Relatively Simple Chain Complexes

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Introduction

The purpose of this short note is to give a characterization of geometric module chain complexes representing the trivial element in the relative $p_X^{-1}(\epsilon)$-controlled Whitehead group $Wh(X, Y; p_X, n, \epsilon)$ of a pair (X, Y) of metric spaces. These groups were introduced in [5]. Here $p_X : W \to X$ is a given control map, and n is the restriction on the dimensions of chain complexes. For different n's, $Wh(X, Y; p_X, n, \epsilon)$ are in genereal different abelian groups, but they are all the same after stabilization (Propositions 4.6, 4.7, and the comment after 4.7 of [5]).

Main Theorem. *There exist a positive constant α such that the following holds: For any chain complex C representing the trivial element of $Wh(X, Y, p_X, n, \epsilon)$, there exist n-dimensional trivial chain complexes T, T' and an n-dimensional free $\alpha\epsilon$ chain complex F on $p_X|Y^{\alpha\epsilon}$ such that $C \oplus T$ and $F \oplus T'$ are $\alpha\epsilon$-simple isomorphic. Actually $\alpha = 500$ works.*

In the first section, we review some facts from [5] and give the proof of the main theorem. The main ingredient of the proof is the restriction of simple isomorphisms described in [3]. In the second section we discuss how this can be used in the theroy of controlled L-theory.

1. Proof of the Main Theorem

We first review the definition of the controlled Whitehead group of a pair. Let (X, Y) be a pair of metric spaces, and $p_X : W \to X$ be a continuous map. $Wh(X, Y; p_X, n, \epsilon)$ is the set of equivalence classes of n-dimensional free ϵ chain complexes on p_X which are strongly ϵ contractible over $X - Y$. The equivalence relation is generated by n-stable 40ϵ-simple equivalences away from $Y^{20\epsilon}$. The following is immediate from [5, 4.1].

Proposition 1. *If $[C] = 0 \in Wh(X, Y, p_X, n, \epsilon)$, then there exist trivial chain complexes T, T' on p_X, free 86ϵ chain complexes D, D' on $p_X|Y^{20\epsilon}$, and an 86ϵ equivalence $f : C \oplus D \oplus T \to D' \oplus T'$.*

For each i, the 86ϵ isomorphism $f_i : C_i \oplus D_i \oplus T_i \to D'_i \oplus T'_i$ is the composition of an 86ϵ deformation

$$C_i \oplus D_i \oplus T_i = G_0 \xrightarrow{g_1} G_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} G_m = D'_i \oplus T'_i \ .$$

Each g_j is either

(1) an elementary automorphism of the form $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} : \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$, or

(2) a geometric isomorphism $\mathbb{Z}[S] \to \mathbb{Z}[S']$ made up of paths with coefficient ± 1 which give a one-to-one correspondence of the basis elements S and S'.

Make a new 86ϵ deformation $G_0 \xrightarrow{g'_1} G_1 \xrightarrow{g'_2} \ldots \xrightarrow{g'_m} G_m$ as follows. Firstly, if g_j is of type (1) above, then define g'_j by the matrix $\begin{pmatrix} 1 & h' \\ 0 & 1 \end{pmatrix}$, where $h' : \mathbb{Z}[S_2] \to \mathbb{Z}[S_1]$ is obtained from h by deleting paths whose starting points are in $p_X^{-1}(Y^{(20+86\times 2)\epsilon})$. Then $g'_j = 1$ over $Y^{192\epsilon}$. Secondly, if g_j is of type (2), then let $g'_j = g_j$. Then the new deformation is geometric over $Y^{106\epsilon}$ and defines a geometric isomorphism of D with a geometric submodule of $D'_i \oplus T'_i$ lying over $Y^{106\epsilon}$. We can delete D_i and the corresponding submodule to get an 86ϵ-simple isomorphism

