EXISTENCE AND CLASSIFICATION RESULTS ON ISOVARIANT MAPS

Abstract. The notion of an isovariant map, i.e, an equivariant map preserving the isotropy subgroups, plays an important role in equivariant topology. In this article, we shall illustrate existence and classification results on isovariant maps and provide some examples.

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1. Isovariant maps

Let G be a compact Lie group. All maps considered are continuous. We first recall the notion of an isovariant map which was introduced by Palais [12].

Definition. (1) A G-map $f : X \to Y$ is called *G-isovariant* if f preserves the isotropy subgroups, i.e., $G_{f(x)} = G_x$ for all $x \in X$.
(2) A G-homotopy $F : X \times I \to Y$ between isovariant maps is called a *G-isovariant homotopy* if F is G-isovariant.

We denote by $[X, Y]_G^{\mathrm{isov}}$ the isovariant homotopy set, i.e., the set of isovariant homotopy classes of G-isovariant maps from X to Y. As usual $[X, Y]_G$ denotes the set of equivariant homotopy classes of G-equivariant maps from X to Y.

Example 1.1. (1) If both X and Y are free G-spaces, then an arbitrary G-map $f : X \to Y$ is isovariant.
(2) An injective G-map f is G-isovariant.
(3) Suppose that X is a G-space with nontrivial action and Y has a G-fixed point. Then a constant map $f : X \to Y^G \subset Y$ is equivariant, but not isovariant.

Let C_n be the cyclic group of order n. Let denote by U_k $(= \mathbb{C})$ the unitary representation of C_n on which a generator $c \in C_n$ acts by $c \cdot z = \xi^k z$, where $z \in U_k$ and $\xi = \exp(2\pi\sqrt{-1}/n)$.

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For simplicity, we consider the case of $G = C_{pq}$, where p, q are distinct primes. Set

$$U_1^r = U_1 \oplus \cdots \oplus U_1 \ (r \text{ times})$$

and

$$W = U_p \oplus U_q.$$

Consider the unit spheres SU_1^r and SW. Then G acts freely on SU_1^r. On the other hand G does not act freely on SW and the singular set (i.e., nonfree part)

$$SW^{>1} = SU_p \coprod SU_q$$

is a Hopf link.

Question. *Does there exist a C_{pq}-isovariant map from SU_1^r to SW?*

In equivariant case, a C_{pq}-map does exist for any $r \geq 1$ as a result from equivariant obstruction theory, see [15]. What about a C_{pq}-isovariant map?

If $r = 1$, then there exists an isovariant map. For example, the equivariant map $f_{0,0} : SU_1 \to SW$ can be defined by

$$f_{0,0}(z) = (z^p, z^q)/\sqrt{2},$$

and it is isovariant. In fact, $f_{0,0}$ is an injective G-map.

If $r \geq 2$, the answer is "*No.*" This is proved by a Borsuk-Ulam type theorem. Many generalizations of the Borsuk-Ulam theorem are known. The following result is one of them.

Theorem 1.2 (mod p Borsuk-Ulam theorem)**.** *Assume that C_p (p: prime) acts freely on mod p homology spheres Σ_1 and Σ_2. If there is a C_p-map $f : \Sigma_1 \to \Sigma_2$, then $\dim \Sigma_1 \leq \dim \Sigma_2$. In other words, if $\dim \Sigma_1 > \dim \Sigma_2$, then there is no C_p-map from Σ_1 to Σ_2.*

The nonexistence of an isovariant map is proved as follows. Suppose that $f : SU_1^r \to SW$ is a C_{pq}-isovariant map. By restricting the action, we regard f as a C_p-isovariant map. Since $f(SU_1^r) \subset SW \setminus SW^{C_p}$, we obtain a C_p-map $f : SU_1^r \to SW \setminus SW^{C_p}$. Since $SW \setminus SW^{C_p}$ is C_p-homotopy equivalent to SU_q, we obtain a C_p-map $g : SU_1^r \to SU_q$ between free C_p-spheres. The mod p Borsuk-Ulam theorem says that

$$2r - 1 = \dim SU_1^r \leq \dim SU_q = 1,$$

however this is a contradiction when $r \geq 2$.

The above example is generalized as follows. Set

$$R_G = \begin{cases} \mathbb{Z}/|G| & \text{if } \dim G = 0, \\ \mathbb{Z} & \text{if } \dim G > 0. \end{cases}$$

Definition. A smooth closed G-manifold Σ is called an *R_G-homologically linear G-sphere* if for every (closed) subgroup H, the H-fixed point set Σ^H is an R_G-homology sphere or the empty set; namely,

$$H_*(\Sigma^H; R_G) \cong H_*(S^{m(H)}; R_G), \ m(H) = \dim \Sigma^H.$$

For convenience, we set $\dim \Sigma^H = -1$ if Σ^H is empty.

