ON THE CONNES-CONSANI-SOULE TYPE ZETA FUNCTION FOR $\mathbb{F}_1$-SCHEMES

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PART I

§1, Background....

Manin (Denninger, Kurokawa, Kapranov-Smironov...) suggested $\exists$ a curve $C = \overline{\operatorname{Spec}\mathbb{Z}}$ "*defined over*" $\mathbb{F}_1$ whose "*zeta function*" $\zeta_C(s)$ is the complete Riemann zeta function $\zeta_{\mathbb{Q}}(s)$:

$$\zeta_C(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) =: \zeta_{\mathbb{Q}}(s)$$

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p:\ \textbf{primes}}\frac{1}{1-p^{-s}} \qquad (\Re(s) > 1)$$

$$\Gamma(s) := \int_0^{\infty} x^{s-1}e^{-x}dx \qquad (\Re(s) > 0)$$

Furthermore, they suggested the Riemann hypothesis may be solved in a fashion similar to the Weil conjecture for smooth schemes defined over a finite field $\mathbb{F}_q$ ($q \to 1$) .

Kato, Kurokawa-Ochiai-Wakayama, Deitmar, Toen-Vaquie, Haran, Durov, Soulé, Connes-Conani... proposed some similar notions of $\mathbb{F}_1$-schemes.

(commutative rings $\to$ commutative monoid with 0)

Deitmar-Kurokawa-Koyama, Kurokawa-Ochiai, Soule, Connes-Consani proposed different kinds of zeta functions of $\mathbb{F}_1$-schemes.

$$\left|X\left(\mathbb{F}_{q^n}\right)\right| \to \left|Y\left(\mathbb{Z}/n\right)\right|$$
$$\left|(\operatorname{Spec} R)\left(\mathbb{F}_{q^n}\right)\right| = \left|\operatorname{Hom}_{\mathbf{rings}}\left(R, \mathbb{F}_{q^n}\right)\right|$$
$$\left|(\operatorname{Spec} A)\left(\mathbb{Z}/n\right)\right| = \left|\operatorname{Hom}_{\mathbf{groups}}\left(A, \mathbb{Z}/n\right)\right|$$

§2, The plan of the paper

PART I: (Soule, Connes-Consani) Rough idea of the $\mathbb{F}_1$-schem and the zeta function for some class of $\mathbb{F}_1$-scheme.
PART II: (Connes-Consani) A similarity between "the counting functions" of the "hypothetical $C = \overline{\operatorname{Spec}\mathbb{Z}}$ ", and an irreducible smooth projective algebraic curve defined over a finite field.
PART III: (Connes-Consani, Deitmar-Kurokawa-Koyama, M) $\mathbb{F}_1$-zeta functions of Deitmar-Kurokawa-Koyama and Kurokawa-Ochiai, some invariants for finite abelian groups, and an expression of the Soule-Connes-Consani zeta function for general, not necessarily torsion free, Noetherian $\mathbb{F}_1$-schemes.

§3, A rough idea of the $\mathbb{F}_1$-scheme
There is a very general theory of $\mathbb{F}_1$-scheme, e.g.

[CC] Alain Connes and Caterina Consani,
"Schemes over F_1 and zeta functions", ArXiv0903.2024

which employs the functor-of-points philosophy for the category $\mathfrak{Ring} \cup_{\mathbf{adjoint}} \mathfrak{Monoid}_0$.

$$\mathfrak{Monoid}_0 \rightleftarrows \mathfrak{Ring}$$
$$\operatorname{Hom}_{\mathfrak{Ring}}(\mathbb{Z}[M], R) \cong \operatorname{Hom}_{\mathfrak{Monoid}_0}(M, R)$$
$$M \mapsto \mathbb{Z}[M] \quad \left(0_M \mapsto 0_{\mathbb{Z}[M]}\right)$$
$$R \;\leftarrow\!\!\!\!+\; R$$

$$\mathrm{Ob}\left(\mathfrak{Ring} \cup_{\textbf{adjoint}} \mathfrak{Monoid}_0\right) = \mathrm{Ob}\left(\mathfrak{Ring}\right) \coprod \mathrm{Ob}\left(\mathfrak{Monoid}_0\right)$$

$$\mathrm{Hom}_{\mathfrak{Ring}\cup_{\textbf{adjoint}}\mathfrak{Monoid}_0}(X,Y) = \begin{cases} \mathrm{Hom}_{\mathfrak{Ring}}(X,Y) & \textbf{if } X,Y \in \mathfrak{Ring} \\ \mathrm{Hom}_{\mathfrak{Monoid}_0}(X,Y) & \textbf{if } X,Y \in \mathfrak{Monoid}_0 \\ \emptyset & \textbf{if } X \in \mathfrak{Ring}, Y \in \mathfrak{Monoid}_0 \\ \mathrm{Hom}_{\mathfrak{Ring}}(\mathbb{Z}[X],Y) & \textbf{if } X \in \mathfrak{Monoid}_0, Y \in \mathfrak{Ring} \\ \cong \mathrm{Hom}_{\mathfrak{Monoid}_0}(X,Y) & \end{cases}$$

A $\underline{\mathbb{F}_1\text{-functor}}$ is by definition a functor

$$\mathfrak{Ring} \cup_{\textbf{adjoint}} \mathfrak{Monoid}_0 \to \mathfrak{Set},$$

which is equivalent to the following data:

