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On the structure of the first homology of the group of equivariant diffeomorphisms of manifolds with smooth torus actions

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§1. Preliminaries

In the previous paper [AF5] we calculate the first homology of the group of equivariant diffeomorphisms of representation spaces of finite groups, and apply to the case of the smooth orbifolds. In this talk we shall consider the case of smooth manifolds with smooth torus actions. First we describe the previous results.

For a finite group G, let M be a smooth connected G-manifold. Let $\mathcal{D}_G(M)$ denote the group of G-equivariant smooth diffeomorphisms of M which are G-isotopic to the identity through the isotopies with compact support. We recall the result the case where M is a finite dimensional G-module V. Let V^G be the subspace of the fixed point set of V. Let $Aut_G(V)$ denote the set of G-invariant automorphisms of V and $Aut_G(V)_0$ the identitiy component of $Aut_G(V)$. Then we have the following.

Theorem 1.1
(1) *If* $\dim V^G > 0$, *then* $\mathcal{D}_G(V)$ *is perfect.*
(2) *If* $\dim V^G = 0$, *then* $H_1(\mathcal{D}_G(V)) \cong H_1(Aut_G(V)_0)$.

We can decompose $V = \oplus_{i=1}^{d} k_i V_i$, where V_i runs over the inequivalent irreducible representation space of G and k_i is a positive integer.
$End_G(V_i)$: the set of G-invariant endmorphisms of V_i
Then $\dim End_G(V_i) = 1, 2$ or 4.

Corollary 1.2

$$H_1(\mathcal{D}_G(V)) \cong \mathbf{R}^d \times \overbrace{U(1) \times \cdots \times U(1)}^{d_2},$$

where d_2 is the number of V_i with $\dim End_G(V_i) = 2$.

If M is a smooth orbifolds, then $p \in M$ is said to be an isolated singular point of M if there exists a local chart (U_i, ϕ_i) around p such that $\tilde{p}$ is an isolated fixed point of $(\Gamma_i, \tilde{U}_i)$ with $\pi_i(\tilde{p}) = p$ Let $\phi_i : U_i \to \tilde{U}_i/\Gamma_i,\ \pi_i : \tilde{U}_i \to U_i$ be the canonical projection. Then we have.

Theorem 1.3 *If a smooth orbifold M has $\{p_1, ..., p_k\}$ as the isolated singular point set. Let $(\Gamma_i,\, V_{p_i})$ $(1 \leq i \leq k)$ be the representaion space associated to the isolated singular points. Then*

$$H_1(\mathcal{D}(M)) \cong H_1(Aut_{\Gamma_{p_1}}(V_{p_1})_0) \times \cdots \times H_1(Aut_{\Gamma_{p_k}}(V_{p_k})_0).$$

§2. Orbit preserving G-diffeomorphisms

Let M be a connected closed G-manifold and B be the orbit space. Then the natural projection $\pi : M \to B$ induces the group homomorphism $P : D_G(M) \to D(B)$.

Let (H_0) be the principal orbit type of M and let $\{(H_i)|\ 0 \leq i \leq \ell\}$ be the other G-orbit types of M. Put

$$M_i = \{p \in M|\ (G_p) = (H_i)\},$$
$$W_i = N(H_i)/H_i, \quad F_i = M_i^{H_i}, \quad B_i = F_i/W_i.$$

Then $q_i : F_i \to B_i$ is the principal W_i-bundle and $\pi_i : M_i \to B_i$ is the associated fiber bundle with the fiber G/H_i. Thus

$$M_i \cong G/H_i \times_{W_i} F_i.$$

Let $\{U_{i,\alpha}\}_{(i,\alpha)\in\Lambda_i}$ be an open covering of B_i such that there exists a local section $\sigma_{i,\alpha}$ on $U_{i,\alpha}$ of q_i. Then we have the transition functions $\{\varphi_{i,\alpha\beta}\}$ of the principal W_i-bundle q_i given by

$$\varphi_{i,\alpha\beta}(b)\sigma_{i,\beta}(b) = \sigma_{i,\alpha}(b) \quad (b \in U_{i,\alpha} \cap U_{i,\beta}).$$

We shall study the group $KerP$ which coincides with the group of orbit preserving equivariant diffeomorphisms of M.

