Decomposition of Link Complements

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1. Introduction

Suppose K is a knot in S^3, and $E(K)$ denotes the exterior of K. Define a 4-manifold $M(K)$ to be $\partial(E(K) \times D^2)$. This 4-manifold has the same fundamental group as $E(K)$, but it is not aspherical. In a talk at the RIMS Conference "Methods of Transformation Group Theory", May 2006, I announced that the TOP surgery obstruction theory works for normal maps to $M(K)$. Later I extended the result to the cases of non-split links and non-split subcomplexes of a triangulation. Actually if X is a connected compact orientable 3-manifold with nonempty boundary such that the assembly map $A : H_4(X; \mathbb{L}_\bullet) \to L_4(\pi_1(X))$ is injective, then we have the same conclusion for $M = \partial(X \times D^2)$.

Then I learned from Jim Davis that, if the 3-manifold X is aspherical, the following theorem of Qayum Khan [3] can be applied to these examples to show that the surgery obstruction theory works even in the $PL = DIFF$ category for normal maps to M:

Theorem. (Khan) *Suppose M is a closed connected orientable PL 4-manifold with fundamental group π such that the assembly map*

$$A : H_4(\pi; \mathbb{L}_\bullet) \to L_4(\pi)$$

is injective, or more generally, the 2-dimensional component of its prime 2 localization

$$\kappa_2 : H_2(\pi; \mathbb{Z}_2) \to L_4(\pi)$$

is injective. Then any degree 1 normal map $(f, b) : N \to M$ with vanishing surgery obstruction in $L_4(\pi)$ is normally bordant to a homotopy equivalence $M \to M$.

So I decided to change the statement. Let X be as above. X has a handle decomposition, and a handle decomposition produces a CW-spine B of X: X is a mapping cylinder of some map $\partial X \to B$. The mapping cylinder structure induces a strong deformation retraction $q : X \to B$. Compose this with the projection $X \times D^2 \to X$ and restrict it to the boundary to get a map

$p : M = \partial(X \times D^2) \to B$. It turns out that, for any choice of the spine B, this map $p : M \to B$ is UV^1 (see [4] for the definition of UV^1-maps). So the following observation of Hegenbarth and Repovš [2] based on [5] can be applied to $p : M \to B$, if the assembly map is injective.

Theorem. (Hegenbarth-Repovš) *Let M be a closed oriented TOP 4-manifold and $p : M \to B$ be a UV^1-map to a finite CW-complex such that the assembly map*

$$A : H_4(B; \mathbb{L}_\bullet) \to L_4(\pi_1(B))$$

is injective. Then the following holds: if $(f, b) : N \to M$ is a degree 1 TOP normal map with trivial surgery obstruction in $L_4(\pi_1(M))$, then (f, b) is TOP normally bordant to a $p^{-1}(\epsilon)$-homotopy equivalence $f' : N' \to M$ for any $\epsilon > 0$. In particular (f, b) is TOP normally bordant to a homotopy equivalence.

For example, we have

Theorem. *If X is a compact connected orientable Haken 3-manifold with boundary, and B is any CW-spine of X, then there is a UV^1-map $p : M(X) \to B$, and the assembly map $A : H_4(B; \mathbb{L}_\bullet) \to L_4(\pi_1(B))$ is an isomorphism. Therefore, if $(f, b) : N \to M$ is a degree 1 TOP normal map with trivial surgery obstruction in $L_4(\pi_1(M))$, then (f, b) is TOP normally bordant to a $p^{-1}(\epsilon)$-homotopy equivalence $f' : N' \to M$ for any $\epsilon > 0$.*

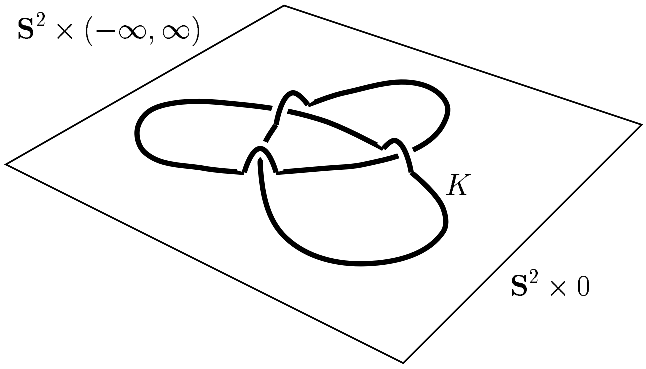
See [8] for details.