- $\underline{X} : \mathfrak{Monoid}_0 \to \mathfrak{Set}$
- $X_{\mathbb{Z}} : \mathfrak{Ring} \to \mathfrak{Set}$
- $e : \underline{X} \to X_{\mathbb{Z}} \circ \beta$, **where** $\beta : \mathfrak{Monoid}_0 \to \mathfrak{Ring},\ M \mapsto \mathbb{Z}[M]\ \left(0_M \mapsto 0_{\mathbb{Z}[M]}\right)$
  ($\iff$ $e : \underline{X} \circ \beta^* \to X_{\mathbb{Z}}$, **where** $\beta^* : \mathfrak{Ring} \to \mathfrak{Monoid}_0,\ R \mapsto R$)

Connes-Consani defined a $\underline{\mathbb{F}_1\text{-scheme}}$ $\mathcal{X}$ to be a $\mathbb{F}_1$-functor $\mathfrak{Ring} \cup_{\textbf{adjoint}} \mathfrak{Monoid}_0 \to \mathfrak{Set}$ s.t.

- **$X_{\mathbb{Z}}$, its restriction to $\mathfrak{Ring}$, is a $\mathbb{Z}$-scheme.**
- **$\underline{X}$, its restriction to $\mathfrak{Monoid}_0$, is a $\underline{\mathfrak{M}_0\text{-scheme}}$.**
- **the natural transformation $e : \underline{X} \circ \beta^* \to X_{\mathbb{Z}}$, associated to a field, is a bijection of sets. In particular,**

$$\begin{array}{ccccccccc} X_{\mathbb{Z}}(\mathbb{F}_q) & \xleftarrow[e]{\cong} & (\underline{X} \circ \beta^*)(\mathbb{F}_q) & = & \underline{X}(\mathbb{F}_q) & = & \underline{X}(\mathbb{F}_1[\mathbb{Z}/(q-1)]) & = & \underline{X}\left(\mathbb{F}_{1^{(q-1)}}\right) \\ \| & & & & & & & & \| \\ \mathrm{Hom}_{\mathbb{Z}-\mathbf{sch}}(\mathrm{Spec}\,\mathbb{F}_q, X_{\mathbb{Z}}) & & & \longrightarrow & & & & & \mathrm{Hom}_{\mathfrak{M}_0-\mathbf{sch}}(\mathrm{Spec}\,\mathbb{F}_{1^{(q-1)}}, \underline{X}) \end{array}$$

Here, $(\lim_{q\to 1} \mathbb{F}_{q^n} \simeq)\ \mathbb{F}_{1^n} := \mathbb{F}_1[\mathbb{Z}/n\mathbb{Z}] := \mathbb{Z}/n\mathbb{Z} \cup \{0\}$

For $\underline{\text{Noetherian } \mathbb{F}_1\text{-scheme}}$ $\mathcal{X}$ (both $X_{\mathbb{Z}}$ and $\underline{X}$ admit a finite open cover by Noetherian affine representables in each category),

(1) **there are just finitely many "points" in $\underline{X}$.**
(2) **at each such a point $x \in \underline{X}$, the "residue field" $\kappa(x) = \mathbb{F}_1\left[\mathcal{O}_x^{\times}\right]$ is a finitely generated abelian group $\mathcal{O}_x^{\times} = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z}$**

(3) $\underline{X}(\mathbb{F}_{1^n}) = \mathrm{Hom}_{\mathcal{M}_0-\mathrm{sch}}(\mathrm{Spec}\,\mathbb{F}_{1^n}, \underline{X}) = \coprod_{x\in\underline{X}} \mathrm{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^\times, \mathbb{Z}/n\mathbb{Z})$, **where**

$$(\lim_{q\to 1} \mathbb{F}_{q^n} \simeq)\ \mathbb{F}_{1^n} := \mathbb{F}_1[\mathbb{Z}/n\mathbb{Z}] := \mathbb{Z}/n\mathbb{Z} \cup \{0\}$$

In general, a $\mathcal{M}_0$-scheme $\underline{\mathbf{X}}$ is caleed <u>torsion free</u> **, if $\mathcal{O}_x^\times$ is a torsion free abelian group for any $x \in \underline{X}$.**

§4, The zeta function for some class of $\mathbb{F}_1$-scheme by Soule, Connes-Consani

(Deitmar, Cones-Consani) For a Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$ with $\underline{X}$ torsion free, $\exists N(u+1) \in \mathbb{Z}_{\geqq}[u]$ s.t.

$$\left|\underline{X}\left(\mathbb{F}_{1^n}\right)\right| = N(n+1), \quad \forall n \in \mathbb{N}$$

In particular,

$$\left|X_{\mathbb{Z}}\left(\mathbb{F}_q\right)\right| = \left|\underline{X}\left(\mathbb{F}_{1^{(q-1)}}\right)\right| = N(q), \quad \forall q, \text{ a prime power}$$

$$\because)\left|\underline{X}\left(\mathbb{F}_{1^n}\right)\right| = \sum_{x\in\underline{X}} \left|\mathrm{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^\times, \mathbb{Z}/n\mathbb{Z})\right| = \sum_{x\in\underline{X}} \left|\mathrm{Hom}_{\mathfrak{Ab}}(\mathbb{Z}^{n(x)}, \mathbb{Z}/n\mathbb{Z})\right| = \sum_{x\in\underline{X}} n^{n(x)}$$