Theorem 1.3 (Isovariant Borsuk-Ulam theorem [7]). *Let G be a solvable compact Lie group. Let Σ_1 and Σ_2 be R_G-homologically linear G-spheres. If there is a G-isovariant map $f : \Sigma_1 \to \Sigma_2$, then the inequality*

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

holds.

Remark. Wasserman first proved this type of result for representation spaces.

If G is nonsolvable, the above result does not hold. In fact, using a result of Oliver [11], we can construct a counterexample for an arbitrary nonsolvable compact Lie group. Thus we obtain that

Theorem 1.4 ([7]). *For homologically linear actions, the isovariant Borsuk-Ulam theorem holds if and only if G is solvable.*

Remark. For (ordinary) linear actions, it is known [16] that there are nonsolvable groups for which the isovariant Borsuk-Ulam theorem holds.

Remark. A similar result is already known in equivariant case. In fact, Bartsch [1] showed that for linear actions, the Borsuk-Ulam theorem holds (in a certain equivariant sense) if and only if G is of prime power order.

Using the isovariant Borsuk-Ulam theorem, we obtain a generalization of the nonexistence result mentioned before.

Corollary 1.5 ([8]). *Let G be a finite group and Σ an R_G-homology sphere with free G-action. Let SW be a representation sphere of G. If*

$$\dim \Sigma \geq \dim SW - \dim SW^{>1},$$

then there is no G-isovariant map form Σ to SW. Here $SW^{>1} = \bigcup_{1 \neq H \leq G} SW^H$ (the singular set of SW).

As seen before, there is a C_{pq}-isovariant map

$$f_{0,0} : SU_1 \to S(U_p \oplus U_q).$$

This is generalized as follows.

Proposition 1.6 ([8]). *Let G be a finite group. Let M be a free G-manifold and SW a representation sphere of G. Suppose*

$$\dim M < \dim SW - \dim SW^{>1}.$$

Then there exists a G-isovariant map from M to SW.

This follows from the fact that the complement of the singular set $SW \setminus SW^{>1}$ is $(d-2)$-connected, where $d = \dim SW - \dim SW^{>1}$.

2. Classification problem

Next we discuss the classification problem on isovariant maps. Historically Hopf showed that the degree of maps provides a bijection $\deg : [M, S^n] \to \mathbb{Z}$, where M is a connected, oriented, closed n-manifold M. We call this kind of result a *Hopf type theorem.* Several equivariant versions of Hopf's theorem (i.e., equivariant Hopf type theorems) have been studied by many researchers, e.g., Rubinsztein [13], tom Dieck [2], Tornehave [14], Laitinen [6], Kushkuley-Balanov [5], Ferrario [3], etc.

We would like to consider an isovariant version. Let $G = C_{pq}$ again. For each $(\alpha, \beta) \in \mathbb{Z} \oplus \mathbb{Z}$, we can define a G-isovariant map

$$f_{\alpha,\beta} : SU_1 \to S(U_p \oplus U_q)$$

by

$$f_{\alpha,\beta}(z) = (z^{p(1+\alpha q)}, z^{q(1+\beta p)})/\sqrt{2}.$$

(The map $f_{0,0}$ is the same one defined previously.) Then we obtain

Proposition 2.1. *If $f_{\alpha,\beta}$ and $f_{\alpha',\beta'}$ are isovariantly homotopic if and only if $(\alpha, \beta) = (\alpha', \beta')$.*

To show this, we introduce the *multidegree* as an isovariant homotopy invariant. Set $SW = S(U_p \oplus U_q)$ and $SW_{\mathrm{free}} = SW \setminus SW^{>1}$. If f is an isovariant map, then we obtain a G-map

$$f : SU_1 \to SW_{\mathrm{free}}$$

Consider the induced homomorphism

$$f_* : H_1(SU_1) \to H_1(SW_{\mathrm{free}}).$$

Since SW_{free} is the complement of a Hopf link in S^3, which is homotopy equivalent to $S^1 \times S^1$, we obtain that

$$\pi_1(SW_{\mathrm{free}}) \cong H_1(SW_{\mathrm{free}}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We define the multidegree $\mathrm{mDeg}\,(f)$ of f by

$$\mathrm{mDeg}\,(f) = f_*([SU_1]) \in \mathbb{Z} \oplus \mathbb{Z}.$$

Lemma 2.2. *The multidegree of $f_{\alpha,\beta}$ is*

$$\mathrm{mDeg}\, f_{\alpha,\beta} = (q(1+\beta p), p(1+\alpha q)).$$

This shows that if $(\alpha,\beta) \neq (\alpha',\beta')$, then $\mathrm{mDeg}\, f_{\alpha,\beta} \neq \mathrm{mDeg}\, f_{\alpha',\beta'}$. Thus the isovariant maps $f_{\alpha,\beta}$ mutually represent different isovariant homotopy classes.

