}{f_2}\varpi\,\theta^2_{24}\right)+\lambda_3\,\frac{f_3-f_4}{f_3}\varpi\,\theta^3_{34}.$$

which is equal to

$$\left\{-\frac{B\begin{pmatrix}0&\star&3\\0&2&3\end{pmatrix}}{B(023)}\theta^3_{34}+\frac{B\begin{pmatrix}0&\star&2\\0&4&2\end{pmatrix}}{B(024)}\theta^2_{34}-(\theta^2_{34}+\theta^2_{24})\frac{B\begin{pmatrix}0&\star&2\\0&3&2\end{pmatrix}}{B(023)}\right\}W_0(234)\varpi.$$

Expanding the RHS with respect to $d\rho^2_{jk}$, we obtain the RHS of (39).

The formula (39) will be derived in the Appendix (see Theorem A.12). □

Corollary 30 *The derivation formula for $\mathcal{J}_\lambda(1)$ is given in invariant form as follows*

$$\begin{aligned}&d_B\mathcal{J}_\lambda(1)=\sum_{j=1}^4\lambda_j\theta_j\mathcal{J}_\lambda(W_0(j))\\&+\sum_{1\le j<k\le 4}\frac{\lambda_j\lambda_k}{\lambda_\infty+2}\theta_{jk}\mathcal{J}_\lambda(W_0(jk))+\sum_{1\le j<k<l\le 4}\frac{\lambda_j\lambda_k\lambda_l}{(\lambda_\infty+2)(\lambda_\infty+1)}\theta_{jkl}\mathcal{J}_\lambda(W_0(jkl))\\&+\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\lambda_\infty+2)(\lambda_\infty+1)\lambda_\infty}\theta_{1234}\mathcal{J}_\lambda(W_0(1234)).\end{aligned}$$

Remark 1 It is noteworthy that all $\theta_j,\theta_{jk},\theta_{jkl},\theta_{1234}$ appearing in Theorem 28 are logarithmic 1-forms in the weak sense, i.e., in the sense of K.Saito (see [23]). In fact $\theta_j,\theta_{jk},\theta_{jkl}$ are all weakly logarithmic as is seen in the formulae (32)-(34). θ_{1234} is also weakly logarithmic owing to the following Proposition.

Proposition 31 *θ_{1234} is weakly logarithmic 1-form i.e., logarithmic 1-form in the sense of K.Saito along its singularity.*

Proof. By definition it is sufficient to prove that

$$\begin{aligned}&\prod_{1\le i<j\le 4}B(0ij)\cdot\prod_{1\le i<j\le k\le 4}B(0ijk)\cdot B(01234)\cdot\theta_{1234},\\&\prod_{1\le i<j\le 4}B(0ij)\cdot\prod_{1\le i<j\le k\le 4}B(0ijk)\cdot B(01234)\cdot d\theta_{1234}\end{aligned}$$

are both holomorphic 1-form in the parameter space of ρ^2_{ij} and r^2_j, i.e., in the whole complex affine space $\mathbf{C}^{10}$. The hypersurfaces

$$\begin{aligned}&Y_{ij}:B(0ij)=0\ (1\le i<j\le 4),\ Y_{ijk}:B(0ijk)=0\ (1\le i<j<k\le 4),\\&Y_{1234}:B(01234)=0\end{aligned}$$

are all irreducible reduced manifolds which are different from each other.

It is obvious from Theorem A.12 in Appendix that θ_{1234} is logarithmic along Y_{ij}.

On the other hand, Corollary A.7 and Lemma A.8 in Appendix show that it is logarithmic along Y_{1234}.

Hence we have only to prove that it is also logarithmic along Y_{ijk}. In the expression (A.1) in Appendix, the union of the poles of $v^1_{14}, v^1_{34}, v^1_{24}, v^2_{24}, v^2_{34}, v^3_{34}$ lie in the analytic hypersurface

$$0 = \prod_{1\le i<j\le 4} B(0ij) \prod_{1<j<k\le 4} B(01jk).$$

i.e., v^k_{ij} is holomorphic along Y_{234}. Hence Y_{234} may be included only in the union of the poles of $\theta^1_{34}, \theta^1_{24}, \theta^1_{14}, \theta^2_{34}, \theta^2_{24}, \theta^3_{34}$. We want to show that $\theta^1_{34}, \theta^1_{24}, \theta^1_{14}, \theta^2_{34}, \theta^2_{24}, \theta^3_{34}$ are all logarithmic along Y_{234}.