So, set $N(u+1) := \sum_{x\in\underline{X}} u^{n(x)} \in \mathbb{Z}_{\geqq}[u]$. □

So, we are naively lead to define the zeta faunction of $\mathcal{X}$ as the Hasse zeta function of $X_{\mathbb{Z}}$, as our first attempt:

$$\zeta\left(s, X_{\mathbb{Z}}\right) := \prod_{p:\text{prime}} \zeta\left(s, X_{\mathbb{Z}}/\mathbb{F}_p\right),$$

where $\zeta\left(s, X_{\mathbb{Z}}/\mathbb{F}_p\right)$ is the congruence zeta function

$$\zeta\left(s, X_{\mathbb{Z}}/\mathbb{F}_p\right) = \exp\left(\sum_{m=1}^{\infty} \frac{\left|X_{\mathbb{Z}}\left(\mathbb{F}_{p^m}\right)\right|}{m} p^{-ms}\right)$$

Bad News. *(Soule, Deitmar, Kurokawa) When* $N(v) = N(u+1) = \sum_{x\in\underline{X}} u^{n(x)} = \sum_{x\in\underline{X}} u^{n(x)} = \sum_{x\in\underline{X}} (v-1)^{n(x)} = \sum_{k=0}^{d} a_k v^k$, $(a_k \in \mathbb{Z})$,

$$\zeta\left(s, X_{\mathbb{Z}}\right) = \prod_{k=0}^{d} \zeta(s-k)^{a_k}, \quad \zeta\left(s, X_{\mathbb{Z}}/\mathbb{F}_p\right) = \prod_{k=0}^{d} \left(1-p^{k-s}\right)^{-a_k}$$

(Too complicated and redundanct for such simple (comparing with $C = \overline{\mathrm{Spec}\,\mathbb{Z}}$) $\mathcal{X}$!)

Good News. *(Soule, predicted by Manin, corected by Kurokawa) For a Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$ with $\underline{X}$ torsion free (so, $\exists N(v) = \sum_{k=0}^{d} a_k v^k \in \mathbb{Z}[v]$ s.t. $\left|\underline{X}(\mathbb{F}_{1^n})\right| = N(n+1), \quad \forall n \in \mathbb{N}$) ,*

$$\zeta_{\mathcal{X}}(s) := \lim_{p\to 1} \zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p)(p-1)^{N(1)} = \prod_{k=0}^{d}(s-k)^{-a_k}$$

(Kurokawa) In an ideal case, for the l-th Betti number b_l of $X_{\mathbb{Z}}/\mathbb{F}_p$,

$$\prod_{k=0}^{d}\left(1-p^k p^{-s}\right)^{-a_k} = \zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p) \overset{\because)\,\text{Weil conj.}}{=} \prod_{l=0}^{m}\left(\prod_{j=1}^{b_l}(1-\alpha_{l,j}p^{-s})\right)^{(-1)^{l+1}}$$

$$(|\alpha_{l,j}| = p^{l/2})$$

$$= \prod_{l=0}^{m}\left(1-p^{l/2}p^{-s}\right)^{-(-1)^l b_l} \qquad \Longrightarrow \qquad b_l = \begin{cases} a_{l/2} & l: \text{ even} \\ 0 & l: \text{ odd} \end{cases}$$

Thus, $N(v) = \sum_{k=0}^{d} a_k \in \mathbb{Z}_{\geq 0}[v]$, $\quad N(1) = \sum_{k=0}^{d} a_k = \sum_{l=0}^{m}(-1)^l b_l$, the Euler characterisitc of $X_{\mathbb{Z}}/\mathbb{F}_p$.

Example (Toric variety)

fan picture: **lattice N:: a group $N \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$.**

convex cone σ in $N_{\mathbb{R}}$:: a convex subset $\sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}_{\geqq 0}\sigma = \sigma$.

A convex cone σ is called:

polyhedral:: if it is finitely generated,

rational:: if the generators lie in the lattice N,

proper:: if it does not contain a non-zero sub vector space of $N_{\mathbb{R}}$.

fan $\triangle$ in N:: a finite collection $\triangle$ of proper convex rational polyhedral cones σ in the real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ s.t.

- **every face of a cone in $\triangle$ is in $\triangle$,**
- **the intersection of two cones in $\triangle$ is a face of each. (Here zero is considered a face of every cone.)**

monoid picture: **dual lattice M:: $M := \mathrm{Hom}(N, \mathbb{Z})$**

dual cone $\check{\sigma}$ in $M_{\mathbb{R}} := \mathrm{Hom}(N, \mathbb{R})$:: $\check{\sigma} := \{\alpha \in M_{\mathbb{R}} \mid \alpha(\sigma) \geqq 0\}$

monoid A_σ:: $A_\sigma := \check{\sigma} \cap M$; face inclusion $\tau \subsetneqq \sigma \implies A_\tau \supsetneqq A_\sigma$

affine open U_σ:: $U_\sigma := \mathrm{Spec}(\mathbb{C}[A_\sigma]) = \mathrm{Spec}(\mathbb{C}[\check{\sigma} \cap M])$

toriv variety $X_\triangle$:: $X_\triangle$ is obtained by glueing $U_\sigma = \mathrm{Spec}(\mathbb{C}[A_\sigma])$ along $U_\tau \to U_\sigma$ for each face inclusion $\tau \subsetneqq \sigma$

This construction allows us to define a $\mathbb{F}_1$-scheme $\overline{\mathcal{X}_\triangle}$.

