Squeezing on a Certain $L$-space

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1. INTRODUCTION

In a joint 2006 paper [2], E. Pedersen and I proved a certain stability result for controlled $L$-groups. The proof depended on a construction called the Alexander trick. In this note I describe a modified Alexander trick which can be used to give a built-in squeezing mechanism of a certain $L$-space. This should replace the “barycentric subdivision argument” used in [4].

2. ITERATED MAPPING CYLINDERS

Let $X$ be a finite polyhedron, and $M$ be a topological space. We are interested in a map $p : M \rightarrow X$ which has an iterated mapping cylinder decomposition in the sense of Hatcher [1]: there is a partial order on the set of the vertices of $X$ such that, for each simplex $\Delta$ of $X$,

1. the partial order restricts to a total order of the vertices of $\Delta$
   
   $$v_0 < v_1 < \cdots < v_n,$$

2. $p^{-1}(\Delta)$ is the iterated mapping cylinder of a sequence of maps
   
   $$F_{v_0} \rightarrow F_{v_1} \rightarrow \cdots \rightarrow F_{v_n},$$

3. the restriction $p|_{p^{-1}(\Delta)}$ is the natural map induced from the iterated mapping cylinder structure of $p^{-1}(\Delta)$ above and the iterated mapping cylinder structure of $\Delta$ coming from the sequence
   
   $$\{v_0\} \rightarrow \{v_1\} \rightarrow \cdots \rightarrow \{v_n\}.$$

To simplify the situation we assume that $X$ is an $n$-simplex $\Delta$ with vertices $v_0$, $v_1$, $\ldots$, $v_n$. The edge $|v_0, v_1|$ is the mapping cylinder $v_0 \times \{0 \leq t_1 \leq 1\}/(v_0, 1) \sim v_1$, the face $|v_0, v_1, v_2|$ is the mapping cylinder $|v_0, v_1| \times \{0 \leq t_2 \leq 1\}/(x, 1) \sim v_2$, $\ldots$, and $\Delta = |v_0, \ldots, v_n|$ is the mapping cylinder $|v_0, \ldots, v_{n-1}| \times \{0 \leq t_n \leq 1\}/(x, 1) \sim v_n$. Thus we can assign a point in $\Delta$ to each $(t_1, \ldots, t_n) \in [0, 1]^n$. $(t_1, \ldots, t_n)$ is pseudo-coordinates of the point in the sense that the coordinates are not uniquely determined.

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by the point. If \((\lambda_0, \ldots, \lambda_n)\) are the barycentric coordinates of a point \(x \in \Delta\), i.e.
\[x = \sum \lambda_i v_i \ (\lambda_0 + \cdots + \lambda_n = 1),\]
then \(t_i\) is equal to \(\lambda_i/(\lambda_0 + \cdots + \lambda_i)\), when defined, and is indeterminate when \(\lambda_0 = \cdots = \lambda_i = 0\).

For each vertex \(v\) of \(\Delta\), define a simplicial map \(s^v : \Delta \to \Delta\) by:
\[
s^v(u) = \begin{cases} 
v & \text{for a vertex } u \text{ with } u < v, \\
u & \text{for a vertex } u \text{ with } u \geq v. \\
\end{cases}
\]
For example, \(s^{v_0}\) is the identity map, and \(s^{v_n}\) is the constant map which sends every point of \(\Delta\) to \(v_n\). A strong deformation retraction \(s^i : \Delta \to \Delta\) is defined by \(s^i(x) = (1 - t)x + ts^v(x)\), where \(x \in \Delta\) and \(t \in [0, 1]\). Note that this strong deformation retraction \(s^i\) is covered by a deformation \(s^i_t\) on \(M\), since \(M\) has an iterated mapping cylinder structure. Also note that \(s^{i_t} (t > 0)\) changes the \(t_j\) pseudo-coordinate but fixes the other pseudo-ordinates \(t_i (i \neq j)\).

3. **Alexander Tricks**

Let \(M\) be an iterated mapping cylinder of maps
\[
F_{v_0} \longrightarrow F_{v_1} \longrightarrow \ldots \longrightarrow F_{v_n},
\]
and \(p : M \to \Delta = [v_0, \ldots, v_n]\) be the projection from \(M\) to the ordered \(n\)-simplex \(\Delta\) as in the previous section. Suppose \(c\) is a quadratic Poincaré \((n + 2)\)-ad on \(p : M \to \Delta\), such that \(\partial_i c\) is a quadratic Poincaré \((n + 1)\)-ad on \(p|p^{-1}(\partial_i \Delta), i = 0, \ldots, n\) ([4] [5]). Such an \((n + 2)\)-ad \(c\) is said to be *proper on \(\Delta\) or simply proper*.

