Squeezing on a Certain $\mathbb{L}$-space

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1. Introduction

In a joint 2006 paper [2], E. Pedersen and I proved a certain stability result for controlled L-groups. The proof depended on a construction called the Alexander trick. In this note I describe a modified Alexander trick which can be used to give a built-in squeezing mechanism of a certain $\mathbb{L}$-space. This should replace the "barycentric subdivision argument" used in [4].

2. Iterated Mapping Cylinders

Let X be a finite polyhedron, and M be a topological space. We are interested in a map $p : M \to X$ which has an iterated mapping cylinder decomposition in the sense of Hatcher [1]: there is a partial order on the set of the vertices of X such that, for each simplex Δ of X,

(1) the partial order restricts to a total order of the vertices of Δ

$$v_0 < v_1 < \cdots < v_n \ ,$$

(2) $p^{-1}(\Delta)$ is the iterated mapping cylinder of a sequence of maps

$$F_{v_0} \longrightarrow F_{v_1} \longrightarrow \dots \longrightarrow F_{v_n} \ ,$$

(3) the restriction $p|p^{-1}(\Delta)$ is the natural map induced from the iterated mapping cylinder structure of $p^{-1}(\Delta)$ above and the iterated mapping cylinder structure of Δ coming from the sequence

$$\{v_0\} \longrightarrow \{v_1\} \longrightarrow \dots \longrightarrow \{v_n\} \ .$$

To simplify the situation we assume that X is an n-simplex Δ with vertices v_0, v_1, $\dots$, v_n. The edge $|v_0, v_1|$ is the mapping cylinder $v_0 \times \{0 \le t_1 \le 1\}/(v_0, 1) \sim v_1$, the face $|v_0, v_1, v_2|$ is the mapping cylinder $|v_0, v_1| \times \{0 \le t_2 \le 1\}/(x, 1) \sim v_2$, $\dots$, and $\Delta = |v_0, \dots, v_n|$ is the mapping cylinder $|v_0, \dots, v_{n-1}| \times \{0 \le t_n \le 1\}/(x, 1) \sim v_n$. Thus we can assign a point in Δ to each $(t_1, \dots, t_n) \in [0, 1]^n$. $(t_1, \dots, t_n)$ is *pseudo-coordinates* of the point in the sense that the coordinates are not uniquely determined by the point. If $(\lambda_0, \dots, \lambda_n)$ are the barycentric coordinates of a point $x \in \Delta$, *i.e.*

$x = \sum \lambda_i v_i$ $(\lambda_0 + \cdots + \lambda_n = 1)$, then t_i is equal to $\lambda_i/(\lambda_0 + \cdots + \lambda_i)$, when defined, and is indeterminate when $\lambda_0 = \cdots = \lambda_i = 0$.

For each vertex v of Δ, define a simplicial map $s^v : \Delta \to \Delta$ by:

$$s^v(u) = \begin{cases} v & \text{for a vertex } u \text{ with } u < v \ , \\ u & \text{for a vertex } u \text{ with } u \geq v \ . \end{cases}$$

For example, s^{v_0} is the identity map, and s^{v_n} is the constant map which sends every point of Δ to v_n. A strong deformation retraction $s^v_t : \Delta \to \Delta$ is defined by $s^v_t(x) = (1-t)x + t\, s^v(x)$, where $x \in \Delta$ and $t \in [0,1]$. Note that this strong deformation retraction s^v_t is covered by a deformation $\tilde{s}^v_t$ on M, since M has an iterated mapping cylinder structure. Also note that $s^{v_j}_t$ $(t > 0)$ changes the t_j pseudo-coordinate but fixes the other pseudo-cordinates t_i $(i \neq j)$.