(Deitmar) GIven a fan $\triangle \subseteqq N \cong \mathbb{Z}^n$, for $j = 0, 1, 2, \ldots, n$, let
f_j be the number of cones in $\triangle$ of dimsension j, and set $c_j := \sum_{k=j}^{n} f_{n-k}(-1)^{k+j}\binom{k}{j}$ Then,

$$\zeta_{\mathcal{X}_\triangle}(s) = \prod_{j=0}^{n}(s-j)^{-c_j}$$

$$\because) N(u+1) = \sum_{x \in \underline{X}_\triangle} u^{n(x)} = \sum_{k=0}^{n} f_k u^{n-k} = \sum_{k=0}^{n} f_{n-k} u^k$$

$$\implies \quad N(v) = \sum_{k=0}^{n} f_{n-k}(v-1)^k = \sum_{k=0}^{n} f_{n-k} \sum_{j=0}^{k} \binom{k}{j} x^j (-1)^{k-j}$$

$$= \sum_{j=0}^{n} x^j \sum_{k=j}^{n} f_{n-k} \binom{k}{j} (-1)^{k-j} \quad \square$$

Question Can we define $\zeta_{\mathcal{X}}(s)$ for more general $\mathbb{F}_1$-scheme $\mathcal{X}$?

Good News. *Connes-Consani proposed two solutions.*

Solution 1: This proceeds as follows:

- **Extend "canonically" $N(n+1) := \left|\underline{X}\left(\mathbb{F}_{1^n}\right)\right|, \ (n \in \mathbb{N})$ to**

  $N : \mathbb{R}_{\geqq 0} \to \mathbb{R}$, **s.t.** $\exists C > 0, \exists k \in \mathbb{N},$ **s.t.** $\left|N(u)\right| \leqq Cu^k$

- **As far as zero points and poles concerns, can characterize $\zeta_N(s)$ (which is supposed to be $\zeta_{\mathcal{X}}(s)$) by**

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u) u^{-s} d^*u, \qquad d^*u = du/u$$

$$\because) \quad \zeta_N(s) := \lim_{q\to 1} \exp\left(\sum_{r\geqq 1} N(q^r)\frac{(q^{-s})^r}{r}\right)(q-1)^{\chi}\,, \quad \chi = N(1)$$

$$\begin{aligned}
\Longrightarrow \qquad \frac{\partial_s \zeta_N(s)}{\zeta_N(s)} &= \lim_{q\to 1} \partial_s \left(\sum_{r\geqq 1} N(q^r)\frac{(q^{-s})^r}{r}\right) \\
&= \lim_{q\to 1} \partial_s \left(\sum_{r\geqq 1} N(q^r)\frac{(q^{-r})^s}{r}\right) \\
&= \lim_{q\to 1} \sum_{r\geqq 1} N(q^r)\frac{(q^{-r})^s}{r}\log\left(q^{-r}\right) \\
&= -\lim_{q\to 1} \sum_{r\geqq 1} N(q^r)(q^r)^{-s}\log q \\
&= -\lim_{q\to 1} \sum_{r\geqq 1} N(q^r)(q^r)^{-s}\left(\log\left(q^r\right) - \log\left(q^{r-1}\right)\right) \\
&= -\int_1^{\infty} N(u)u^{-s} d\log u = -\int_1^{\infty} N(u)u^{-s} du/u
\end{aligned}$$

Solution 2: Rather than extending to $N : \mathbb{R}_{\geq 0} \to \mathbb{R}$, consider $\zeta_N^{\mathbf{disc}}(s)$, whose zero points and poles are characterized by

$$\frac{\partial_s \zeta_N^{\mathbf{disc}}(s)}{\zeta_N^{\mathbf{disc}}(s)} = -\sum_{n\geqq 1} N(n)n^{-s-1}$$

Good News. *(Connes-Consani) For any Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$,*

$\exists h(z)$, an entire function, s.t.

$$\zeta_N^{disc}(s) = \zeta_N(s)\exp\left(h(z)\right)$$

Therefore, $\zeta_N^{disc}(s)$ and $\zeta_N(s)$ have the same zero points and poles including multiplicities.

$\zeta_N^{disc}(s)$ may be defined for more general, not necesarily Noetherian, $\mathbb{F}_1$-schemes...