From Proposition 26, θ^3_{34} is holomorohpic along Y_{234}, while $\theta^2_{34}, \theta^2_{24}, \theta^1_{14}$ are all logarithmic along Y_{234}.

It remains to prove that θ^1_{24} and θ^1_{34} are logarithmic along Y_{234}.

We first prove that θ^1_{24} is logarithmic along Y_{234}. Proposition 26 shows that $B(0234)\,\theta^1_{24}$ is holomorphic along Y_{234}. We must prove that $B(0234)\,d\theta^1_{24}$ is also holomorphic along Y_{234}. This fact can be proved in the following way: $B(0234)\,d\theta^1_{24}$ can be expressed as

$$\begin{aligned}
&B(0234)\,d\theta^1_{24} = U_0 + U_1,\\
&\quad U_0 = -d\log B(0234) \wedge \Bigg\{ dB\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}\\
&\quad -\frac{1}{2}B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix} d\log B(034) - \frac{1}{2}B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix} d\log B(01234)\Bigg\}\\
&\quad = -\frac{1}{2}B\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix} d\log B(0234) \wedge d\log\left(-\frac{B^2\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(01234)B(034)} \right),\\
&\quad U_1 = -\frac{1}{2}dB\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix} \wedge \{d\log B(034) + d\log B(01234)\}.
\end{aligned}$$

U_1 is obviously holomorphic along Y_{234}. On the other hand, since

$$B(01234)B(034) = B(0134)B(0234) - B^2\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix},$$

we have

$$-\frac{B^2\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{pmatrix}}{B(01234)B(034)} = 1 - \frac{B(0134)B(0234)}{B(01234)B(034)}.$$

This implies

$$d\log\left(-\frac{B^2\begin{pmatrix}0&1&3&4\\0&2&3&4\end{pmatrix}}{B(01234)B(034)}\right)=d\log\left(1-\frac{B(0134)B(0234)}{B(01234)B(034)}\right)$$
$$=\eta_0 B(0234)+\eta_1 dB(0234),$$

where η_0 is a holomorphic 1-form and η_1 is a holomorphic function along $Y_{234.}$ Since we have

$$U_0=-\frac{1}{2}B\begin{pmatrix}0&1&3&4\\0&2&3&4\end{pmatrix}dB(0234)\wedge\eta_0,$$

U_0 is holomorphic along Y_{234}. We have shown that θ^1_{24} is logarithmic along Y_{234}.

θ^1_{14} and θ^1_{24} being now logarithmic along Y_{234}, θ^1_{34} is also logarithmic along Y_{234} from the last equality in Lemma A.1 in Appendix.

Thus we can conclude that θ_{1234} is logarithmic along Y_{234}. Since θ_{1234} is symmetric with respect to the suffices $1, 2, 3, 4$, it is also logarithmic along $Y_{ijk}\,(1\le i<j<k\le 4)$. □

Remark 2 In the case where $n=1, m=2$ and $n=2, m=3$ under the conditions $(\mathcal{H}1), (\mathcal{H}2)$, the analogues to the identity (11) together with Theorem 24 and Theorem 29 are valid. Indeed we can define similarly the twisted cycles $\mathfrak{z}_J\,(J\in\mathcal{B})$ for general n. The expression of Cayley-Menger determinants, the differential forms $W_0(J)\varpi$ and θ_j, θ_{jk} do not depend on dimension n.

In case of $n=1$, by using the same notation as in §5-6, we have

$$\mathfrak{z}_\infty\sim-\frac{1}{\cos\pi\lambda_\infty}\mathfrak{z}_{12}-\frac{\cos\pi\lambda_2}{\cos\pi\lambda_\infty}\mathfrak{z}_1-\frac{\cos\pi\lambda_1}{\cos\pi\lambda_\infty}\mathfrak{z}_2, \tag{41}$$

$$(2\lambda_\infty+1)\varpi\sim-\sum_{j=1}^{2}\lambda_j W_0(j)\varpi+\frac{\lambda_1\lambda_2}{\lambda_\infty}W_0(12)\varpi, \tag{42}$$

$$\nabla_B\varpi\sim\sum_{j=1}^{2}\lambda_j W_0(j)\varpi\,\theta_j+\frac{\lambda_1\lambda_2}{\lambda_\infty}W_0(12)\varpi\,\theta_{12}. \tag{43}$$

