On Nukaga's Theorem

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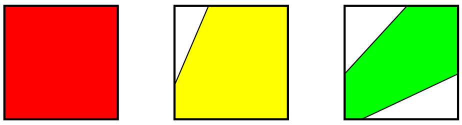
1. Nukaga's theorem on the area of lattice polygons

This is an excerpt of our survey article [2] on formulae of the area of lattice polygons and the volume of lattice polyhedra. More specifically we give a proof of a Nukaga's theorem [1] which gives a method of computing the area of lattice polygons. The area of a lattice polygon X will be denoted $S(X)$.

Definition 1.1. The **lattice frame** F is the union of the horizontal lines and vertical lines which go through lattice points. A lattice square whose edges have length 1 is called a **unit square**, and its boundary is called a **a unit frame**. The subsets obtained by cutting a lattice polygon X along the lattice frame are called **pieces** of X; *i.e.* a piece is the topological closure of a connected component of $X \setminus F$. A piece P have several edges, and the number of edges that are slant (*i.e.* neither horizontal nor vertical) is called the **type** of P. There are no pieces with type greater than 2. See the remark at the end of this section. The pictures below are examples of pieces. The types are 0, 1, and 2 from left.

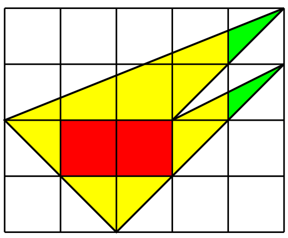


Definition 1.2. For a lattice polygon X, the number of its type 0 pieces is denoted $m(X)$, and the number of its type 1 pieces is denoted $n(X)$. The **Nukaga number** $N(X)$ of X is defined by $N(X) = m(X) + \dfrac{n(X)}{2}$.

The area of a type 0 piece is 1. On the other hand, the area varies from piece to piece when the type is 1 or 2. The Nukaga number $N(X)$ of a lattice polygon X can be thought of as an estimate of its area $S(X)$ obtained by regarding the area of type 1 pieces to be $\frac{1}{2}$ and the area of type 2 pieces to be 0, but surprisingly it turns out that $N(X)$ is exactly equal to $A(X)$!

Theorem 1.3 (Nukaga's Theorem)**.** *For each lattice polygon X, the equality $S(X) = N(X)$ holds.*

Example 1.4. In the lattice polygon below, there are two type 0 pieces (painted red), eleven type 1 pieces (painted yellow), and two type 2 pieces (painted green). Therefore its Nukaga number is $2 + \dfrac{11}{2} = 7.5$ and is equal to the area.



We first prove the theorem for lattice trainagles. For a lattice polygon with more than three vertices, we split it using a diagonal into polygons with fewer vertices and use induction. Here a diagonal is a line segment connecting two vertices that meet the boundary only at the ends. Although the fact that any polygon has a diagonal and that it can be split into triangles by diagonals is well-known, but we review its proof since we will need to extend this result to a more general situation.

[1] http://www.das.ous.ac.jp/masayuki/blogn/

Proposition 1.5. *If $n \geqq 4$, then any n-gon has a diagonal.*

Proof. Let Y be the convex hull (*i.e.* the smallest convex set containing X) of the given n-gon X. Y is also a plolygon. Choose a vector $\vec{a}$ which is not perpendicular to any of the edges of Y. Then there is a unique vertex v of X which lies farthest in the direction of the vector $\vec{a}$. Let u and w be the two vertices of X next to v. If the line segment uw lies in the interior of X except for the endpoints, then it is a diagonal of X. If not, then there exist vertices of X in the interior of $\triangle uvw$ or on the interior of the linesegment uw. Let v' be the farthest from the line uw among them. Then vv' is a diagonal of X. □

Proposition 1.6 (Additivity of Nukaga numbers)**.** *If we split a lattice polygon X into two lattice polygons A and B by a diagonal, then the equality $N(X) = N(A) + N(B)$ holds.*

Proof. Let l be the diagonal which cuts X into A and B. If l is either horizontal or vertical, there are no changes of the pieces and the equality is obvious. So let us assume that l is neither horizontal nor vertical. If l passes through the interior of a piece P of X, then P is split into pieces P_A, P_B of A, B respectively. There are three possible cases:

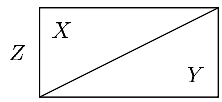
(1) if P is of type 0, then both P_A, P_B are of type 1.
(2) if P is of type 1, then one of P_A, P_B is of type 1, and the other is of type 2.
(3) if P is of type 2, both P_A, P_B are of type 2.