Let $h \in KerP$. Then h induces the bundle isomorphisms h_i of $\pi_i \quad (0 \leq i \leq \ell)$. We have smooth maps
$s_{i,\alpha} : U_{i,\alpha} \to W_i \quad ((i,\alpha) \in \Lambda_i)$ satisfying

$$h_i(\sigma_{i,\alpha}(b)) = s_{i,\alpha}(b)\sigma_{i,\alpha}(b) \quad (b \in U_{i,\alpha}).$$

It follows that, for $b \in U_{i,\alpha} \cap U_{i,\beta}$, we have

$$s_{i,\alpha}(b)\varphi_{i,\alpha\beta}(b) = \varphi_{i,\alpha\beta}(b)s_{i,\beta}(b).$$

Then h_i corresponds to the collections $S_i(h) = \{s_{i,\alpha}\}_{(i,\alpha)\in\Lambda_i}$ satisfying

$$(3.1) \qquad s_{i,\alpha} \cdot \varphi_{i,\alpha\beta} = \varphi_{i,\alpha\beta} \cdot s_{i,\beta} \quad \text{on } U_{i,\alpha} \cap U_{i,\beta}.$$

Put $\mathcal{S}(h) = \{\mathcal{S}_i(h) |\ 0 \leq i \leq \ell\}$.

Definition 2.1 *A cocycle of an orbit preserving G-diffeomorphism of M is a collection $\mathcal{S} = \{\mathcal{S}_i |\ 1 \leq i \leq \ell\}$ such that*

(1) *$\mathcal{S}_i$ is the set of smooth functions $\{s_{i,\alpha}\}_{(i,\alpha)\in\Lambda_i}$ from $U_{i,\alpha}$ to W_i satisfying the condition* (3.1)*,*

(2) *Let V be a slice at $p \in M$. Then $\psi_V : V \to G \cdot V$ is a smooth map. Here the map ψ_V is given by $\psi_V(v) = s_{i,\alpha}(\pi(v)) \cdot v$ if $\pi(v) \in U_{i,\alpha}$.*

By definition $\mathcal{S}(h)$ is a cocycle of an orbit preserving G-diffeomorphism of M for each $h \in Ker\ P$. Let $\mathcal{S} = \{\mathcal{S}_i |\ 1 \leq i \leq \ell\}$ be a cocycle of an orbit preserving G-diffeomorphism of M. Then we have a map $h : M \to M$ defined by

$$h([gH_i,\ x]) = [g \cdot s_{i,\alpha}(q_i(x)),\ x]$$

for $g \in G,\ x \in q_i^{-1}(U_{i,\alpha})$.

Lemma 2.2 *$h \in Ker\ P$.*

Let $\mathcal{S}(M)$ be the set of all cocycles of an orbit preserving G-diffeomorphism of M.

Corollary 2.3 *Let $S : Ker\ P \to \mathcal{S}(M)$ be a map which assigns each $h \in Ker\ P$ to $\mathcal{S}(h)$. Then S is bijective.*

Remark 2.4 *M. Davis* [DA] *introduced the $\mathcal{G}$-normal system of smooth G-manifolds to classify the set of G-manifolds and suggested that the orbit preserving G-diffeomorphisms are expressed by using this system. We can express them more easy way by using the above cocycles.*

Let $M(m,\mathbf{R})$ denote the set of all $m\times m$-matrices. Let $f:\mathbf{R}^n\backslash\mathbf{R}^m\to M(m,\mathbf{R})$ be a smooth map. Define a map $\hat{f}:\mathbf{R}^n\to\mathbf{R}^m$ by

$$\hat{f}(x,y)=\begin{cases} f(x,y)x & x\in\mathbf{R}^m, y\in\mathbf{R}^{n-m}\ (y\neq 0)\\ 0 & x\in\mathbf{R}^m, y=0.\end{cases}$$

Lemma 2.5 *If $\hat{f}$ is a smooth map, then f can be extended to a smooth map $F:\mathbf{R}^n\to M(m,\mathbf{R})$.*

If H_i is a subgroup of H_j, let $r_{i,j}:G/H_i\to G/H_j$ be the canonical projection.