Similarly in case of $n = 2$, we have

$$\mathfrak{z}_\infty \sim -\sum_{j=1}^{3} \frac{\sin \pi\lambda_j}{\sin \pi\lambda_\infty}\mathfrak{z}_{\widehat{j}} - \sum_{j<k} \frac{\sin \pi(\lambda_j + \lambda_k)}{\sin \pi\lambda_\infty}\mathfrak{z}_{\widehat{jk}} \tag{44}$$

$$(2\lambda_\infty + 2)\varpi \sim -\sum_{j=1}^{3} \lambda_j W_0(j)\varpi + \sum_{1\le j<k\le 3} \frac{\lambda_j\lambda_k}{\lambda_\infty + 1} W_0(jk)\varpi - \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_\infty + 1)\lambda_\infty} W_0(123)\varpi, \tag{45}$$

$$\nabla_B\varpi \sim \sum_{j=1}^{3} \lambda_j W_0(j)\varpi\,\theta_j + \sum_{1\le j<k\le 3} \frac{\lambda_j\lambda_k}{\lambda_\infty + 1} W_0(jk)\varpi\,\theta_{jk} + \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_\infty + 1)\lambda_\infty} W_0(123)\varpi\,\theta_{123}, \tag{46}$$

where $\lambda_\infty = \lambda_1 + \lambda_2, \sum_{j=1}^{3} \lambda_j$ for $n = 1, 2$ respectively, and $\widehat{J}$ denotes the complement $\{1, 2, 3\} - J$. The proofs of (41)-(46) can be accomplished in the same way as (11), Theorem 24 and Theorem 29. They are much simpler and omitted here.

So far we have restricted ourselves only to the case where $n \le 3$. It is possible to formulate them for arbitrary n in a similar way. However its proof is not yet available and remains to be done (see [10]).

Appendix The variational formula for the volume of a spherically faced simplex

In this appendix, we give the variational formula for the volume of the spherically faced simplex D which is defined in $\mathbf{R}^3$

$$D : f_1 \le 0, f_2 \le 0, f_3 \le 0, f_4 \le 0.$$

.

The simplex D can be identified as the cycle $\mathfrak{z}_{1234}$ defined in Proposition 9.

$S_j, S_j \cap S_k, S_j \cap S_k \cap S_l$ are spheres of dimension two, one and zero respectively.

$$S_4 : |x|^2 = r_4^2,$$
$$S_3 \cap S_4 : x_2^2 + x_3^2 = r_{34}^2 \ (r_{34} > 0),$$
$$S_2 \cap S_3 \cap S_4 : \{\text{two points}\}.$$

The residues of the forms $\frac{\varpi}{f_j}, \frac{\varpi}{f_jf_k}, \frac{\varpi}{f_jf_kf_l}$ along $S_j, S_j\cap S_k, S_j\cap S_k\cap S_l$ can be computed explicitly as follows.

Lemma A.1 We have

$$\mathrm{Res}\left[\frac{\varpi}{f_j}\right]_{S_j} = \left[\frac{\varpi}{df_j}\right]_{S_j} = -\frac{1}{\sqrt{2B(0\star j)}}\varpi_j, \tag{A.1}$$

$$\mathrm{Res}\left[\frac{\varpi}{f_jf_k}\right]_{S_j\cap S_k} = \left[\frac{\varpi}{df_j\wedge df_k}\right]_{S_j\cap S_k} = \frac{1}{\sqrt{-4B(0\star jk)}}\varpi_{jk}, \tag{A.2}$$

$$\mathrm{Res}\left[\frac{\varpi}{f_jf_kf_l}\right]_{S_j\cap S_k\cap S_l} = \left[\frac{\varpi}{df_j\wedge df_k\wedge df_l}\right]_{S_j\cap S_k\cap S_l} = -\frac{1}{\sqrt{8B(0\star jkl)}}\varpi_{jkl}, \tag{A.3}$$

where $\varpi_j, \varpi_{jk}, \varpi_{jkl}$ denote the standard sphere elements on $S_j, S_j\cap S_k, S_j\cap S_k\cap S_l$ respectively :

$$\begin{aligned}
\varpi_4 &= \frac{x_2dx_3\wedge dx_4 - x_3dx_2\wedge dx_4 + x_4dx_2\wedge dx_3}{r_4},\\
\varpi_{34} &= \frac{x_3dx_4 - x_4dx_3}{r_{34}},\\
\varpi_{234} &= \text{the standard point measure.}
\end{aligned}$$

and r_{34} denotes the radius of the circle $\Re S_3\cap\Re S_4$.