$$\bar{f}_i : C_i \oplus T_i \to E_i \ ,$$

where E_i is the submodule of $D'_i \oplus T'_i$ generated by the basis elements corresponding to the basis elements of $C_i \oplus T_i$. We define the boundary map $d_E : E_i \to E_{i-1}$ by the 173ϵ morphism $\bar{f}_{i-1} \circ (d_C \oplus d_T) \circ \bar{f}_i^{-1}$. The composition d_E^2 is $(86 \times 3 + 2)\epsilon$ homotopic to 0, and therefore (E, d_E) is a 173ϵ chain complex. Note that $f'_i = f_i$ outside of $Y^{(20+86\times 3)\epsilon}$; therefore, d_E is equal to $f_{i-1} \circ (d_C \oplus d_T) \circ f_i$ outside of $Y^{(20+86\times 4)\epsilon}$, and it is 172ϵ homotopic to the boundary map of T' there, since f is an 86ϵ chain map. Replace the boundary map d_E by the boundary map of T' outside of $Y^{(20+86\times 4)\epsilon}$. Now E splits as the sum of a free chain complex $F = \{E_i(Y^{(20+86\times 5)\epsilon})\}$ and a trivial chain complex $T'' = \{E_i(X - Y^{(20+86\times 5)\epsilon})\}$, and $\bar{f}$ is a $(1 + 86 \times 3)\epsilon$-simple isomorphism between $C \oplus T$ and $F \oplus T''$. This completes the proof. □

2. Controlled L-theory

In [2], Pedersen, Quinn, and Ranicki established the controlled surgery exact sequence when the local fndamental group of the control map is trivial. One of the key ingredients was the splitting of quadratic Poincaré complexes. In this section we discuss an obstruction for splitting in the general case.

Fix a control map p_B on a metric space B and fix $n \geq 3$. Let (D, ψ) be an n-dimensional quadratic Poincaré complex on p_B of radius $< \delta$ ([4][6]), and let W be a subset of B. One can construct pairs $(C \to D')$, $(C \to D'')$ such that D' and D'' lie over $B - W$ and W^γ respectively and the union $D' \cup_C D''$ is equivalentto D. C is Ranicki's algebraic boundary of D''. As it stands, it has two flaws:

(1) It may lie all over $B - W$.

(2) It may be non-trivial in degrees n and -1.

The second flaw is easy to remedy. Homologically, it is $(n-1)$-dimensional, and one can use the usual folding argument to make it into a strictly $(n-1)$-dimensional complex. Here we need $n \geq 3$. The first flaw is harder to remedy, but it is strongly contractible away from $V = W^\gamma \cap (B - W)$, and defines an element $\xi \in Wh(B - W, V; p, n-1, \gamma)$, where $p = p_B|(B - W)$. Recall from [5] that there is a constant $\kappa > 1$ and a homomorphism

$$\partial : Wh(B - W, V; p, n-1, \gamma) \longrightarrow \widetilde{K}_0(V^{\kappa\gamma}, p|V^{\kappa\gamma}, n-1, \kappa\gamma) \ .$$

The image $\partial(\xi)$ is the obstruction for splitting. Roughly speaking C is equivalent to a projective chain complex P lying over $V^{\kappa\gamma}$, and this represents $\partial(\xi)$. If this element is 0, then C is equivalent to a free chain complex lying over $V^{\kappa\gamma}$, and we get the desired splitting.

When the local fundamental group of the control map is trivial, the absolute Wh groups and $\widetilde{K}_0$ groups are stably trivial (see 8.1 and 8.2 of [5] when p=1). Therefore there is no obstruction. In fact, since the sequence

$$\ldots Wh(B - W; p, n-1, \gamma) \longrightarrow Wh(B - W, V; p, n-1, \gamma) \longrightarrow \widetilde{K}_0(V^{\kappa\gamma}, p|V^{\kappa\gamma}, n-1, \kappa\gamma)$$

is stably exact (5.3 of [5]), the relative Whitehead group also vanishes stably, and hence $[C] = 0 \in Wh(B - W, V^{\lambda\gamma}; p, n-1, \lambda\gamma)$ for some $\lambda > 0$. By the main theorem, there exist $(n-1)$-dimensional trivial chain complexes T and T' and an $(n-1)$-dimensional free $\alpha\lambda\gamma$ chain complex F on $p_B|V^{(1+\alpha)\lambda\gamma}$ such that $C \oplus T$ and $F \oplus T'$ are $\alpha\lambda\gamma$-simple isomorphic. This F can be used as the center piece for the splitting. This is essentially the proof for splitting in [2]. And from this the stability (squeezing) of controlled L-theory follows.

In the general case the spitting obstruction may not vanish. Nevertheless, the stability theorem for controlled L-groups may hold true. Several years ago Pedersen claimed that an argument like the one in [1] should work also for the controlled L-groups. To actually carry out his program, we need to study the relative torsions arising in the construction carefully to construct splitting. It seems working with the relative torsion is easier than stdying the splitting obstruction itself. As mentioned above splitting in the category of free complexes is in general impossible, so we need to squeeze a big piece at some stage.

References

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