In the talk at RIMS, I used an ideal cell decomposition of link complements to construct a spine for $X = E(K)$. This is now obsolete. But it may be of some interest, so I will discuss the construction in this note.

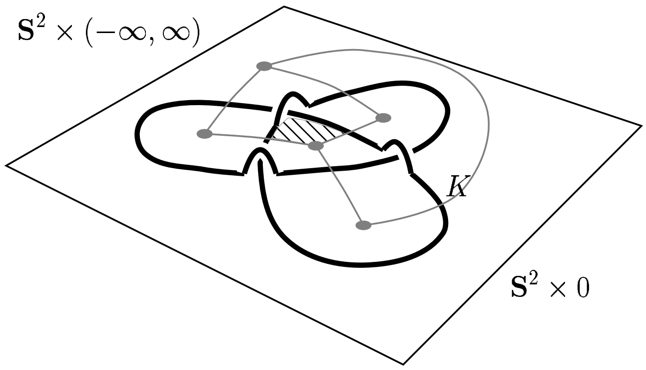
2. Ideal Cell Decomposition of Link Complements

Let K be a knot in S^3. We show that $S^3 - K$ decomposes into ideal 3-cells (= 3-cells whose vertices are removed). The following construction works equally well when K is a link.

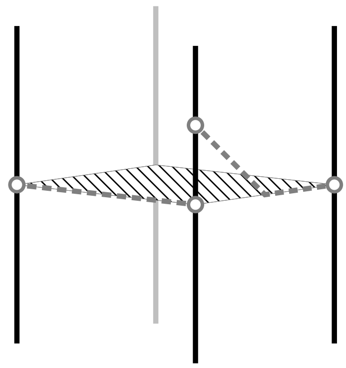
Identify S^3 with $S^2 \times (-\infty, \infty) \cup \{\pm\infty\}$, and consider a knot projection to $S^2 \times 0$, with n crossings. We assume that $n \geq 1$ and that K stays in $S^2 \times 0$ except at the overcrossings as in the next picture:



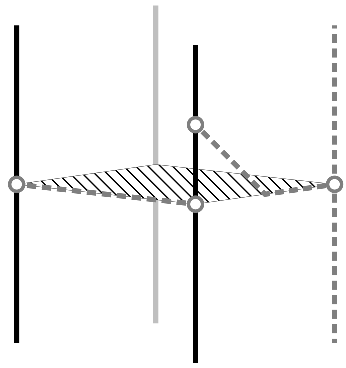
Consider the dual graph of the knot diagram:



The dual graph and the knot diagram together decompose $S^2 \times 0$ into $4n$-many quadrangles R_i. One such quadrangle is indicated in the picture above. Roughly speaking, $R_i \times (-\infty, \infty) - K$ are the desired ideal 3-cells:



Unfortunately their union is not $S^3 - K$, but $S^3 - \{\pm\infty\} - K$. So pick an intersection point of K and the dual graph, and dig tunnels from that point to $\pm\infty$ along the edges. This affects four of the 3-cells as in the picture below and gives a decomposition of $S^3 - K$ into ideal cells:



Remark. A knot/link complement has a decomposition into ideal tetrahedra. Discussions on this topic can be found in [1][6][7][9], but these are all quite technical.

The dual spine of the ideal cell decomposition can be defined in the following way: Take one point from each 1-cell; the union of these points is the dual spine of the 1-skeleton and there is a collapsing map from the 1-skeleton to the spine. Next, take one point from the interior of each 2-cell, and take the topological join of the point and the the spine of the boundary. The union of these joins is the spine of the 2-skeleton. The collapsing map of the 1-skeleton extends to the collapsing map of the 2-skeleton to the spine. Finally, take one point from the interior of each 3-cell, take the join of the point and the spine of the boundary. The union of these joins is the desired spine B, and the collapsing map of the 2-skeleton extends to a collapsing map $q : S^3 - K \to B$.

References

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