FUNCTORIALITY OF ISOVARIANT STRUCTURE SETS AND THE GAP HYPOTHESIS

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Section 1. Introduction

In 1987, W. Browder [Br] claimed a fundamental theorem relating equivariant vs. isovariant homotopy equivalences, under the Gap Hypothesis. More than twenty years have passed since then, but the claim is still "folklore", despite the fact that many people (cf. [We 1]) have developed theories under the assumption that Browder's claim is true. The current author's earlier works [N 6], [N 5] also relied on it.

In 2006, R. Schultz [Sch] published a proof of Browder's theorem for semi-free actions. He used homotopy theoretic methods, and built a new obstruction theory in order to construct an isovariant homotopy equivalence from an equivariant homotopy equivalence in the semi-free situation. However, for general (non-semi-free) cases, the situation is not settled yet. If one wants to generalize Schultz' proof for non-semi-free cases, one would have to construct even more complicated obstruction theories, which do not look so straightforward.

In 2009, S. Cappell, S. Weinberger and M. Yan published a paper [CWY 2] claiming the functoriality of the isovariant structure set $\mathcal{S}_G\left(M, \operatorname{rel} M_s\right)$ "under mild conditions." That is, they claim that the isovariant structure set is functorial with respect to equivariant maps. But they never provide fine details regarding the isovariance vs. equivariance problems, especially for non-semi-free cases that we are mainly interested in. They mainly give a proof of the "stable version" and rely on the equivariant periodicity of $\mathcal{S}_G\left(M, \operatorname{rel} M_s\right)$ ([WY 1] , [WY 2]) for which the "destabilization" is highly non-canonical.

In this note, we will generalize the "diagram cohomology obstruction theory" developed by Dula and Schultz [DS] to more general group actions. We try to construct one such obstruction theory, and test it in some particular group actions.

Here we note a phenomenon, via one particular example, that although there are nontrivial classes (as pointed out in [N 8] and [N 4]) in the equivariant homotopy groups, the obstruction class of which will vanish if we go to the "diagram obstruction" groups for calculation of the "isovariant versus equivariant" obstruction, if we assume the strong Gap Hypothesis. That will mean that the Browder's claim holds true, which states that equivariant homotopy equivalences between smooth G-manifolds are equivariantly homotopic to a G-isovariant homotopy equivalence if the strong Gap Hypothesis holds, in one particular situation for one particular group G. We hope to generalize it into more grup actions, to support the Browder's claim in more general situations, in a future work.

Section 2. Definition and the Basic Example

Let G be a finite group. Let M be a closed, connected, G-oriented smooth G-manifold. For any subgroup H of G, let M^H be the fixed-point set, which may consist of submanifolds of various dimension. A G-manifold M is said to satisfy the Gap Hypothesis if the following holds:

The Gap Hypothesis. *For any pair of subgroups $K \lneqq H$ of G, and for any pair of connected components $B \subset M^H$ and $C \subset M^K$ such that $B \subsetneqq C$, the inequality $2\dim B + 2 \leq \dim C$, in other words, $\dim B < [\frac{1}{2}\dim C]$, holds.*

The Gap Hypothesis provides general position arguments and transversality between each isotropy type pieces, thus making it possible to provide various geometric constructions in the equivariant settings. Madsen and Rothenberg ([MR 2]) constructed a beautiful surgery exact sequence in an equivariant category, and used it to classify spherical space forms.

Browder's insight told us to use this condition to construct isovariant homotopy equivalences from equivariant homotopy equivalences. And that is what we would like to consider here.

Definition. *A map $f : X \to Y$ between G-sapces X and Y is called equivariant if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. In other words, the isotropy subgroup G_x is included in the isotropy subgroup $G_{f(x)}$ for all $x \in X$. The map f is called isovariant if G_x is equal to $G_{f(x)}$ for all $x \in X$.*

Browder [Br] claimed the folowing:

Theorem (Browder). *Let M and N be closed, connected, G-oriented smooth G-manifolds. Assume that M satisfies the Gap Hypothesis. Then, any G-homotopy equivalence $f : M \to N$ is G-equivariantly homotopic to a G-isovariant homotopy equivalence f'. Moreover, if $M \times I$ satisfies the Gap Hypothesis, then the f' is unique up to G-homotopy.*

Here is an example, given by Browder, that illustrates the principal obstruction in deforming an equivariant map into an isovariant map:

Let G be a cyclic group of prime order, and let it act on the sphere S^q by rotation, with 2 fixed points 0 and ∞. Let $Y = S^k \times S^q$ where G acts trivially on the first coordinate S^k, thus the fixed point set is $Y^G = \left(S^k \times 0\right) \cup \left(S^k \times \infty\right)$. Let $X = \left(S^k \times S^q\right) \sharp_G \, G\left(S^k \times S^q\right)$, the equivariant connected sum of $Y = S^k \times S^q$ and

$|G|$ copies of G-trivial $\left(S^k \times S^q\right)$ with G freely acting by circulating the $|G|$ copies, and the equivariant connected sum is made on a free orbit.