Proof. Because of symmetry it sufficient to prove (A.1),(A.2),(A.3) in the case where $j=4; j=3, k=4$ and $j=2, k=3, l=4$ respectively. The identity (A.1) is obvious since

$$df_4\wedge(x_2dx_3\wedge dx_4 - x_3dx_2\wedge dx_4 + x_4dx_2\wedge dx_3) = 2|x|^2\varpi.$$

and $f_j = |x|^2 - r_4^2,\ 2r_4^2 = B(0\star 4)$.

On the other hand we have from Lemma 7

$$df_2\wedge df_3\wedge(x_3dx_4 - x_4dx_3) = 4\alpha_{31}(x_2^2+x_3^2)\varpi,$$

and

$$\begin{aligned}
x_2^2+x_3^2 = r_{34}^2 &= -\frac{4\alpha_{40}\alpha_{31}^2 + (\alpha_{30}-\alpha_{40})^2}{4\alpha_{31}^2}\\
&= -\frac{B(0\star 34)}{2B(034)}.
\end{aligned}$$

This gives (A.2).

To prove (A.3) first remark the identity.

$$df_2 \wedge df_3 \wedge df_4 = -8\alpha_{22}\alpha_{31}x_3\varpi. \tag{A.4}$$

Furthermore

$$\begin{aligned} x_1 &= -\frac{\alpha_{30}-\alpha_{40}}{2\alpha_{31}}, \\ x_2 &= \frac{\alpha_{21}(\alpha_{30}-\alpha_{40})-\alpha_{31}(\alpha_{20}-\alpha_{40})}{2\alpha_{22}\alpha_{31}}. \end{aligned}$$

Hence from $f_4 = 0$

$$x_3 = \sqrt{-\frac{B(0\star 234)}{2B(0234)}}. \tag{A.5}$$

In fact

$$\begin{aligned} x_3^2 &= -(x_1^2+x_2^2)-\alpha_{40} \\ &= -\frac{\alpha_{22}^2(\alpha_{30}-\alpha_{40})^2+\{\alpha_{21}(\alpha_{30}-\alpha_{40})-\alpha_{31}(\alpha_{20}-\alpha_{40})\}^2+4\alpha_{40}\alpha_{22}^2\alpha_{31}^2}{4\alpha_{22}^2\alpha_{31}^2} \\ &= \frac{\eta}{\xi} \end{aligned}$$

where from Lemma 7

$$\begin{aligned} \xi &= -B(0234), \\ \eta &= -\frac{1}{2}B(024)B^2\begin{pmatrix} 0 & 3 & 4 \\ 0 & \star & 4 \end{pmatrix} - \frac{1}{2}B(034)B^2\begin{pmatrix} 0 & 2 & 4 \\ 0 & \star & 4 \end{pmatrix} \\ &+B\begin{pmatrix} 0 & 2 & 4 \\ 0 & 3 & 4 \end{pmatrix}B\begin{pmatrix} 0 & 2 & 4 \\ 0 & \star & 4 \end{pmatrix}B\begin{pmatrix} 0 & 3 & 4 \\ 0 & \star & 4 \end{pmatrix} - \frac{1}{2}B(0\star 4)B(0234) \\ &= \frac{1}{2}B(0\star 234). \end{aligned}$$

We may assume $x_3 < 0$ so that we get the identity (A.4). (A.4)-(A.5) finally show (A.3). $\Box$.

The following Theorem is an immediate consequence of Theorem 29 applying to the case where $\mathfrak{z}$ is taken $\mathfrak{z}_{1234} = D$ and tending $\lambda_j \to 0$ for all positive λ_j.