PART II

§5, Hypothetical computation of $N(n+1) = \left| \left(\overline{\operatorname{Spec}\mathbb{Z}}\right)(\mathbb{F}_{1^n}) \right|$ (Connes-Consani)

Using results of Ingham, Connes-Consani observed:

- **Regard** $w(u) = \sum_{\rho\in Z} \operatorname{order}(\rho)\frac{u^{\rho+1}}{\rho+1}$ **as a distribution on** $[1,\infty)$.
- **Then,**

$$N(u) := u - \frac{d}{du}w(u) + 1 = u - \frac{d}{du}\left(\sum_{\rho\in Z}\operatorname{order}(\rho)\frac{u^{\rho+1}}{\rho+1}\right) + 1, \tag{1}$$

where the derivative is in the sense of distributions, enjoys

$$-\frac{\partial_s\zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \int_1^\infty N(u)\, u^{-s} d^*u \tag{2}$$

- **The evaluation** $\omega(1)$ "$=$" $\lim_{s\to 1} w(s) = \sum_{\rho\in Z}\operatorname{order}(\rho)\frac{1}{\rho+1} = \frac{1}{2}+\frac{\gamma}{2}+\frac{\log 4\pi}{2}-\frac{\zeta'(-1)}{\zeta(-1)}$, **plays an essential role in establishing** (2).

Connes-Consani further pointed out the following analogue:

- X, **an irreducible, smooth projective algebraic curve over** $\mathbb{F}_p$,
- **If** X **comes from** $\mathbb{F}_1$ **by "scalar extension" ,**

$$N(q) = \left|X(\mathbb{F}_q)\right| = q - \sum_\alpha \alpha^r + 1, \qquad q = p^r,$$

where α**'s are the complex roots of the characteristic polynomial of the Frobenius on** $H^1_{et}(X\otimes\bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$ $(\ell \neq p)$
- **Expressing these roots in the form** $\alpha = p^\rho$, **for** $\rho \in Z'$, **the set of zeros of the Hasse-Weil zeta function of** X,

$$N(q) = \left|X(\mathbb{F}_q)\right| = q - \sum_{\rho\in Z'}\operatorname{order}(\rho)\, q^\rho + 1. \tag{3}$$

- **Now, compare** (3) **with the formal differantiation of (1):**

$$N(u) \sim u - \sum_{\rho\in Z}\operatorname{order}(\rho)u^\rho + 1$$

PART III

§6, Invariants $\mu_r(A)$ for an abelian group A

For a finite abelian group A, define the r-th μ-invariant $\mu_r(A)$ $(r \in \mathbb{N})$ by

$$\mu_r(A) := \frac{1}{|A|^r} \sum_{k_1,\dots,k_r=1}^{|A|} \Big| \mathrm{Hom}_{\mathfrak{Ab}} \left(A, \mathbb{Z}/(k_1 k_2 \cdots k_r)\mathbb{Z} \right) \Big|.$$

$\mu_r(A)$ is essentially the average of the random variable

$$\tilde{X}_r(A) : \tilde{\Omega} := \mathbb{N}^r \to \mathbb{N}$$
$$(k_1, k_2, \dots, k_r) \mapsto \Big| \mathrm{Hom}_{\mathfrak{Ab}} \left(A, \mathbb{Z}/(k_1 k_2 \cdots k_r)\mathbb{Z} \right) \Big|,$$

when the infinite set $\tilde{\Omega} = \mathbb{N}^r$ is given the homogeneous measure.

$$\Longrightarrow \quad \mu_r(A) = E\left[\tilde{X}_r(A)\right], \quad E\left[\tilde{X}_r(A)^w\right] = \mu_r\left(A^w\right) \quad (w \in \mathbb{N})$$

The invariants $\mu_r(A)$ were first considered by Deitmar-Kurokawa-Koyama and Kurokawa-Ochiai, throuth their study of, what they call, multiplicative Igusa-type zeta functions of $\mathbb{F}_!$-scheme, which we review by comparing with the Connes-Consani modified zeta function.

(i) The modified zeta function $\zeta_{\mathcal{X}}^{\mathrm{disc}}(s)$ for a Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$, defined and studied by Connes-Consani [CC] is characterized by the following property:

$$\begin{cases} -\frac{\zeta_{\mathcal{X}}^{\mathbf{disc}}(s)'}{\zeta_{\mathcal{X}}^{\mathbf{disc}}(s)} & \equiv \sum_{x \in \underline{X}} \sum_{m \geq 1} \Big| \mathrm{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^*, \mathbb{Z}/m\mathbb{Z}) \Big| (m+1)^{-s-1} \ (\textbf{mod. constant } \mathbf{N(1)}) \\ \zeta_{\mathcal{X}}(s) & = e^{h(z)} \zeta_{\mathcal{X}}^{\mathbf{disc}}(s) \quad (\zeta_X(s) : \textbf{Soulé zeta function}, \ h(z) : \textbf{entire}) \end{cases}$$

(ii) The multivariable (r variable) Igusa type zeta function $Z_{\mathcal{X}}^{Igusa}(s_1, \dots, s_r)$ for a Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$ ([DKK] for $r = 1$ and [KO] for general $r \in \mathbb{N}$) is given by

$$Z_{\mathcal{X}}^{\mathbf{Igusa}}(s_1, \dots, s_r) := \sum_{x \in \underline{X}} \sum_{m_1, \cdots, m_r \geq 1}^{\infty} \Big| \mathrm{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^*, \mathbb{Z}/m_1 \cdots m_r \mathbb{Z}) \Big| m_1^{-s_1} \cdots m_r^{-s_r}$$

[DKK] Anton Deitmar, Shin-ya Koyama and Nobushige Kurokawa, "*Absolute zeta functions.*" **Proc. Japan Acad. Ser. A Math. Sci. 84 (2008), no. 8, 138–142**
[KO] Nobushige Kurokawa and Hiroyuki Ochiai, "*A multivariable Euler product of Igusa type and its applications,*" **Journal of Number Theory, 12 pages, Available online 10 March 2009.**