Define $f : X \to Y$ to be the identity on the first component $S^k \times S^q$, and via the composition of the projection $G\left(S^k \times S^q\right) \to GS^q$ and the canonical G-map $GS^q \to S^q$ on the second component of the equivariant connected sum.

By construction, f is a degree 1 equivariant map. But it is not an isovariant map, because the fixed point set X^G is just the "central" $\left(S^k \times 0\right)$ on the first component, thus $f^G : X^G \to Y^G$ is just the identity, but the free part of X is $X - X^G = S^k \times \left(S^{q-1} \times \mathbb{R}\right) \sharp_G \, G\left(S^k \times S^q\right)$, which contains all the S^q-cycles on the $|G|$ copies of $\left(S^k \times S^q\right)$. When mapped onto Y, this free part must intersect with the fixed-point set Y^G in Y, thus f could not be deformed in any way to an isovariant map.

Note that both X and Y satisfy the Gap Hypothesis if $q \geq k+2$, thus it is a serious obstruction in considering Browder's deformation of equivariant things into isovariant things. The Gap Hypothesis and degree 1 maps are not enough; being an equivariant homotopy equivalence is an essential condition, and so this is really a deep geometrical problem.

Section 3. The Methods of Schultz

Schultz [Sch] gave a proof of Browder's theorem under the additional assumption that the G-action is semi-free (that is, $M - M^G$ is G-free) everywhere. In the semi-free case, the only possible isotropy types are G-free and trivial types, so one can do the construction considering only those two distinct types. Thus, Schultz (and Dula and Schultz [DS]) constructed an obstruction theory in a form of equivariant cohomology, which they called "diagram cohomology", of triads of the form (manifold; regular neighborhood of the fixed-point set, and the free-part).

Since the fixed point sets $N^G = \coprod_\alpha N_\alpha$ and $M^G = \coprod_\alpha M_\alpha$ with $M_\alpha = f^{-1}\left(N_\alpha\right) \cap M^G$ is in one-to-one correspondence component-wise, one can first deform f inside the regular neighborhood of each of the components M_α of the fixed-point set. The normal bundles of M_α and N_α are stably fiber homotopy equivalent, but thanks to the Gap Hypothesis, it is unstably fiber homotopy equivalent. Therefore, it is possible to deform f to be isovariant in the regular neighborhood of M_α for each α, by using standard construction.

Next one pushes down the non-isovariant points into the system of tubular neighborhoods of M_α. That is, deform the map f so that any non-isovariant point is contained in a closed tubular neighborhood W_α of M_α for some α. (See Proposition 4.2 of [Sch].) Here, the deformation is done via the "diagram cohomology" obstruction theory. One notes that the map $f : X \to Y$ in the example of the previous section cannot be deformed this way, since the "diagram cohomology" detects its non-trivial obstruction.

Finally, one deforms the result map into a G-isovariant map. Again, one uses the "diagram cohomology" to detect the deformation obstruction. First, one uses G-transversality (due to the Gap Hypothesis) to construct appropriate "diagram maps" that have necessary local isovariancy properties (which they call "almost isovariant maps,") and then apply the "diagram cohomology" obstruction theory to see that the obstruction vanishes, producing the desired deformation, to get a global G-isovariant map. (See Proposition 5.3 of [Sch].)

Schultz has successfully built an appropriate obstruction theory just enough for proving the theorem in the semi-free case. As he remarks in the last section in his paper, he seems to be interested in applying the obstruction theory to situations where the Gap Hypothesis fails, and to build a new framework of applications of equivariant homotopy theory into equivariant surgery. However, in non-semi-free cases, the "diagram cohomology" obstruction theory (of [DS]) does not seem to be directly applicable, and things seem to be much complicated if one pursues to reduce them into algebraic topology methods. Here we try to investigate what happens in such complicated situations, by doing calculation in some particular example situation, to see if their methods can be generalized, and to see how it can be done if it is possible.

Thus, a more generalized version of obstruction theory is needed here, and so we first work out a new form of "diagram cohomology" in the style of Dula and Schultz [DS].

Claim. *The diagram cohomology obstruction theory of Dula and Schultz can be directly generalized to non-semi-free actions of metacyclic groups. In particular, Theorem 4.5 of* [DS] *still holds for an arbitrary action of any metacyclic group.*

In order to prove this, we go back to Serre-type spectral sequence of Bredon cohomology with twisted coefficients, as developed by J. M. Møller [Mo] and I. Moerduk and J.-A. Svensson [MoS]. Working parrarel to Dula and Schultz for such group actions using Bredon cohomology with twisted coefficients, Dula and Schultz' arguments can be directly generalized to our cases, too, and Theorem 4.5 of [DS] can be proved in such cases, providing recognition principle for a diagram map to produce an isovariant map.