3. Alexander Tricks

Let M be an iterated mapping cylinder of maps

$$F_{v_0} \longrightarrow F_{v_1} \longrightarrow \ldots \longrightarrow F_{v_n} \ ,$$

and $p : M \to \Delta = |v_0, \ldots, v_n|$ be the projection from M to the ordered n-simplex Δ as in the previous section. Suppose c is a quadratic Poincaré $(n+2)$-ad on $p : M \to \Delta$, such that $\partial_i c$ is a quadratic Poincaré $(n+1)$-ad on $p|p^{-1}(\partial_i \Delta)$, $i = 0, \ldots, n$ ([4] [5]). Such an $(n+2)$-ad c is said to be *proper on* Δ or simply *proper*.

We will describe a version of Alexander trick for such a proper $(n+2)$-ad c. First fix a positive integer N ("*height*") and pick up a vertex $v = v_j$ of Δ toward which we try to squeeze the objects. Triangulate the closed interval $I_N = [0, N]$ using unit intervals and represent each simplex by its barycenter. Use these points to construct the symmetric Poincaré triad e of $(I_N; 0, N)$. Take the tensor product of c and e and denote it by $c \times I_N$. This is a geometric object on $M \times I_N$ which gives a cobordism between $c \times 0$ and the $(n+2)$-ad c' defined by:

$$c' = c \times N \cup \partial_j c \times I_N,$$

$$\partial_i c' = \begin{cases} \partial_i c \times N \cup \partial_{j-1} \partial_i c \times I_N & \text{if } i < j, \\ \partial_j c \times 0 & \text{if } i = j, \\ \partial_i c \times N \cup \partial_j \partial_i c \times I_N & \text{if } i > j. \end{cases}$$

So this construction does not change the j-th face $\partial_j c = \partial_j c \times 0$. If $i \neq j$, then one can perform the same construction to $\partial_i c$ to get $(\partial_i c)'$, which coincides with $\partial_i c'$.

Define maps $S_N^v : \Delta \times I_N \to \Delta \times I_N$ and $\widetilde{S}_N^v : M \times I_N \to M \times I_N$ by

$$S_N^v(x,t) = (s_{t/N}^v(x), t) \ \text{ and } \ \widetilde{S}_N^v(w,t) = (\tilde{s}_{t/N}^v(w), t) \ .$$

Define an ordered $(n+1)$-simplex Δ^{n+1} $(\subset \Delta \times I_N)$ by

$$\Delta^{n+1} = (\langle v_0, \ldots, v_j \rangle \times 0) * (\langle v_j \rangle \times N) * (\langle v_{j+1}, \ldots, v_n \rangle \times 0) \ .$$

Here $*$ denotes the join of simplices. Note that

$$S_N^v(\Delta \times I_N) = \bigcup_{0 \le t \le N} (s_{t/N}^v(\langle v_0, \ldots, v_j \rangle \times t) * (\langle v_{j+1}, \ldots, v_n \rangle \times t) \ ,$$
$$\Delta^{n+1} = \bigcup_{0 \le t \le N} (s_{t/N}^v(\langle v_0, \ldots, v_j \rangle \times t) * (\langle v_{j+1}, \ldots, v_n \rangle \times 0).$$

Therefore, the obvious vertical retraction

$$\langle v_{j+1}, \ldots, v_n \rangle \times I_N \longrightarrow \langle v_{j+1}, \ldots, v_n \rangle \times 0$$

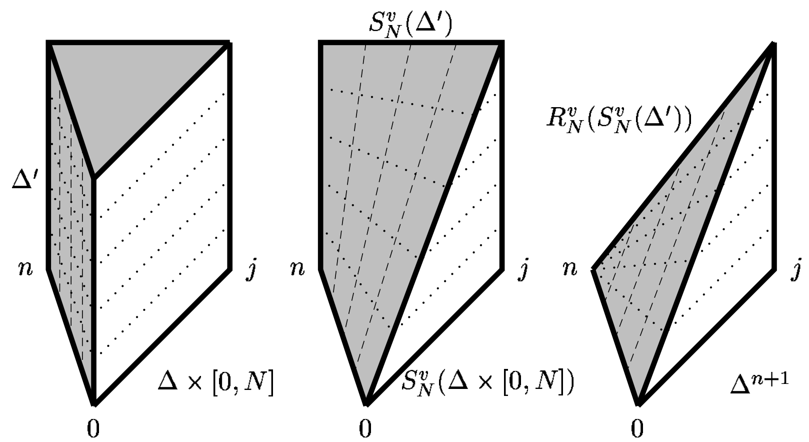
induces a map R_N^v from the image $S_N^v(\Delta \times I_N)$ to Δ^{n+1}. Let