Analyzing analytic properties of

$$Z^{\textbf{Igusa}}_{\mathrm{Spec}\,\mathbb{F}_1[A]}(s_1,\ldots,s_r)=\sum_{m_1,\cdots,m_r\geq 1}^{\infty}\Big|\mathrm{Hom}_{\mathfrak{Ab}}(A,\mathbb{Z}/m_1\cdots m_r\mathbb{Z})\Big|m_1^{-s_1}\cdots m_r^{-s_r},$$

some $\widehat{\text{v}}$ery mysterious looking idnetity of elementary number theory , which expresses $\mu_r(A)$ in two different ways, was obtained in the following two cases:

[DKK] $r=1$ **and arbitrary finite abelian group** A.
[KO] **Cyclic groups** $A=\mathbb{Z}/n\mathbb{Z}$ **and arbitray** $r\in\mathbb{N}$.

I reported a purely elementary proof of some slight generalization of these identities at the Vanderbilt conference in May, 2009:

[M1] Norihiko Minami, "*On the random variable* $\mathbb{N}^r \ni (k_1,k_2,...,k_r)\mapsto \gcd(n,k_1k_2...k_r)\in\mathbb{N}$, "**arXiv:0907.0916.**
[M2] Norihiko Minami, "*On the random variable* $\mathbb{N}\ni l\mapsto \gcd(l,n_1)\gcd(l,n_2)...\gcd(l,n_k)\in\mathbb{N}$,"**arXiv:0907.0918.**

Theorem of [DKK] type. *For a finite abelian group* $A = \prod_{j=1}^{k} (\mathbb{Z}/n_j\mathbb{Z})$,

$$\mu_1(A) = \mu_1\left(\prod_{j=1}^{k} (\mathbb{Z}/n_j\mathbb{Z})\right)$$

$$:= \frac{1}{\mathrm{lcm}(n_1, n_2, \ldots, n_k)} \sum_{l=1}^{\mathrm{lcm}(n_1,n_2,\ldots,n_k)} \gcd(l, n_1)\gcd(l, n_2)\cdots\gcd(l, n_k)$$

$$= \prod_{p \mid \mathrm{lcm}(n_1,n_2,\ldots,n_k)} \left[p^{\nu_{p,0}+\cdots+\nu_{p,k-1}} + \left(1 - \frac{1}{p}\right)\sum_{j=0}^{k-1} p^{\nu_{p,0}+\cdots+\nu_{p,j}} \sum_{l=\nu_{p,j}}^{\nu_{p,j+1}-1} p^{(k-j)\nu_{p,j}-l}\right]$$

Here, for each prime $p \mid n$,

$$\{\nu_{p,1}, \nu_{p,2}, \ldots, \nu_{p,k-1}, \nu_{p,k}\} = \{\mathrm{ord}_p(n_1), \mathrm{ord}_p(n_2), \ldots, \mathrm{ord}_p(n_{k-1}), \mathrm{ord}_p(n_k)\}$$

$$\nu_{p,0} := 0 \leq \nu_{p,1} \leq \nu_{p,2} \leq \ldots \leq \nu_{p,k-1} \leq \nu_{p,k}$$

Set ${}_nH_r := {}_{n+r-1}C_r$. **Then, we have:**

Theorem of [KO] type. *For* $n, r \in \mathbb{N}$, $w \in \mathbb{C}$,

$$\frac{1}{n^r}\sum_{k_1,\ldots,k_r=1}^{n} \gcd(n, k_1\cdots k_r)^{w}$$

$$= \begin{cases} \prod_{p|n}\left[\left(\frac{1-p^{-1}}{1-p^{w-1}}\right) + p^{\mathrm{ord}_p(n)(w-1)}\sum_{l=0}^{r-1} {}_{\mathrm{ord}_p(n)}H_l\left\{\left(1-p^{-1}\right)^l - \left(\frac{1-p^{-1}}{1-p^{w-1}}\right)^r (1-p^{w-1})^l\right\}\right] & (\textit{if } w \neq 1) \\ \prod_{p|n}\left[\sum_{l=0}^{r} {}_{\mathrm{ord}_p(n)}H_l\left(1-\frac{1}{p}\right)^l\right] & (\textit{if } w = 1) \end{cases}$$

$$= \begin{cases} \prod_{p|n}\left[\left(\frac{1-p^{-1}}{1-p^{w-1}}\right) + p^{\mathrm{ord}_p(n)(w-1)}(1-p^{-1})^r\sum_{l=0}^{r-1} {}_{\mathrm{ord}_p(n)}H_l\left\{\left(1-p^{-1}\right)^{l-r} - (1-p^{w-1})^{l-r}\right\}\right] & (\textit{if } w \neq 1) \\ \prod_{p|n}\left[\sum_{l=0}^{r} {}_{\mathrm{ord}_p(n)}H_l\left(1-\frac{1}{p}\right)^l\right] & (\textit{if } w = 1) \end{cases}$$

Corollary [KO]. *For* $n, r \in \mathbb{N}$,

$$\mu_r(\mathbb{Z}/n\mathbb{Z}) = \prod_{p|n}\left[\sum_{l=0}^{r} {}_{\mathrm{ord}_p(n)}H_l\left(1-\frac{1}{p}\right)^l\right]$$

§7 Motivation of the rest of talk

When we play with $\mu_r(A)$, the following questions seem to be very natural:

- **Is there any more conceptual interpretation or description of $\mu_r(A)$?**
- **Is $\mu_r(A)$, whose origin is the Igusa-type zeta functions for $\mathbb{F}_1$-schemes of Kurokawa and his collaboratiors, useful to study $\mathbb{F}_1$-scheme?**
- **Is there any relationship between the zeta functions of Soullé, Connes-Consani, and the Igusa-type zeta functions, which was the orgin of $\mu_r(A)$?**

§8 $\mu_1(A)$and the zeta functions of Soullé, Connes-Consani.