Section 4. The Fixed-Point Homomorphism for Nonabelian Group Actions

In this section we compute the normal data in an equivariant surgery exact sequence for one particular, easiest nontrivial example which could produce an exotic equivariant obstruction class. Let us consider the metacyclic group $G = G_{21} = \mathbb{Z}/7 \underset{\alpha}{\rtimes} \mathbb{Z}/3$:

$$1 \longrightarrow H = \mathbb{Z}/7 \longrightarrow G \longrightarrow \mathbb{Z}/3 \longrightarrow 1$$

Here $\alpha : \mathbb{Z}/3 \to \operatorname{Aut}\mathbb{Z}/7$ is defined by multiplication by 2. The system $\underset{=}{RO}$ of real representation rings is well-known. We fix notation as follows. Let A be a subgroup of order 3. All such are conjugate to each other.

Here the system $\underline{\underline{RO}}$ consists of

$$\begin{aligned} RO(e) &= \mathbb{Z} \ni 1 \\ RO(H) &= \mathbb{Z}^4 \ni 1, z_1, z_2, z_4 \\ RO(A) &= \mathbb{Z}^2 \ni 1, w \\ RO(G) &= \mathbb{Z}^3 \ni 1, w, P \end{aligned}$$

where

$$\begin{aligned} &\mathrm{Res}_e^H(1) = 1, \mathrm{Res}_e^H(z_i) = 2, \\ &\mathrm{Res}_e^A(1) = 1, \mathrm{Res}_e^A(w) = 2, \\ &\mathrm{Res}_H^G(1) = 1, \mathrm{Res}_H^G(w) = 2, \mathrm{Res}_H^G(P) = z_1 + z_2 + z_4, \\ &\mathrm{Res}_A^G(1) = 1, \mathrm{Res}_A^G(w) = w, \mathrm{Res}_A^G(P) = 2 + 2w. \end{aligned}$$

Note that Res_H^G is not surjective but is onto the WH-invariant submodule of $RO(H)$, and therefore we cannot have a decomposition for this system.

We remark that any metacyclic group has a similar system $\underline{\underline{RO}}$.

In ([N 8]), we determined the term $\widetilde{\mathcal{N}}_G(X)$ of the equivariant surgery exact sequence, that is, the set of equivariant normal maps, localized at 2. More precisely, we have

$$\begin{aligned} \widetilde{\mathcal{N}}_G(X)_{(2)} &= [x, F/PL]^G \\ &= [X^*, E^{\underline{\underline{s}}}]_{\mathcal{O}_G} \times \bigoplus_{i \geq 6} H_G^i\left(X; L_i(e)^{\underline{\underline{s}}}\right) \times \bigoplus_{i \geq 2} H_G^i\left(X; \underline{\underline{\hat{\mathcal{L}}}}_i\right). \end{aligned}$$

where

$$\hat{\mathcal{L}}_i(H) = \bigoplus_{(\Gamma) \subset H} \widetilde{L}_i\left(N_H\Gamma/\Gamma\right)$$

is the system (that is, the Mackey functor structure, in the notation of [E]) of the L-group term in the equivariant surgery exact sequence.

Thus we express $\widetilde{\mathcal{N}}_G(X)_{(2)}$ as the product of Bredon cohomology groups and a certain group of homotopy classes of maps between systems, which in turn can be calculated by a natural spectral sequence.

Together with Madsen-Rothenberg's description of $\widetilde{\mathcal{N}}_G(X)$ localized away from 2 as a product of equivariant K-theories, this gives us an algorithm of calculation of the group $\widetilde{\mathcal{N}}_G(X)$.

We now consider the non-injectivity of the fixed-point homomorphism of:

$$(*) \qquad \bigoplus \mathrm{Res}_H^G : H_G^m\left(X; \underline{\underline{M}}\right) \longrightarrow \bigoplus_{(\Gamma)} H^m\left(X^\Gamma \underline{\underline{M}}(G/\Gamma)\right)$$

with $\underline{\underline{M}} = \underline{\underline{\pi}}_n(F/PL)$. This would in turn detect the equivariant k-invariant of F/PL, as investigated in ([N 8]). Non-triviality of the k-invariant would imply the existence of some new information hiding in the Mackey structure of the terms of the equivariant surgery exact sequence that we are interested in.

Assumption. We assume that the homomorphism (*) is injective on the group

$$H_G^{i+1}\left(F/PL\langle i-2\rangle; \underline{\underline{\pi}}(F/PL)\right)$$

in which the i-th equivariant k-invariant of F/PL lies, for $i < n$.

Under this assumption, the k-invariants in dimension less than n are all detected by the nonequivariant k-invariants, and therefore produce a map

$$F/PL \longrightarrow \mathcal{E} \times \prod_{i=2}^{n-1} \mathcal{K}\left(\underline{\underline{\hat{\mathcal{L}}}}_i, i\right)$$

which is an $(n-1)$-equivalence.

In particular, we identify the $(n-1)$-st Postnikov component of F/PL as

$$X = F/PL\langle n-1\rangle = \mathcal{E}_0 \times \mathcal{K}\left(\underline{\underline{\hat{\mathcal{L}}}}_2, 2\right) \times \mathcal{K}\left(\underline{\underline{\hat{\mathcal{L}}}}_4, 4\right) \times \prod_{i=6}^{n-1} \mathcal{K}\left(\underline{\underline{\mathcal{L}}}_i, i\right),$$

which we denote by X throughout this section.