Theefore the desired equality holds. □

Proposition 1.7. *If a lattice polygon X has only pieces of type 0, then $S(X) = N(X)$.*

Proof. Let m be the number of the pieces of X. Then obviously $S(X) = m = N(X)$. □

Proposition 1.8. *If X is a lattice triangle with a horizontal edge and a vertical edge, then $S(X) = N(X)$.*



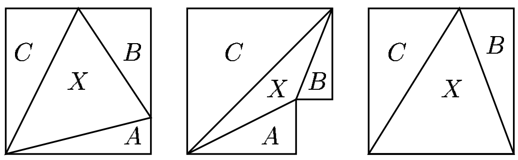
Proof. The triangle X is a right triangle. Rotate it 180 deg around the midpoint of its hypotenuse to obtain a right triangle Y, and let Z be the union of X and Y. By the symmetry of Z and the additivities of $S(\ )$ and $N(\ )$, we have the following equalities:

$$S(X) = S(Y) = \frac{S(Z)}{2}, \qquad N(X) = N(Y) = \frac{N(Z)}{2}.$$

Proposition 1.7 applies to Z, and we have $S(Z) = N(Z)$; therefore, we obtain $S(X) = N(X)$. □

Proposition 1.9. *Nukaga's theorem holds for any lattice triangle.*

Proof. For each edge γ of X, construct a right triangle outside of X whose hypotenuse is γ and the other edges are horizontal or vertical as in the picture below. Let A, B, C be the three of them. If the edge being considered is either horizontal or vertical, then the edge itself is regarded to be a degenerate right triangel and its area and its Nukaga number are both assumed to be 0. Let Y be the union of X, A, B, C.



Now we have the following equality:

$$S(Y) = S(X) + S(A) + S(B) + S(C) \ .$$

And, by the additivitiy of Nukaga number, we have the following:

$$N(Y) = N(X) + N(A) + N(B) + N(C) \ .$$

Since Y is lattice polygon with only pieces of type 0, we can apply Proposition 1.7 to Y, and we have

$$S(Y) = N(Y) \ .$$

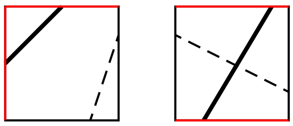
On the other hand, by Proposition 1.8, we have the equalities

$$S(A) = N(A), \qquad S(B) = N(B), \qquad S(C) = N(C) \ .$$

$S(X)$=$N(X)$ follows from these. □

Now we are ready to prove Nukaga's theorem for an arbitrary lattice polygon. Let n be the number of vertices. The $n = 3$ case was handled in Proposition 1.9. So, let us assume that Nukaga's theorem is true for $3 \leqq n \leqq k$, and let us consider the case when $n = k+1$ ($k \geqq 3$). A given lattice $(k+1)$-gon X can be split into two lattice polygons A, B by a diagonal. The numbers of vertices of A, B are both less than or equal to k, and so, by induction hypothesis, the equalities $S(A) = N(A)$, $A(B) = N(B)$ hold. On the other hand, by additivity, we have $S(X) = S(A)+S(B)$ and $N(X) = N(A) + N(B)$. Therefore we obtain $S(X) = N(X)$, and Nukaga's theorem is proved.

Remark. To see that there are no pieces of type greater than 2, first note that the slopes of the slant edges of a piece are either all positive or all negative. Suppose that a piece has a positive slope edge e and a negative slope edge f. They cannot meet the same edge of the unit square, because such a situation forces the relevant edges of the polygon to meet at a non-lattice point. The endpoints of the edge e may lie either on the neighboring edges or on the opposite edges of the unit frame as shown in the figures below.



The endpoints of f cannot lie on the red edges of the unit frame, so they have to lie on the black edges of the unit frame. But this forces the edge f to have a negative slope in the first case, and forces f to meet e at an interior point in the second case; and we have contradictions.