Theorem A.1 We have the variational formula

$$d_Bv(D) = -\sum_{j=1}^{4}\sqrt{\frac{B(0\star j)}{2}}\,v_j\theta_j + \sum_{1\le j<k\le 4}\sqrt{\frac{-B(0\star jk)}{4}}\,v_{jk}\frac{\theta_{jk}}{2}$$
$$-\sum_{1\le j<k<l\le 4}\sqrt{\frac{B(0\star jkl)}{8}}\,v_{jkl}\,\frac{\theta_{jkl}}{2} - \sqrt{\frac{B(01234)}{8}}\,\frac{\theta_{1234}}{2}.$$

Here v_j, v_{jk}, v_{jkl} denote the lower dimensional volumes corresponding to the boundaries $S_j \cap \partial D, S_j \cap S_k \cap \partial D,\ S_j \cap S_k \cap S_l \cap \partial D$ respectively :

$$v_j = \int_{S_j\cap\partial D} \varpi_j,$$
$$v_{jk} = \int_{S_j\cap S_k\cap\partial D} \varpi_{jk},$$
$$v_{jkl} = \int_{S_j\cap S_k\cap S_l\partial D} \varpi_{jkl}.$$

(Remark that all the signs should be taken in an opposite way, because the domain D is defined for all f_j negative.)

This is just an extension of the classical variational formula due to L.Schläfli concerning the volume of a three dimensional geodesic simplex in the unit sphere (see [6], [13], [15], [22], [24], [25] etc).

One may present a conjectural formula similar to Theorem A.1 to a general case where $m = n + 1$ as follows.

Because of symmetry we may assume that f_j can be described as

$$f_j = Q(x) + \sum_{\nu=1}^{n+1-j} 2\alpha_{j\nu}x_\nu + \alpha_{j0} \quad (1 \le j \le n+1).$$

Let D be a n-simplex with spherical faces in $\mathbf{R}^n$ defined by

$$D : f_1 \le 0,\ \ldots,\ f_{n+1} \le 0.$$

Denote by $v(D)$ the volume of D with respect to the standard n-form ϖ :

$$v(D) = \int_D \varpi.$$

Then $v(D)$ can be regarded as an analytic functionn of the Cayle-Menger matrix B.

We define the covariant differentiation with respect to B as follows:

$$\nabla_B(\varphi\varpi) = d_B(\varphi)\varpi + (d_B \log \Phi)\varpi \quad (\varphi\,\varpi \in \Omega(*S))$$

.

where d_B denotes the total differentiation

$$d_B\psi = \sum_{j=1}^{n} \sum_{\nu=1}^{n+1-j} d\alpha_{j\nu} \frac{\partial \psi}{\partial \alpha_{j\nu}}$$

The following conjecture is an immediate consequence from Conjecture II.

Conjecture III.
The variations formula for $v(D)$ is given as

$$d_B v(D) = \sum_{p=1}^{n} (-1)^p \sum_{J \in \mathcal{B}, |J|=p} \theta_J \frac{1}{\prod_{\nu=1}^{p-1}(n-\nu)} v_J^*(D) \sqrt{(-1)^{p+1} \frac{B(0 \star J)}{2^p}}$$
$$+ \theta_{123\ldots n+1} \frac{(-1)^{\frac{(n-1)(n-2)}{2}}}{(n-1)!} \sqrt{\frac{(-1)^{n+1} B(0123\ \ldots\ n+1)}{2^n}}.$$

where $v_J^*(D)$ denotes the volume of the $n-p$ dimensional hypersphere $\bigcap_{j \in J} \Re S_j$ with respect to the lower dimensional standard form ϖ_J:

$$v_J^*(D) = \int_{\bigcap_{j \in J} \Re S_j} \varpi_J,$$

such that

$$\left[\frac{\varpi}{df_{j_1} \wedge \cdots \wedge df_{j_p}}\right]_{\bigcap_{j \in J} S_j} = \frac{1}{\sqrt{(-1)^{p+1} 2^p B(0 \star J)}} \varpi_J \quad (p < n).$$

for $J = \{j_1, \ldots, j_p\}$ and

$$\left[\frac{\varpi}{df_2 \wedge \cdots \wedge df_{n+1}}\right]_{\bigcap_{j=2}^{n+1} \Re S_j} = \frac{(-1)^{\frac{(n-1)(n-2)}{2}+1}}{\sqrt{(-1)^{n+1} 2^p B(0 \star J)}}.$$

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