The next k-invariant lies in the group

$$H_G^{n+1}\left(X; \underline{\underline{\pi}}_n(F/PL)\right) \qquad \text{with } \underline{\underline{\pi}}_n(F/PL) = \underline{\underline{\mathcal{L}}}_n.$$

Proposition. *For the group $G = G_{21}$ and X as above, the homomorphism*

$$\bigoplus \mathrm{Res}_\Gamma^G : H_G^{n+1}\left(X; \underline{\underline{\mathcal{L}}}_n\right) \longrightarrow \bigoplus_{(\Gamma)} H^{n+1}\left(X^\Gamma; \mathcal{L}_n(\Gamma)\right)$$

is not injective for some choice of n.

Our tool of computation will be the Bredon spectral sequence ([Bre, I.10.4]):

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{C}_G}\left(\underline{\underline{H}}_q(X), \underline{\underline{M}}\right) \implies H_G^{p+q}\left(X; \underline{\underline{M}}\right),$$

where $\underline{\underline{H}}_q(X)$ is the system $G/\Gamma \mapsto H_q(X^\Gamma)$ and $\mathcal{C}_G$ is the category of systems (contravariant functors on $\mathcal{O}_G$). All homology is understood to be with $\mathbb{Z}_{(2)}$-coefficients.

The proof of the Proposition will occupy the rest of this section.

Lemma. *For the group $G = G_{21}$, the homomorphism*

$$\bigoplus \mathrm{Res}^G_\Gamma : H^k_G\left(\mathcal{K}(\underset{=}{RO}, m); \underset{=}{RO}\right) \longrightarrow \bigoplus_{(\Gamma)} H^k\left(K(RO(\Gamma), m); RO(\Gamma)\right)$$

is not injective for some k with $m + 4 \leq k < 2m$.

Proof. Let $Y = \mathcal{K}\left(\underset{=}{RO}, m\right)$ and $\underset{=}{M} = \underset{=}{RO}$. Consider the Bredon spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{C}_G}\left(\underset{=}{H}_q(Y), \underset{=}{M}\right) \implies H^{p+q}_G\left(Y; \underset{=}{M}\right).$$

Since $RO(\Gamma)$ is a free abelian group, Y^Γ is a product of $K(\mathbb{Z}, m)$'s.

We construct a projective resolution of $\underset{=}{H}_q(Y)$ in the category $\mathcal{C}_G$ of systems. Bredon [Bre] pointed out that $\mathcal{C}_G$ has enough projectives and a projective resolution can be condtructed using the projective objects F_S:

$$F_S(G/\Gamma) = \mathbb{Z}[S^\Gamma]$$

for finite G-sets S.

In the stable range $m \leq q < 2m$, generators of $H_q(K(\mathbb{Z}, m); \mathbb{Z})$ are explicitly written down by H. Cartan in [C, 11.6., Théorème 2]. Also in the stable range Künneth theorem implies that generators of $H_q\left(Y^\Gamma; \mathbb{Z}_{(2)}\right)$ are just images of Cartan's elements. More precisely,

$$\begin{aligned}
H_m\left(Y^\Gamma\right) &\cong RO(\Gamma)_{(2)},\\
H_{m+1}\left(Y^\Gamma\right) &= 0,\\
H_{m+2}\left(Y^\Gamma\right) &\cong RO(\Gamma) \otimes \mathbb{Z}/2,\\
H_{m+3}\left(Y^\Gamma\right) &= 0,\\
H_{m+4}\left(Y^\Gamma\right) &\cong RO(\Gamma) \otimes \mathbb{Z}/2, \qquad \text{etc.}
\end{aligned}$$

If we let F and $F_{(q)}$ respectively denote a projective resolution of $\underset{=}{RO}$ in $\mathcal{C}_G$, and of $\underset{=}{RO} \otimes \mathbb{Z}/2$ in $\mathcal{C}_G$ with shifted dimension starting from q, respectively, then a projective resolution of $\underset{=}{H}_q(Y)$ can be obtained by F or sum of $F_{(q)}$'s, one for each Cartan generator in dimension q, as long as we consider matters below dimension $2m$.

Now $\underset{=}{RO}$ being the system as in (5.2), its projective resolution F can be given as follows:

$$\begin{cases}
F^0 &= \left(F_{G/G}\right)^3 \oplus F_{G/H},\\
F^1 &= F_{G/H} \oplus F_{G/A},\\
F^t &= F_{G/H} \oplus F_{G/e} \qquad (t \geq 2),
\end{cases}$$

where

$$
\begin{aligned}
&F_{G/G}(G/-) = \mathbb{Z},\\
&F_{G/H}(G/e) = F_{G/H}(G/H) = \mathbb{Z}^3, F_{G/H}(G/A) = F_{G/H}(G/G) = 0,\\
&F_{G/A}(G/e) = \mathbb{Z} \oplus \mathbb{Z}^6, F_{G/A}(G/A) = \mathbb{Z}, F_{G/A}(G/H) = F_{G/A}(G/G) = 0\\
&F_{G/G}(G/e) = \mathbb{Z}^{21}, F_{G/G}(G/H) = F_{G/G}(G/A) = F_{G/G}(G/G) = 0.
\end{aligned}
$$