So, let us assume that all the slant edges have positive (or negative) slopes. Since they possibly meet only at endpoints, they are aligned from top to bottom, and, if there are more than one edge, the top two edges together with a subset of the unit frame form the boundary of a piece; so there cannot be a third edge.

2. Nukaga's theorem for lattice polygons with holes

Nukaga's theorem is known to hold also for lattice polygons with holes, and we give a proof of this in this section. Here a lattice polygon with holes is a set obtained from a lattice polygon, say Y, by removing the interiors of mutually disjoint lattice polygons lying in the interior of Y. The Nukaga number $N(X)$ of a lattice polygon X with holes is defined by

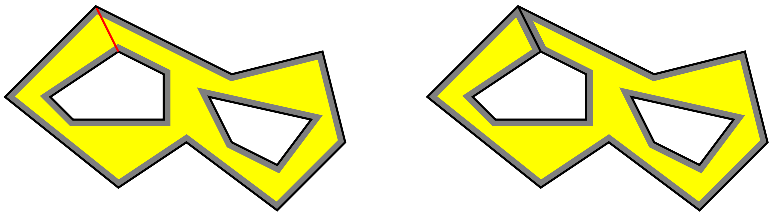
$$N(X) = m(X) + \frac{n(X)}{2},$$

where we cut X into pieces using the lattice frame as before and define $m(X)$ and $n(X)$ to be the numbers of pieces of type 0 and 1.

Theorem 2.1. *For an arbitrary lattice polygon X with holes, its area $S(X)$ is equal to the Nukaga number $N(X)$.*

The strategy of the proof is the same as in the case of poloygons. We can show that any lattice polygon X with holes has a diagonal as before. But, contrary to the polygon case, a cut of X along a diagonal may not split it into two lattice polygons with holes. There are two cases:

(1) The endpoints of the diagonal lie on the same boundary component.
(2) The endpoints of the diagonal lie on different boundary components (as below).



In the first case, X splits into two figures, and in the second case it does not. To carry out an inductive argument we need to introduce a new notion.

Definition 2.2. A “figure”[2] obtained from a lattice polygon with holes by performing cutting operation along a diagonal connecting different boundary components finitely many times is called a **lattice polygon with holes and cuts**. A cutting operation along such a diagonal connects the two boundary components as in the picture above. The **complexity** of a lattice polygon with holes and cuts X is defined to be the pair $(k(X), v(X))$ of the number of the boundary components $k(X)$ and the number of vertices $v(X)$ of X. We give the lexicographic order to the complexities. In the case of the example above, the complexity is $(3, 17)$ before the cut operation, and is $(2, 17)$ after the cut operation.

Theorem 2.3 below follows immediately from the observations below, and Theorem 2.1 is a corollary to this.

(1) Any lattice polygons with holes and cuts that has more than three vertices has a diagonal. The proof is the same as in the case of polygons.
(2) If a cut of a lattice polygon X with holes and cuts along a diagonal splits X into A and B, then $N(X) = N(A) + N(B)$ and the complexities of A and B are strictly smaller than that of X, because the numbers of boundary components are less than or eaual to that of X and the number of vertices are strictly smaller than that of X.
(3) If a cut of a lattice polygon X with holes and cuts along a diagonal does not split X, then the Nukaga number does not change and the complexitiy becomes strictly smaller because the number of boundary components decreases.
(4) A cut operation along a diagonal does not increase the number of vertices. Therefore, after a finitly many cut operations, a lattice polygon with holes and cuts is changed to lattice polygons, for which Nukaga's theorem is known to hold.

Theorem 2.3. *For any lattice polygons with holes and cuts, its area $S(X)$ is equal to its Nukaga number $N(X)$.*

References

[1] H. Nukaga, *A new theorem on the area of lattice polygons*, `http://www10.plala.or.jp/h-nukaga/math/English-math.htm|`

[2] S. Wakisaka, and H. Watanabe and J. Yoshinaga, *On the area of lattice polygons and the volume of lattice Polyhedra*, a senior thesis, Department of Applied Science, Okayama University of Science, February 2009.

[2] The two edges introduced by a cut along a diagonal are overlapping each other, and so it is not a subset of the plane. We can avoid this difficulty if we use “open” polygons (with holes) instead of the usual “closed” polygons (with holes).