We will describe a version of Alexander trick for such a proper \((n + 2)\)-ad \(c\). First fix a positive integer \(N\) ("height") and pick up a vertex \(v = v_j\) of \(\Delta\) toward which we try to squeeze the objects. Triangulate the closed interval \(I_N = [0, N]\) using unit intervals and represent each simplex by its barycenter. Use these points to construct the symmetric Poincaré triad \(e\) of \((I_N; 0, N)\). Take the tensor product of \(c\) and \(e\) and denote it by \(c \times I_N\). This is a geometric object on \(M \times I_N\) which gives a cobordism between \(c \times 0\) and the \((n + 2)\)-ad \(c'\) defined by:
\[
c' = c \times N \cup \partial_j c \times I_N,
\]
\[
\partial_i c' = \begin{cases} 
\partial_i c \times N \cup \partial_{j-1} \partial_i c \times I_N & \text{if } i < j, \\
\partial_j c \times 0 & \text{if } i = j, \\
\partial_i c \times N \cup \partial_j \partial_i c \times I_N & \text{if } i > j.
\end{cases}
\]
So this construction does not change the $j$-th face $\partial_j c = \partial_j c \times 0$. If $i \neq j$, then one can perform the same construction to $\partial_i c$ to get $(\partial_i c)'$, which coincides with $\partial_i c'$.

Define maps $S_N^v : \Delta \times I_N \to \Delta \times I_N$ and $\tilde{S}_N^v : M \times I_N \to M \times I_N$ by

$$S_N^v(x, t) = (s_{i/N}(x), t) \quad \text{and} \quad \tilde{S}_N^v(w, t) = (\tilde{s}_{i/N}(w), t).$$

Define an ordered $(n + 1)$-simplex $\Delta^{n+1} (\subset \Delta \times I_N)$ by

$$\Delta^{n+1} = (\langle v_0, \ldots, v_j \rangle \times 0) \ast (\langle v_j \rangle \times N) \ast (\langle v_{j+1}, \ldots, v_n \rangle \times 0).$$

Here $\ast$ denotes the join of simplices. Note that

$$S_N^v(\Delta \times I_N) = \bigcup_{0 \leq t \leq N} (s_{i/N}^{v}(\langle v_0, \ldots, v_j \rangle \times t) \ast (\langle v_{j+1}, \ldots, v_n \rangle \times t),$$

$$\Delta^{n+1} = \bigcup_{0 \leq t \leq N} (s_{i/N}^{v}(\langle v_0, \ldots, v_j \rangle \times t) \ast (\langle v_{j+1}, \ldots, v_n \rangle \times 0).$$

Therefore, the obvious vertical retraction

$$\langle v_{j+1}, \ldots, v_n \rangle \times I_N \longrightarrow \langle v_{j+1}, \ldots, v_n \rangle \times 0$$

induces a map $R_N^v$ from the image $S_N^v(\Delta \times I_N)$ to $\Delta^{n+1}$. Let

$$q = p \times 1_{I_N} : M_{\Delta^{n+1}} = (p \times 1_{I_N})^{-1}(\Delta^{n+1}) \to \Delta^{n+1}$$

denote the pull-back of $p : M \to \Delta$ by the projection map

$$\pi : \Delta^{n+1} \xrightarrow{\text{inclusion}} \Delta \times I_N \xrightarrow{\text{projection}} \Delta.$$

The map $R_N^v$ is covered by a map $\tilde{R}_N^v : \tilde{S}_N^v(\Delta \times I_N) \to M_{\Delta^{n+1}}$.

Let us look at the relation between $c$ and $c'$ (and its functorial image $(\tilde{R}_N^v \circ \tilde{S}_N^v)_*(c')$) more closely. As in the pictures above, define a subset $\Delta'$ of $\partial(\Delta \times I_N)$ by

$$\Delta' = \Delta \times N \cup \partial_j \Delta \times I_N.$$
The \((n+2)\)-ad \(c'\) lies over \(\Delta'\). By gluing some of the faces, let us regard \(c \times I_N\) as an \((n+3)\)-ad whose faces are

\[
\partial_0 c \times I_N, \ldots, \partial_{j-1} c \times I_N, c', c \times 0, \partial_{j+1} c \times I_N, \ldots, \partial_n c \times I_N.
\]

The functorial image of this \((n+3)\)-ad by the composition \(\tilde{R}_N^0 \circ \tilde{S}_N^0\) defines a proper quadratic Poincaré \((n+3)\)-ad \(C_N^0(c)\) on \(q : M_{\Delta^{n+1}} \to \Delta^{n+1}\).