where the nontrivial maps are the identity maps, except the $\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}^6$, which is the inclusion onto the first component. The maps are given as follows:

$$
\begin{aligned}
\phi^0 : F^0 \longrightarrow \underline{\underline{RO}} :& (F_{G/G})^3(G/G) \ni a_1, a_2, a_3 \mapsto 1, w, P\\
& F_{G/H}(G/H) \ni b_1, b_2, b_3 \mapsto z_1, z_2, z_3\\
\phi^1 : F^1 \longrightarrow F^0 :& F_{G/H}(G/H) \ni c_1, c_2, c_3 \mapsto a_2 - 2a_1, a_3 - b_1 - b_2 - b_3, 0\\
& F_{G/A}(G/A) \ni d \mapsto a_3 - 2a_1 - 2a_2\\
& F_{G/A}(G/e) \ni d_2, \ldots, d_7 \mapsto\\
& b_1 - 2a_1, b_2 - 2a_1, b_3 - 2a_1, 0, 0, 0\\
\phi^2 : F^2 \longrightarrow F^1 :& F_{G/H}(G/H) \ni e_1, e_2, e_3 \mapsto 0, 0, c_3\\
& F_{G/e}(G/e) \ni f_1, \ldots, f_{21} \mapsto\\
& c_2 - d + d_2 + d_3 + d_4 - 2c_1, d_5, d_6, d_7, 0, \ldots, 0\\
\phi^{2s-1} : F^{2s-1} \longrightarrow F^{2s-2} :& F_{G/H}(G/H) \ni e_1, e_2, e_3 \mapsto e_1, e_2, 0\\
& F_{G/e}(G/e) \ni f_1, \ldots, f_{21} \mapsto 0, 0, 0, 0, f_5, \ldots, f_{21}\\
\phi^{2s} : F^{2s} \longrightarrow F^{2s-1} :& F_{G/H}(G/H) \ni e_1, e_2, e_3 \mapsto 0, 0, e_3\\
& F_{G/e}(G/e) \ni f_1, \ldots, f_{21} \mapsto f_1, f_2, f_3, f_4, 0, \ldots, 0
\end{aligned}
$$

where $s \geq 2$.

Next we consider the system $\underline{\underline{RO}} \otimes \mathbb{Z}/2$. It is

$$
\begin{aligned}
\underline{\underline{RO}} \otimes \mathbb{Z}/2 &= \left(\mathbb{Z}/2 \oplus \underline{\underline{R}}^-\right) \otimes \mathbb{Z}/2\\
&= \underline{\underline{\mathbb{Z}/2}} \oplus \underline{\underline{w}} \oplus \underline{\underline{P}},
\end{aligned}
$$

where

$$
\begin{aligned}
&\underline{\underline{\mathbb{Z}/2}}(G/-) = \mathbb{Z}/2;\\
&\underline{\underline{w}}(G/e) = \underline{\underline{w}}(G/H) = 0,\\
&\underline{\underline{w}}(G/A) = \underline{\underline{w}}(G/G) = \mathbb{Z}/2,\\
&\underline{\underline{P}}(G/e) = \underline{\underline{P}}(G/A) = 0,\\
&\underline{\underline{P}}(G/H) = \mathbb{Z}/2^3, \underline{\underline{P}}(G/G) = \mathbb{Z}/2,
\end{aligned}
$$

where the nontrivial maps are the identity maps, except the $\mathbb{Z}/2 \longrightarrow \mathbb{Z}/2^3$, which is the diagonal map.

Therefore its projective resolution $F_{(q)}$ can be given as follows:

$$F_{(q)} = F_{(\mathbb{Z}/2)} \oplus F_{(w)} \oplus F_{(P)}$$

with dimension shifted, where

$$\begin{aligned}
F^0_{(\mathbb{Z}/2)} &= F^1_{(\mathbb{Z}/2)} = F_{G/G}, \\
F^t_{(\mathbb{Z}/2)} &= 0 \qquad (t \geq 2); \\
F^0_{(w)} &= F_{G/G}, \\
F^1_{(w)} &= F_{G/G} \oplus F_{G/H}, \\
F^2_{(w)} &= F_{G/H}, \\
F^t_{(w)} &= 0 \qquad (t \geq 3); \\
F^0_{(P)} &= F_{G/G} \oplus F_{G/H}, \\
F^1_{(P)} &= F_{G/G} \oplus (F_{G/H})^2 \oplus F_{G/A}, \\
F^2_{(P)} &= F_{G/e}, \\
F^t_{(P)} &= 0 \qquad (t \geq 4),
\end{aligned}$$

where the morphisms are easily computed by the explicit description of the maps ϕ^i in the above.