The converse is proved by equivariant obstruction theory. There is a commutative diagram:

$$\begin{array}{ccc}
[SU_1, SW_{\mathrm{free}}]_G & \xrightarrow[\cong]{\gamma_G} & H^1(SU_1/G, \pi_1) = \mathbb{Z}\oplus\mathbb{Z} \\
\varepsilon \big\downarrow & & \big\downarrow \pi^* \\
[SU_1, SW_{\mathrm{free}}] & \xrightarrow[\gamma]{\cong} & H^1(SU_1, \pi_1) = \mathbb{Z}\oplus\mathbb{Z},
\end{array}$$

where $\pi_1 = \pi_1(SW_{\mathrm{free}}) \cong \mathbb{Z}\oplus\mathbb{Z}$. The vertical map ε is the forgetful map. The horizontal maps are defined by

$$\gamma_G([f]) = \mathfrak{o}_G(f, f_{0,0}) \quad \text{and} \quad \gamma([f]) = \mathfrak{o}(f, f_{0,0})$$

by using the (equivariant) obstruction to construct a G-homotopy between f and $f_{0,0}$. The homomorphism π^* is the induced homomorphism of the orbit map π, which is multiplication by $|G| = pq$. In particular π^* is injective. By calculation of the obstruction class, we obtain

Proposition 2.3.

$$\pi^*(\mathfrak{o}_G(f, f_{0,0})) = \mathfrak{o}(f, f_{0,0}) = \mathrm{mDeg}\, f - \mathrm{mDeg}\, f_{0,0} \in pq(\mathbb{Z}\oplus\mathbb{Z}).$$

We define a map

$$D : [SU_1, SW_{\mathrm{free}}]_G \to \mathbb{Z}\oplus\mathbb{Z}$$

by

$$D([f]) = (\mathrm{mDeg}\, f - \mathrm{mDeg}\, f_{0,0})/pq.$$

Since $D([f_{\alpha,\beta}]) = (\beta,\alpha)$, D is surjective. Furthermore D is injective. In fact, if $D([f]) = D([g])$, then it follows from the above proposition that $\mathfrak{o}_G(f, f_{0,0}) = \mathfrak{o}_G(g, f_{0,0})$. This implies $\mathfrak{o}_G(f,g) = 0$, and hence f and g are G-homotopic. Thus D is a bijection. Since $[SU_1, SW]_G^{\mathrm{isov}} = [SU_1, SW_{\mathrm{free}}]_G$, we obtain the following classification result.

Proposition 2.4. *There is a bijection*

$$D : [SU_1, SW]^{isov}_{C_{pq}} \to \mathbb{Z}\oplus\mathbb{Z}.$$

In particular, the maps $f_{\alpha,\beta}$ represent all isovariant homotopy classes.

The above proposition is generalized as follows. We assume the following.

- G is a finite group.
- M is a connected, closed *free* G-manifold.

- SW is a *unitary* representation sphere of G.
- $\dim M = \dim SW - \dim SW^{>1} - 1$.

We set

$$\mathcal{A} = \{\, H \in \operatorname{Iso} W \mid \dim SW^H = \dim SW^{>1} \,\},$$

and

$$\mathcal{A}/G = \{\, (H) \mid H \in \mathcal{A} \,\}.$$

We have

Theorem 2.5 (Orientable case [9]). *If M is orientable and the G-action on M is orientation-preserving, then there is a one-to-one correspondence*

$$[M, SW]_G^{isov} \cong \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}.$$

Every isovariant homotopy class is determined by the multidegree.

Remark. If the G-action is not orientation-preserving, the multidegree does not always determine isovariant homotopy classes. Indeed, $\mathbb{Z}/2$ components appear in $[M, SW]_G^{\mathrm{isov}}$, see [9] for the detail.

Theorem 2.6 (Nonorientable case [10]). *If M is nonorientable, then there is a one-to-one correspondence*

$$[M, SW]_G^{isov} \cong \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}/2.$$

If G is of odd order, then every isovariant homotopy class is determined by the mod 2 multidegree.

Remark. If G is not of odd order, then the mod 2 multidegree does not always determine isovariant homotopy classes.

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