The face \((\tilde{R}_N^0 \circ \tilde{S}_N^0)^* (c')\) is a proper quadratic Poincaré \((n+2)\)-ad on \(q|q^{-1}(R_N^0(S_N^0(\Delta'))),\) and is denoted \(A_N^0(c)\). Its functorial image \(\pi_* (A_N^0(c))\) will be denoted \(a_N^0(c)\). It is a proper on \(\Delta\). The functorial image \(\pi_* (C_N^0(c))\) can be regarded as a Poincaré cobordism between \(c\) and \(a_N^0(c)\). The operation described above is called the Alexander trick (of height \(N\)) at the vertex \(v = v_j\). Note that \(a_N^0(c)\) has a fine control in the \(t_j\) pseudo-coordinate. Also note that \(\partial_j a_N^0(c) = a_N^0(\partial_j c) = \partial_j c\), where \(v = v_j\).

If we successively apply the Alexander tricks at \(v_n, \ldots, v_1, v_0\) to the given proper quadratic Poincaré \((n+2)\)-ad \(c\), then we get finely controlled object which is cobordant to \(c\). This process is called “squeezing” of “shrinking”. When we use the same height \(N\) at every vertex, then the squeezed object obtained from \(c\) will be denoted \(S_N(c)\):

\[
S_N(c) = a_N^0(a_N^1(\ldots (a_N^0(c)) \ldots ))
\]

The cobordism between \(c\) and \(S_N(c)\) constructed above is called the standard cobordism. The squeezing operation \(S_N\) preserves the face relation:

**Proposition 3.1.** \(\partial S_N(c)\) is equal to \(S_N(\partial c)\). Furthermore, the standard cobordism between \(\partial c\) and \(\partial S_N(c)\) is equal to the standard cobordism between \(\partial c\) and \(S_N(\partial c)\).

4. **\(L\)-spaces**

The squeezing operation seems to justify the following simple definition of the coefficient \(L\)-space \(\mathbb{L}_n(p : M \to X)\) for the generalized homology \(H_*(X; \mathbb{L}(p))\), where \(p : M \to X\) is a map from a space to a finite polyhedron which has an iterated mapping cylinder decomposition and \(n\) is an integer. It is a \(\Delta\)-set; a \(k\)-simplex is an \((n+k)\)-dimensional proper quadratic Poincaré \((k+2)\)-ad \((c; \partial_0 c, \ldots, \partial_k c)\) on the pull-back \(\pi^* M \to (\Delta; \partial_0 \Delta, \ldots, \partial_k \Delta)\), where \(\Delta\) is a \(k\)-simplex and \(\pi : \Delta \to \Delta^l\) is an affine surjection from \(\Delta\) to an \(l\)-dimensional simplex \(\Delta^l\) of \(X\) \((l \leq k)\) induced by an order\((\leq)\) preserving map between the vertices.

Two such simplices \((c, \pi : \Delta \to \Delta^l)\) and \((c', \pi' : \Delta' \to \Delta^l)\) are identified when there is an affine homeomorphism \(\phi : \Delta \to \Delta'\) of ordered simplices such that \(\pi = \pi' \circ \phi\) and \(\phi_*(c) = c'\).
Note that the squeezing operation $S_N$ defines a simplicial homotopy of the identity map of $\mathbb{L}_n(p : M \to X)$ to a simplicial map whose image is contained in a subset made up of simplices of ‘small radius’ measured on $X$, if $N$ is large. Thus this space has a built-in ‘squeezing’ mechanism.

Let us consider the special case when $X$ is a single point. There is a similar $\Delta$-set $\mathbb{L}'_n(M)$ whose $k$-simplex is an $(n+k)$-dimensional quadratic Poincaré $(k+2)$-ad $c$ on $M$ that is special, i.e. $\partial_0 \partial_1 \ldots \partial_k c = 0$. $\pi_0(\mathbb{L}'_n(p : M \to *))$ is isomorphic to $L^h_0(\mathbb{Z}\pi_1(M))$.

There is a map $\mathbb{L}_n(M \to *) \to \mathbb{L}'_n(M)$ that sends a $k$-simplex $(c, \pi)$ to its functorial image $\pi_*(c)$. A map in the reverse direction can be constructed as follows. Let $c$ be a $k$-simplex of $\mathbb{L}'_n(M)$. It is made up of three type of things: (1) ‘points’ in $M$ (generators of free modules), (2) paths with coefficients connecting the generators, and (3) homotopies of certain paths. Since $c$ is special, one can make a 1–1 correspondence between its faces (including $c$ itself) and the faces of a standard $k$-simplex $\Delta$ (including $\Delta$ itself), and can make copies of the faces of $c$ on the sets $\{\text{barycenters}\} \times M \subset \Delta \times M$ and realizing the morphisms between adjacent pieces by using the original paths in $c$ in the $M$-direction and the path connecting two adjacent barycenters in the $\Delta$-direction as components. Similarly for homotopies of paths. These are homotopy inverses of each other.
Therefore, $\mathbb{L}_n(\rho : M \to X)$ defined above may give a convenient description of $\mathbb{L}$-homology groups.

REFERENCES