Now, a direct computation shows that

$$\begin{aligned}
E_2^{p,q} &= \operatorname{Ext}^p_{\mathcal{C}_G}\left(\underline{\underline{H}}_q(Y), \underline{\underline{M}}\right) \\
&= \begin{cases} H^p\left(\operatorname{Hom}_{\mathcal{C}_g}(F, \underline{\underline{M}})\right) & \text{if } q = m \\ \left\{H^p\left(\operatorname{Hom}_{\mathcal{C}_G}\left(F_{(\mathbb{Z}/2)} \oplus F_{(w)} \oplus F_{(P)}, \underline{\underline{M}}\right)\right)\right\}^{A(q,m)} & \text{if } m < q < 2m, \end{cases}
\end{aligned}$$

where $A(q,m)$ is the number of Cartan generators on $H_q(K(\mathbb{Z}, m); \mathbb{Z})$, and

$$H^p\left(\operatorname{Hom}_{\mathcal{C}_g}(F, \underline{\underline{M}})\right) = \begin{cases} \mathbb{Z}^{10} & \text{if } p = 0 \\ \mathbb{Z}^2 & \text{if } p = 1 \\ 0 & \text{if } p \geq 2, \end{cases}$$

$$H^p\left(\operatorname{Hom}_{\mathcal{C}_g}(F_{(\mathbb{Z}/2)}, \underline{\underline{M}})\right) = \begin{cases} 0 & \text{if } p = 0 \\ (\mathbb{Z}/2)^3 & \text{if } p = 1 \\ 0 & \text{if } p \geq 2, \end{cases}$$

$$H^p\left(\mathrm{Hom}_{\mathcal{C}_g}(F_{(w)}, \underline{\underline{M}})\right) = \begin{cases} 0 & \text{if } p=0 \\ \mathbb{Z}/2 & \text{if } p=1 \\ (\mathbb{Z}/2)^2 = \mathbb{Z}^3/\Delta + 2\mathbb{Z}^3 & \text{if } p=2 \\ 0 & \text{if } p\geq 3, \end{cases}$$

$$H^p\left(\mathrm{Hom}_{\mathcal{C}_g}(F_{(P)}, \underline{\underline{M}})\right) = \begin{cases} 0 & \text{if } p=0 \\ (\mathbb{Z}/2)^3 & \text{if } p=1 \\ 0 & \text{if } p\geq 2. \end{cases}$$

The unique elements of homological degree 2 in $H^2\left(\mathrm{Hom}_{\mathcal{C}}\left(F_{(w)}, \underline{\underline{M}}\right)\right)$ are produced by the relation

$$\phi^2_{(w)}(c_1) = a - 2b_1 \in F_{G/H}(G/H)$$

in $F_{(w)}$, and the map

$$\mathrm{Res}^G_H(P) = z_1 + z_2 + z_4 \in RO(H)$$

in $\underline{\underline{M}} = \underline{\underline{RO}}$. Both of them reflect the fact that Res^G_H is not surjective in the system.

Let us turn to the image of the map $\oplus\mathrm{Res}^G_H$. Given any $\mathcal{C}_G$-resolution F_* of $\underline{\underline{H}}_q(Y)$, if we restrict it to the values of G/Γ, it forms a module resolution $F_*(G/\Gamma)$ of the module $\underline{\underline{H}}_q(Y) = H_q\left(Y^\Gamma\right)$. Also this correspondence gives a cochain map

$$\mathrm{Hom}_{\mathcal{C}_G}\left(F_*, \underline{\underline{M}}\right) \longrightarrow \mathrm{Hom}\left(F_*(G/\Gamma), \underline{\underline{M}}(G/\Gamma)\right)$$

and hence a map of spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{C}}\left(\underline{\underline{H}}_q(Y), \underline{\underline{M}}\right) \longrightarrow {}^{\backprime}E_2^{p,q} = \mathrm{Ext}^p_{\mathbb{Z}}\left(\underline{\underline{H}}_q\left(Y^\Gamma\right), \underline{\underline{M}}(G/\Gamma)\right).$$

The right hand side forms the usual universal coefficient spectral sequence for the space Y^Γ, and hence collapses since

$$H_q\left(Y^\Gamma\right) = \begin{cases} \mathbb{Z}^t & \text{if } q=m \\ (\mathbb{Z}/2)^s & \text{if } q>m. \end{cases}$$

Now that we know

$$\begin{aligned} E_2^{p,q} &= 0 \qquad \text{if } p\geq 3, \\ E_2^{0,q} &= 0 \qquad \text{if } q\geq m+1, \\ E_2^{2,q} &= (\mathbb{Z}/2)^{2A(q,m)}, \\ {}^{\backprime}E_2^{p,q} &= 0 \qquad \text{if } p\geq 2, \end{aligned}$$

and the differentials are

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1},$$

we see that there is no room for nontrivial differentials, so both of the spectral sequences collapse.