The logarithmic derivative of the deformed modified zeta function of Soulé type $\zeta_{\mathcal{X}}^{\mathbf{disc}}(s;w)$:

$$\frac{\partial_s \zeta_{\mathcal{X}}^{\mathbf{disc}}(s;w)}{\zeta_{\mathcal{X}}^{\mathbf{disc}}(s;w)} \equiv -\sum_{x\in\underline{X}}\sum_{m\geq 1}\left|\mathrm{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^*,\mathbb{Z}/m\mathbb{Z})\right|^w (m+1)^{-s-1} \quad \text{(mod. constant)}$$

is a meromorphic function of s with all of its poles simple.

This gives us the following expression of the deformed modified zeta function of Soulé type:

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$$
\begin{aligned}
&\zeta_{\mathcal{X}}^{\mathbf{disc}}(s; w) \\
&= e^{h(s;w)} \prod_{x \in \underline{X}} \left(\left(\prod_{j=0}^{n(x)w} (s-j)^{\left(-\binom{n(x)w}{j}(-1)^{n(x)w-j}\right)} \right)^{\left(\frac{\sum_{k=1}^{l(x)} \left| \mathrm{Hom}_{\mathfrak{Ab}}(A_x, \mathbb{Z}/k\mathbb{Z}) \right|^{w}}{l(x)} \right)} \right),
\end{aligned}
$$

where, for each $x \in \underline{X}$**,** $\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x$**,** $l(x) = \mathrm{lcm}\{m_j(x)\}$**, amd** $h(s; w)$ **is some entire function of** s **depending upon** $w \in \mathbb{N}$**.**

Restricting to the case $w = 1$ **further, we obtain the following:**

For a Noetherian F_1**-scheme** $\mathcal{X}$**, there are some entire functions** $h_1(s), h_2(s)$ **s.t.**

$$
\begin{aligned}
\zeta_{\mathcal{X}}(s) &= e^{h_1(s)} \zeta_{\mathcal{X}}^{\mathbf{disc}}(s) \\
&= e^{h_2(s)} \prod_{x \in \underline{X}} \left(\left(\prod_{j=0}^{n(x)} (s-j)^{\left(-\binom{n(x)}{j}(-1)^{n(x)-j}\right)} \right)^{\mu_1(A_x)} \right),
\end{aligned}
$$

where, for each $x \in \underline{X}$**,** $\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x$**,**

Message:

- μ_1 **measures "local contribution of ramification"!.**
- **locally, torsion does not creat any new singularity.**

An outline of the proof of the $\zeta_{\mathcal{X}}^{disc}(s;w)$ formula.

$$-\sum_{x\in\underline{X}}\sum_{m=1}^{\infty}\left|\mathrm{Hom}_{\mathfrak{Ab}}\left(\mathcal{O}_x^*,\mathbb{Z}/m\mathbb{Z}\right)\right|^{w}(m+1)^{-s-1}$$

$$(4)\qquad =-\sum_{x\in\underline{X}}\sum_{j=0}^{n(x)w}\binom{n(x)w}{j}(-1)^{n(x)w-j}l(x)^{-(s+1-j)}$$

$$\times\sum_{k=1}^{l(x)}\left|\mathrm{Hom}_{\mathfrak{Ab}}\left(A_x,\mathbb{Z}/k\mathbb{Z}\right)\right|^{w}\zeta\left(s+1-j,\frac{k+1}{l(x)}\right),$$

where the Hurwitz zeta function

$$\zeta(s,q):=\sum_{n\geq 0}(n+q)^{-s}\ (\Re(s)>1,\Re(q)>0)$$

only has a pole of residue 1 at $s=1$ Thus, the singularities of (4) are poles at $s=j\in\cup_{x\in X}\{0,\cdots,n(x)\}$ with residue

$$-\sum_{x\in\underline{X}}\sum_{j=0}^{n(x)w}\binom{n(x)w}{j}(-1)^{n(x)w-j}l(x)^{-(1)}\sum_{k=1}^{l(x)}\left|\mathrm{Hom}_{\mathfrak{Ab}}\left(A_x,\mathbb{Z}/k\mathbb{Z}\right)\right|^{w}$$

$$=\sum_{x\in\underline{X}}\sum_{j=0}^{n(x)w}\left(-\binom{n(x)w}{j}(-1)^{n(x)w-j}\frac{\sum_{k=1}^{l(x)}\left|\mathrm{Hom}_{\mathfrak{Ab}}\left(A_x,\mathbb{Z}/k\mathbb{Z}\right)\right|^{w}}{l(x)}\right)$$

$$=\sum_{x\in\underline{X}}\sum_{j=0}^{n(x)w}\left(-\binom{n(x)w}{j}(-1)^{n(p)w-j}\right)\frac{\sum_{k=1}^{l(x)}\left|\mathrm{Hom}_{\mathfrak{Ab}}\left(A_x,\mathbb{Z}/k\mathbb{Z}\right)\right|^{w}}{l(x)}$$

$$=\sum_{x\in\underline{X}}\sum_{j=0}^{n(x)w}\left(-\binom{n(x)w}{j}(-1)^{n(x)w-j}\right)\mu_1({A_x}^{w})$$

Now the claim follows immediately. □

§9. The conceptual interpratation of $\mu_1(A)$.