Corollary 2.6 *Let $h\in KerP$. Assume that H_i is a subgroup of H_j and $U_{j,\beta}$ is contained in the closure $\overline{U_{i,\alpha}}$ of $U_{i,\alpha}$. Then $s_{i,\alpha}$ is extended to a map $\bar{s}_{i,\alpha}$ on $\overline{U_{i,\alpha}}$ satisfying*

$$r_{i,j}(\bar{s}_{i,\alpha}(b))=s_{j,\beta}(b)\quad \text{for } b\in U_{j,\beta}.$$

Example 2.7

(1) *Assume that $q_0:F_0\to B_0$ is a trivial W_0-bundle. It follows from Corollary 2.6 that each $h\in Ker\,P$ corresponds to a smooth map $s:B\to W_0$ satisfying the following condition. If $b\in B_i$ $(1\leq i\leq\ell)$, then*

$$s(b)\in r_{0i}^{-1}(W_i)=(N(H_0)\cap N(H_i))/H_0.$$

(2) *If M is a $(2n-1)$-dimensional homotopy sphere with a smooth $O(n)$-action with $B=D^2$. Then*

$$H_0=O(n-2),\ H_1=O(n-1),\quad W_0=O(2),\ W_1=O(1).$$

Thus $Ker\,P$ is one to one correspondence with the smooth maps from $s:D^2\to SO(2)$ such that $s(\partial D^2)=1$.

Example 2.8 *Let M be a $2n$-dimensional torus manifold with the local standard action. Note that $N(H)/H=T^n/H\cong H^c$ for each toral subgroup H in T^n, where H^c is the complementary torus subgroup of H. Let $\mathcal{F}(M)$ be the set of smooth maps $s:B\to T^n$ such that $s(\pi(p))\in(T^n_p)^c$ for each $p\in M$. Then $Ker\,P$ is isomorphic to $\mathcal{F}(M)$ as a group.*

Let V be a slice at $p \in M$ with $\pi(p) \in U_{i,\alpha}$. Let $P_V : D_G(G \cdot V) \to D(G \cdot V/G)$ be the natural group homomorphism. Note that $\dim U_{i,\alpha} = \dim F_i/W_i$.

Proposition 2.9
If $\dim U_{i,\alpha} > 0$, then

$$Ker\ P_V \subset [Ker\ P_V,\ D_G(G \cdot V)].$$

§3. Appliction to torus actions

M: $2n$-dimensional torus manifold with local standard action
Let p be a fixed point of M. Let $Aut_{T^n}(T_p(M))$ denote the set of T^n-equivariant linear automorphisms of $T_p(M)$. We have a group homomorphism $\Phi_p : D_G(M) \to Aut_{T^n}(T_p(M))_0$ assighning each $h \in D_G(M)$ to the differential dh_p at p. Set the homomorphism

$$\Phi = \{\Phi_p\} : D_G(M) \to \prod_{p \in F(M)} Aut_{T^n}(T_p(M))_0.$$

Here $F(M)$ is the fixed point set of M.

Since $T_p(M)$ is the standard representation space of T^n, $Aut_{T^n}(T_p(M))_0$ is isomorphic to $(\mathbf{C}^*)^n$. Define the group homomorphism

$$\Theta = (P, \Phi) : D_{T^n}(M) \to D(M/T^n) \times \prod_{p \in F(M)} (\mathbf{C}^*)^n.$$

Since M/T^n has a structure of an orbifold and is locally diffeomorphic to $\tilde{\mathbf{R}}^n/\mathbf{Z}_2{}^n$ around the isolated singular point of M/T^n, where $\tilde{\mathbf{R}}$ is the non-trivial 1-dimensional $\mathbf{Z}_2$-module. By Corollary 1.2 we have.