The nontrivial term $E^{2,q}$ is in the kernel of the spectral sequence morphism, and hence is a nontrivial kernel in the $E^{2,q}$. But since $E_\infty^{p,q} = 0$ for $p \geq 3$, this kernel lies in the highest (i.e., smallest) filtration term, thus produces a nontrivial kernel of

$$\mathrm{Res}_\Gamma^G : H_G^{p+q}\left(Y; \underline{\underline{M}}\right) \longrightarrow H^{p+q}\left(Y^\Gamma; \underline{\underline{M}}(G/\Gamma)\right).$$

Since the same $E_2^{p,q}$ is in the kernel for any Γ, it produces a nontrivial kernel of

$$\bigoplus \mathrm{Res}_H^G : H_G^{p+q}\left(Y; \underline{\underline{M}}\right) \longrightarrow \bigoplus_{(\Gamma)} H^{p+q}\left(Y^\Gamma; \underline{\underline{M}}(G/\Gamma)\right).$$

This completes the proof of the Lemma.

Remark. $A(q,m) = \frac{1}{2}\,\mathrm{rank}\, E_2^{2,q}$ is non-zero if

$$q - m = 2, 4, 6, 8, 10, 12, 14, 16, 17, \ldots .$$

(See Cartan's formula in [C].)

We also remark that similar proof works for

$$Y = \mathcal{K}\left(\underline{\underline{RO}}, m\right) \quad \text{or} \quad \mathcal{K}\left(\mathbb{Z}/2 \oplus \underline{\underline{R}}^-, m\right),$$
$$\underline{\underline{M}} = \underline{\underline{RO}} \quad \text{or} \quad \mathbb{Z}/2 \oplus \underline{\underline{R}}^-,$$

and an analogue of the Lemma holds.

We return to the proof of the Proposition, where

$$X = \mathcal{E}_0 \times \mathcal{K}\left(\underline{\underline{\hat{\mathcal{L}}}}_2, 2\right) \times \mathcal{K}\left(\underline{\underline{\hat{\mathcal{L}}}}_4, 4\right) \times \prod_{i=6}^{n-1} \mathcal{K}\left(\underline{\underline{\mathcal{L}}}_i, i\right),$$

and the coefficient system is $\underline{\underline{\mathcal{L}}}_n$.

If we take n to be a multiple of 4, we can choose m in such that m is also a multiple of 4, $m + 4 \leq n + 1 < 2m$ and such that

$$A(n-1, m) \neq 0 \qquad \text{for such } m,$$

by the above remark.

Therefore it suffices to show that there is a natural homomorphism

$$P^* : H_G^*\left(Y; \underline{\underline{RO}}\right) \longrightarrow H_G^*\left(X; \underline{\underline{\mathcal{L}}}_n\right)$$

which is injective. This follows from the next Lemma, which implies that Y is a direct factor of X as a G-space:

Lemma. *The system $\underline{\underline{RO}}$ is included in the system $\underline{\underline{\mathcal{L}}}_n$ as a direct summand of system, if $n \equiv 0 \mod 4$.*

Proof. $\underline{\underline{\mathcal{L}}}_n(G/\Gamma) = \mathcal{L}_n(\Gamma) = \bigoplus_{(\Lambda)\subset\Gamma} L_n(N_\Gamma\Lambda/\Lambda)$ includes $L_n(\Gamma/e) = RO(\Gamma)$ as a "top summand". The system structure of $\underline{\underline{\mathcal{L}}}_n$ splits this collection of $RO(\Gamma)$'s as a direct summand of system, because the "top summand" and the complementary summand are both preserved by the structure. Thus the proof of the Proposition is complete.

Finally we remark that the same situation occurs for actions of general non-abelian metacyclic group G of odd order. In the similar way as above, the non-surjectivityof Res_H^G in the system $\underline{\underline{RO}}$ produces a nontrivial kernel of the fixed-point homomorphism inside the Bredon cohomology group.

The result of the Proposition implies that the Bredon cohomology group in which the euqivariant k-invariant of F/PL lies is not detected by the nonequivariant cohomology of the fixed-point setsm for the group $G = G_{21}$, or more generally, by the above remark, of any nonabelian metacyclic group G of odd order.

This fact, together with a spectral sequence argument (similar to the one in Section 6 of [DS]) shows the existence of an exotic k-invariant of F/PL, in the sense that it is nontrivial, but vanishes after one maps it to nonequivariant data. We hope to construct an explicit geometric invariant which could detect these exotic elements in future work.

Section 5. Diagram Obstruction, an Example

Using the concrete example of the previous section, we will construct "diagram obstruction" groups à la Dula and Schultz [DS] in certain situations of G-homotopy equivalences of G-manifolds, where $G = G_{21} = \mathbb{Z}/7 \underset{\alpha}{\rtimes} \mathbb{Z}/3$ as in the previous section.