$$q = p \times 1_{I_N}| : M_{\Delta^{n+1}} = (p \times 1_{I_N})^{-1}(\Delta^{n+1}) \to \Delta^{n+1}$$

denote the pull-back of $p : M \to \Delta$ by the projection map

$$\pi : \Delta^{n+1} \xrightarrow{\text{inclusion}} \Delta \times I_N \xrightarrow{\text{projection}} \Delta \ .$$

The map R_N^v is covered by a map $\widetilde{R}_N^v : \widetilde{S}_N^v(M \times I_N) \to M_{\Delta^{n+1}}$.



Let us look at the relation between c and c' (and its functorial image $(\widetilde{R}_N^v \circ \widetilde{S}_N^v)_*(c')$) more closely. As in the pictures above, define a subset Δ' of $\partial(\Delta \times I_N)$ by

$$\Delta' = \Delta \times N \cup \partial_j \Delta \times I_N \ .$$

The $(n+2)$-ad c' lies over Δ'. By glueing some of the faces, let us regard $c \times I_N$ as an $(n+3)$-ad whose faces are

$$\partial_0 c \times I_N,\ \ldots,\ \partial_{j-1} c \times I_N,\ c',\ c \times 0,\ \partial_{j+1} c \times I_N,\ \ldots,\ \partial_n c \times I_N\ .$$

The functorial image of this $(n+3)$-ad by the composition $\widetilde{R}_N^v \circ \widetilde{S}_N^v$ defines a proper quadratic Poincaré $(n+3)$-ad $\mathcal{C}_N^v(c)$ on $q : M_{\Delta^{n+1}} \to \Delta^{n+1}$.

The face $(\widetilde{R}_N^v \circ \widetilde{S}_N^v)_*(c')$ is a proper quadratic Poincaré $(n+2)$-ad on $q|q^{-1}(R_N^v(S_N^v(\Delta')))$, and is denoted $A_N^v(c)$. Its functorial image $\pi_*(A_N^v(c))$ will be denoted $a_N^v(c)$. It is a proper on Δ. The functorial image $\pi_*(\mathcal{C}_N^v(c))$ can be regarded as a Poincaré cobordism between c and $a_N^v(c)$. The operation described above is called the *Alexander trick* (*of height* N) *at the vertex* $v = v_j$. Note that $a_N^v(c)$ has a fine control in the t_j pseudo-coordinate. Also note that $\partial_j a_N^v(c) = a_N^v(\partial_j c) = \partial_j c$, where $v = v_j$.

If we successively apply the Alexander tricks at $v_n, \ldots, v_1, v_0$ to the given proper quadratic Poincaré $(n+2)$-ad c, then we get finely controlled object which is cobordant to c. This process is called "squeezing" of "shrinking". When we use the same height N at every vertex, then the squeezed object obtained from c will be denoted $S_N(c)$:

$$S_N(c) = a_N^{v_0}(a_N^{v_1}(\ldots(a_N^{v_n}(c))\ldots))\ .$$

The cobordism between c and $S_N(c)$ constructed above is called the *standard cobordism*. The squeezing operation S_N preserves the face relation:

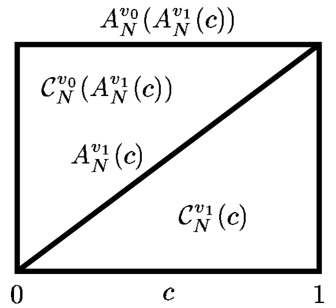
Proposition 3.1. *$\partial_i S_N(c)$ is equal to $S_N(\partial_i c)$. Furthermore, the standard cobordism between $\partial_i c$ and $\partial_i S_N(c)$ is equal to the standard cobordism between $\partial_i c$ and $S_N(\partial_i c)$.*

4. $\mathbb{L}$-spaces

The squeezing operation seems to justify the following simple definition of the **coefficient L-space** $\mathbb{L}_n(p : M \to X)$ for the generalized homology $H_*(X; \mathbb{L}(p))$, where $p : M \to X$ is a map from a space to a finite polyhedron which has an iterated mapping cylinder decomposition and n is an integer. It is a Δ-set; a k-simplex is an $(n+k)$-dimensional proper quadratic Poincaré $(k+2)$-ad $(c; \partial_0 c, \ldots, \partial_k c)$ on the pull-back $\pi^* M \to (\Delta; \partial_0\Delta, \ldots, \partial_k\Delta)$, where Δ is a k-simplex and $\pi : \Delta \to \Delta^l$ is an affine surjection from Δ to an l-dimensional simplex Δ^l of X ($l \le k$) induced by an order($\le$) preserving map between the vertices.

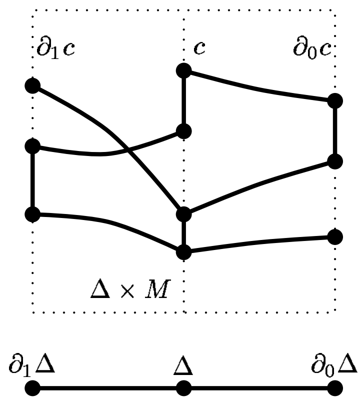
Two such simplices $(c, \pi : \Delta \to \Delta^l)$ and $(c', \pi' : \Delta' \to \Delta^l)$ are identified when there is an affine homeomorphism $\phi : \Delta \to \Delta'$ of ordered simplices such that $\pi = \pi' \circ \phi$ and $\phi_*(c) = c'$.

Note that the squeezing operation S_N defines a simplicial homotopy of the identity map of $\mathbb{L}_n(p : M \to X)$ to a simplicial map whose image is contained in a subset made up of simplices of 'small radius' measured on X, if N is large. Thus this space has a built-in 'squeezing' mechanism.



Let us consider the special case when X is a single point. There is a similar Δ-set $\mathbb{L}'_n(M)$ whose k-simplex is an $(n+k)$-dimensional quadratic Poincaré $(k+2)$-ad c on M that is *special*, *i.e.* $\partial_0\partial_1 \ldots \partial_k c$ is 0. $\pi_0(\mathbb{L}'_n(p : M \to *))$ is isomorphic to $L^h_n(\mathbb{Z}\pi_1(M))$.

There is a map $\mathbb{L}_n(M \to *) \to \mathbb{L}'_n(M)$ that sends a k-simplex (c, π) to its functorial image $\pi_*(c)$. A map in the reverse direction can be constructed as follows. Let c be a k-simplex of $\mathbb{L}'_n(M)$. It is made up of three type of things: (1) 'points' in M (generators of free modules), (2) paths with coefficients connecting the generators, and (3) homotopies of certain paths. Since c is special, one can make a 1–1 correspondence between its faces (including c itself) and the faces of a standard k-simplex Δ (including Δ itself), and can make copies of the faces of c on the sets $\{\text{barycenters}\} \times M \subset \Delta \times M$ and realizing the morphisms between adjacent pieces by using the original paths in c in the M-direction and the path connecting two adjacent barycenters in the Δ-direction as components. Similarly for homotopies of paths. These are homotopy inverses of each other.



Therefore, $\mathbb{L}_n(p : M \to X)$ defined above may give a convenient description of $\mathbb{L}$-homology groups.

References

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