For any finite abelian group A,

$$(5)\qquad \mu_1(A):=\frac{1}{|A|}\sum_{k=1}^{|A|}\left|\mathrm{Hom}_{\mathfrak{Ab}}\left(A,\mathbb{Z}/k\mathbb{Z}\right)\right|=\sum_{a\in A}\frac{1}{|a|}$$

If we interpret that $\frac{1}{|a|}$ " = " $\frac{1}{\infty}$ " = " 0 for an element a of infinite order, we may generalize the definition of $\mu_1(A)$ to finitely generalized abelian groups, as well as to finite (not necessary commutative) groups.

Proof of $\mu_1(A) = \sum_{a\in A}\frac{1}{|a|}$.

$$\frac{1}{|A|}\sum_{l=1}^{|A|}\Big|\mathrm{Hom}_{\mathfrak{Ab}}(A,\mathbb{Z}/l\mathbb{Z})\Big| = \frac{1}{|A|}\sum_{l=1}^{|A|}\Big|\mathrm{Hom}_{\mathfrak{Ab}}(\mathbb{Z}/l\mathbb{Z},A)\Big|$$
$$= \frac{1}{|A|}\sum_{l=1}^{|A|}\sum_{cyclic\ C\subset A}\Big|\mathrm{Epi}_{\mathfrak{Ab}}(\mathbb{Z}/l\mathbb{Z},C)\Big| = \frac{1}{|A|}\sum_{cyclic\ C\subset A}\sum_{l=1}^{|A|}\Big|\mathrm{Epi}_{\mathfrak{Ab}}(\mathbb{Z}/l\mathbb{Z},C)\Big|$$
$$= \frac{1}{|A|}\sum_{cyclic\ C\subset A}\sum_{l=1}^{|A|}\Big|\mathrm{Mono}_{\mathfrak{Ab}}(C,\mathbb{Z}/l\mathbb{Z})\Big| = \frac{1}{|A|}\sum_{cyclic\ C\subset A}\frac{|A|}{|C|}\phi(|C|)$$
$$= \sum_{cyclic\ C\subset A}\frac{\phi(|C|)}{|C|} = \sum_{h\in\mathrm{Hom}(\mathbb{Z},A)}\frac{1}{|h(1)|} = \sum_{a\in A}\frac{1}{|a|}$$

□

§10, $\mu_r(A)$ for general $r\in\mathbb{N}$.

For any abelian group A and $r\in\mathbb{N}$, we have

$$\mu_r(A) = \sum_{a\in A}\frac{KO_{r-1}(|a|)}{|a|} = \sum_{a\in A}\frac{\prod_{p\big||a|}\left[\sum_{l=0}^{r-1}{}_{\mathrm{ord}_p(|a|)}H_l\left(1-\frac{1}{p}\right)^l\right]}{|a|}$$
$$= \prod_{p\big||A|}\sum_{a\in A_p}\frac{KO_{r-1}(|a|)}{|a|} = \prod_{p\big||A|}\left(\sum_{a\in A_p}\frac{\sum_{l=0}^{r-1}{}_{\mathrm{ord}_p(|a|)}H_l\left(1-\frac{1}{p}\right)^l}{|a|}\right)$$

where KO stands for Kurokawa-Ochiai [KO]:

$$KO_r(n) := \begin{cases} 1 & (r=0) \\ \mu_r(\mathbb{Z}/n\mathbb{Z}) = \prod_{p|n}\left[\sum_{l=0}^{r}{}_{\mathrm{ord}_p(n)}H_l\left(1-\frac{1}{p}\right)^l\right] & (r\geq 1)\end{cases}$$

§11, Connes-Consani modified Soulé type zeta function, again

To recap, let us combine the two theorem:

For a Noetherian F_1-scheme $\mathcal{X}$, there are some entire functions $h_1(s), h_2(s)$ s.t.

$$\zeta_{\mathcal{X}}(s) = e^{h_1(s)}\zeta_{\mathcal{X}}^{\mathbf{disc}}(s)$$

$$= e^{h_2(s)} \prod_{x \in \underline{X}} \left(\left(\prod_{j=0}^{n(x)} (s-j)^{\left(-\binom{n(x)}{j}(-1)^{n(x)-j}\right)} \right)^{\sum_{a \in A_x} \frac{1}{|a|}} \right)$$

where, for each $x \in \underline{X}$, $\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x$,

Once again, the above result is in the following:

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I would like to end this paper with the following question to transformation group theorists:

Is there any application of the invariants $\mu_r(A)$ to the transformation group theory?