Corollary 3.1 *If M has m fixed points, then $H_1(D(M/T^n)) \cong \mathbf{R}^{mn}$.*

Proposition 3.2 *$Ker\,\Theta$ is contained in the commutator subgroup of $D_{T^n}(M)$.*

There exists the following group exact sequence.

$$Ker\,\Theta/[Ker\,\Theta,\ D_{T^n}(M)] \xrightarrow{\iota_*} H_1(D_{T^n}(M)) \xrightarrow{\Theta_*} H_1(D_{T^n}(M) \times \prod_{p \in F_n} (\mathbf{C}^*)^n) \to 1.$$

By Proposition 3.2 $\iota_* = 0$ and Θ_* is isomorphic. From Corollary 3.1 we have.

Theorem 3.3 *Let M be a $2n$-dimensional torus manifold with local standard action. If M has m fixed points, then*

$$H_1(D_{T^n}(M) \cong (\mathbf{R} \times \mathbf{C}^*)^{mn}.$$

In order to prove Proposition 3.2, we need the following lemmas.

Lemma 3.4 (*Fragmentation lemma*)
Let M be a smooth G-manifold and $\{V_i|\ 1 \leq i \leq n\}$ be a G-invariant finite open covering of M. Let N be an open neighborhood of the identity in $D_G(M)$. Then there exists an open neighborhood $N_0 \subset N$ of the identity with the following properties: For any $f \in N_0$, there exist $f_i \in N$, $1 \leq i \leq n$, such that

a) *f_i is G-isotopic to the identity through an equivariant C^∞ isotopy whose support is contained in V_i, and*

b) *$f = f_n \circ f_{n-1} \circ \cdots \circ f_1$.*

Theorem 3.5 (Bierstone [BI1], Schwarz [SC2])
Let N be a smooth G-manifold. Then each smooth isotopy on N/G with compact support lifts to a smooth G-equivariant isotopy on N.

Proof of Proposition 3.2 :
Combining Fragmentation lemma and Theorem 3.5, the proof of Proposition 3.2 is reduced to the case $M = T^n \cdot V$, where V is a slice of a point $p \in M$. Then $M = T^n \cdot V \cong T^n \times_{H_p} V$. Let $P_V : T^n \cdot V \to V/H_p$ be the natural projection. Then it is enough to prove that

$$Ker\, \Theta \subset [Ker\, P_V,\ D_{T^n}(T^n \cdot V)].$$

By Proposition 2.9, if p is not a isolated fixed point of T^n, then Proposition 3.2 is valid. Assume that p is a isolated fixed point of M. Let $T^n \cdot V = V$ and Θ be the compsosition

$$\Theta = (P_V, \Phi_p) : D_{T^n}(V) \to D(V/T^n) \times Aut_{T^n}(T_p(M))_0 \cong D(V/T^n) \times (\mathbf{C}^*)^n.$$

Let $h \in Ker\ \Theta$. Then h is an orbit preserving equivariant diffeomorphism of V with compact support and $dh_p = 0$. From the linearlization theorem by Sternberg we can prove that $h \in [Ker\, \Phi_p,\ D_{T^n}(V)]$ by using the contraction map.

§4. S^1-action on 3-manifolds

Let M be a smooth closed 3-manifold with a smooth $U(1)$-action. Let n_1 and $n_2 = m - n_1$ be the numbers of the exceptional orbits $U(1) \cdot p$ with $U(1)_p = \mathbf{Z}_2$ and $U(1)_p = \mathbf{Z}_k$ $(k \geq 3)$, respectively. The we have the following.

Proposition 4.1

$$H_1(\mathcal{D}(M/U(1))) \cong \overbrace{\mathbf{R} \times \cdots \times \mathbf{R}}^{n_1+n_2} \times \overbrace{U(1) \times \cdots \times U(1)}^{n_2}.$$

Theorem 4.2

$$H_1(\mathcal{D}_{U(1)}(M)) \cong \overbrace{\mathbf{R} \times \cdots \times \mathbf{R}}^{n_1+n_2} \times \overbrace{U(1) \times \cdots \times U(1)}^{n_1+2n_2}.$$

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