Dula and Schultz [DS] constructed their "diagram obstruction" groups via orbit-type stratification of G-manifolds, and the basis of their calculation is Barratt-Federer Spectral Sequence [DS, Theorem 1.3]. It is a spectral sequences of the following form:

$$E^2_{i,j} = {}_{BR}H_G^{-i}\left(X; {}_G\pi_j(Y)\right) \implies \pi_{i+j}\left(F_G(X,Y)\right)$$

for finite dimensional G-CW complex X, where $F_G(X,Y)$ is the set of G-maps from X to Y, and ${}_{BR}H_G^{-i}\left(X; {}_G\pi_j(Y)\right)$ is Bredon Cohomology with equivariant twisted coefficients. Based on this tool, they compute the equivariant cohomology of orbit-type stratification, as follows:

Theorem (Dula-Schultz, [DS, Theorem 1.5]**).** *Let X be a finite simplicial complex with a simplicial action of the finite group G, and let Y be a G-CW complex satisfying certain additional conditions. Choose an indexing $\{(K_i)\}$ for the conjugacy classes of isotropy subgroups such that $i < j$ if a representative for (K_j) is contained in a* *representative for (K_i), let F_ℓ be the G-subcomplex of points whose isotropy subgroups represent (K_ℓ), and let $X_\ell = F_1 \cup \cdots \cup F_\ell$. Then there is a spectral sequence such that*

$$E^2_{p,q} \subset \bigoplus_i H^i\left(X_{i-p}/G, X_{i-p-1}/G; \pi_{p+q-i}(\mathrm{Fix}(K_{i-p}, Y))\right)$$

(the coefficient on the right may be twisted), with equiality if $p+q \geq 2$, and such that $E^\infty_{p,q}$ gives a series for $\pi_{p+q}\left(F_G(X,Y)\right)$.

In our case of metacyclic group action, their "certain additional conditions" are not quite satisfied, but since we explicitly know the non-linear orbit category for this simple example of metacyclic group $G = G_{21} = \mathbb{Z}/7 \underset{\alpha}{\rtimes} \mathbb{Z}/3$, we can manage to construct a similar spectral sequence in our situation.

Assume that we are given a G-equivariant homotopy equivalence $f : M \to N$ of conected, compact, closed, oriented smooth G-manifolds, and assume the Gap Hypothesis for M and N as in Section 2. In trying to build a G-homotopy from f to a G-isovariant homotopy equivalence, Dula and Schultz defined the notion of "almost isovariant" maps, in terms of cohomology classes (where "isovariant" maps are defined by a strict point-wise condition, which is hard to detect by obstruction theories) and proved that the isovariance condition can be replaced with the almost isovariance condition in many important cases. That is discussed in Section 4 of [DS].

So, we are reduced to computing the obstruction classes defined by the orbit type stratification as in Dula-Schultz' Theorem 1.5 above, and try to construct a homotopy that deforms the given homotopy equivalence into an almost isovariant homotopy equivalence.

Now we try to generalize their methods into our particular metacyclic group G, which does not satisfy Dula-Schultz' "certain additional conditions". Here is our main claim:

Proposition. *For the group $G = G_{21}$, the "treelike isotropy structure" of* [DS, Proposition 3.6] *can be generalized to the orbit-type structure of G in a weak sense. That is, it is not quite "tree-like", but after moving to the cohomology level for the calculation in Theorem 1.5 above, the difficulty vanishes if the Gap Hypothesis is satisfied, and we can proceed to make use of the theorem.*

In dealing with cohomology groups

$$\bigoplus_i H^i\left(X_{i-p}/G, X_{i-p-1}/G; \pi_{p+q-i}(\mathrm{Fix}(K_{i-p}, Y))\right)$$

as in Theorem 1.5 on which the orbit-type category acts, we construct diagrams of cohomology groups and homomorphisms that reflect the orbit-type structure of our $G = G_{21}$, which is described in the previous section. A spectral sequence similar to the one in there can be constructed for our strata-wise cohomology groups, quite in the same way as in the proof of Lemma in the previous section, and we find that there is a nontrivial class in the E^2-term level of the spectral sequence of Theorem 1.5. However, given the Gap Hypothesis, we can estimate the deviation from the "treelike"-ness will vanish in the E^∞ level, and thus the argument of Dula-Schulz' paper [DS] can be applied to our situation, too.

The key to the vanishing under the Gap Hypothesis is the following:

Lemma. *The orbit category of $G = G_{21}$ has only one "non-treelike" path, that is the $1 \longrightarrow H = \mathbb{Z}/7 \longrightarrow G$ path as shown in Section 4, that creates an extra relation in the diagram of cohomology groups that does not align linearly as required by the "treelike" condition of Dula-Schultz. Thus, we can keep track of the extra information as diagrams of cohomology groups, and can confirm that it does not produce any additional obstruction for constructing the Dula-Schultz type "diagram obstruction" classes, provided that the relevant spaces all satisfy the Gap Hypothesis.*

The computation involves dimension estimates in the spectral sequences, which provides vanishing under the Gap Hypothesis, and examination of the way the "extra relation" affects the cohomology classes. Under our assumption, we can follow the arguments of Dula and Schultz, keeping track of the action of the subgroup $H = \mathbb{Z}/7$ in our system. That in turn results in the construction of isovariant maps in exactly the same way as the one in Proposition 5.3 of [Sch], and we get the desired global G-isovariant map, confirming the functoriality as stated in [CWY 2], in our particular example.

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