A definable strong G retract of a definable G set in a real closed field

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Ordered fields

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  (1) For any $x, y, z \in R$, if $x < y$, then $x + z < y + z$.
  (2) For any $x, y, z \in R$, if $x < y$ and $z > 0$, then $xz < yz$.

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  (2) For any $y_1, \dots, y_m \in R$, $y_1^2 + \cdots + y_m^2 = 0 \Rightarrow y_1 = \cdots = y_m = 0$.

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  A real field $(R, +, \cdot, <)$ is a *real closed field* if it satisfies one of the following two equivalent conditions.
  (1) [Intermediate value property] For every $f(x) \in R[x]$, if $a < b$ and $f(a) \neq f(b)$, then $f([a, b]_R)$ contains $[f(a), f(b)]_R$ if $f(a) < f(b)$ or $[f(b), f(a)]_R$ if $f(b) < f(a)$, where $[a, b]_R = \{x \in R | a \leq x \leq b\}$.
  (2) The ring $R[i] = R[x]/(x^2 + 1)$ is an algebraically closed field.

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  (2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where f ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \longrightarrow \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.

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  (3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where S is a subset of $\mathbb{R}$, f ranges over all restricted analytic functions as in (2)., and the function $x^r : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

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(5) $\mathbf{R}_{an,exp} := (\mathbb{R}, +, \cdot, <, (f), exp)$, where (f) and exp denote as above.

- An ordered structure $(R, <)$ with a dense linear order $<$ without endpoints is *o-minimal (order minimal)* if every definable set of R is a finite union of open intervals and points, where open interval means $(a, b), -\infty \leq a < b \leq \infty$.

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  If $(R, +, \cdot, <)$ is a real closed field, then it is o-minimal and the collection of definable sets coincides that of semialgebraic sets.
  The topology of R is the interval topology and the topology of R^n is the product topology.
  In this presentation, everything is considered in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, \dots,)$ of a real closed field $(R, +, \cdot, <)$ unless otherwise stated.

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  1. A set R called the underlying set or universe of $\mathcal{N}$.
  2. A collection of functions $\{f_i | i \in I\}$, where $f_i : R^{n_i} \to R$ for some $n_i \geq 1$.
  3. A collection of relations $\{L_j | j \in J\}$, where $L_j \subset R^{m_j}$ for some $m_j \geq 1$.
  4. A collection of distinguished elements $\{c_k | k \in K\} \subset R$, and each c_k is called a constant.

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  Any (or all) of the sets I, J, K may be empty. We refer n_i and m_j as the arity of f_i and L_j.
  We say that f (resp. L) is an m-place function symbol (resp. an m-place relation symbol) if $f : R^m \to R$ is a function (resp. $L \subset R^m$ is a relation).

Terms

- A term is a finite string of symbols obtained by repeated applications of the following three rules:
  1. Constants are terms.
  2. Variables are terms.
  3. If f is an m-place function symbol of $\mathcal{N}$ and $t_1, \dots, t_m$ are terms, then the concatenated string $f(t_1, \dots, t_m)$ is a term.

Formulas

- A formula is a finite string of symbols $s_1 \dots s_k$, where each s_i is either a variable, a function symbol, a relation symbol, one of the logical symbols $=, \neg, \vee, \wedge, \exists, \forall$, one of the brackets (,), or comma ,.

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  Arbitrary formulas are generated inductively by the following three rules:
  1. For any two terms t_1 and t_2, $t_1 = t_2$ and $t_1 > t_2$ are formulas.
  2. If L is an m-place relation symbol and $t_1, \ldots, t_m$ are terms, then $L(t_1, \ldots, t_m)$ is a formula.
  3. If ϕ and ψ are formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are formulas. If ϕ is a formula and v is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

Definable sets

- A subset X of R^n is definable (in $\mathcal{N}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and elements $b_1, \dots, b_m \in R$ such that $X = \{(a_1, \dots, a_n) \in R^n | \phi(a_1, \dots, a_n, b_1, \dots, b_m)$ is true in $\mathcal{N}\}$.

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  In this case, we say that X is a definable set.

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- The development of o-minimality has been influenced by real analytic geometry and it is based on the following four things.
  (1) Adaptation of methods of real analytic geometry and Nash setting to the o-minimal setting.
  (2) Construction of new interesting examples of o-minimal structures.
  (3) New insights originated from model-theoretic methods into the real analytic setting and Nash setting.
  (4) O-minimal structures give a generalization, a uniform treatment and new tools.

- The field $\mathbb{R}[X]^\wedge$ of Puiseux series with real coefficients, namely the set of expressions $\sum_{i=k}^{\infty} a_i X^{\frac{i}{q}}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$.
  $\mathbb{R}[X]^\wedge$ is non-Archimedian.

Facts on o-minimal structures

Theorem

(1) The characteristic of every real closed field is $\mathbf{0}$.
(2) For any cardinality $\boldsymbol{\kappa} \geq \boldsymbol{\aleph_0}$, *there exist* $\mathbf{2}^{\boldsymbol{\kappa}}$ *many non-isomorphic real closed fields with cardinality* $\boldsymbol{\kappa}$.
(3) There exists uncountably many o-minimal expansions of the field $\mathbb{R}$ *of real numbers.*

Definably compact and definably connected

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  A definable subset X of R^n is definably connected if there do not exist two non-empty definable open subsets Y, Z of X such that $X = Y \cup Z$ and $Y \cap Z = \emptyset$.
  A compact definable set is definably compact, but a definably compact set is not necessarily compact. A connected definable set is definable connected, but a definably connected set is not necessarily connected. For example if
  $R = \mathbb{R}_{alg} := \{x \in \mathbb{R} | x \text{ is algebraic over } \mathbb{Q}\}$, then
  $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \le x \le 1\}$ is definably compact and definably connected but neither compact nor connected.

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Definition

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Proposition

Let $X \subset R^n$, $Y \subset R^m$ be definable sets and $f : X \longrightarrow Y$ a definable map. If X is definably compact (resp. definably connected), then $f(X)$ is definably compact (resp. definably connected).

Facts on definable maps

Theorem (Intermediate value property)

For every definable function $f(x)$ *defined on* $[a, b]$ *with* $f(a) \neq f(b)$, $f([a, b]_R)$ *contains* $[f(a), f(b)]_R$ *if* $f(a) < f(b)$ *or* $[f(b), f(a)]_R$ *if* $f(b) < f(a)$.

Definable groups and definable G sets

Definition

(1) A definable subset G of R^n is a definable group if G is a group and the group operations $G \times G \longrightarrow G$ and $G \longrightarrow G$ are definable.
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Definition

Let G be a definably compact definable group. A group homomorphism from G to some $O_n(R)$ is a representation if it is definable, where $O_n(R)$ means the nth orthogonal group of R. A representation space of G is R^n with the orthogonal action induced from a representation of G. A definable G set means a G invariant definable subset of some representation space of G.

Definable homeomorphisms and definable C^r diffeomorphisms

- Let $X \subset R^n, Z \subset R^m$ be definable sets and $f : X \to Z$ a definable map. We say that f is a definable homeomorphism if there exists a definable map $h : Z \to X$ such that $f \circ h = id_Z$ and $h \circ f = id_X$. We call f definably proper if for every definably compact subset C of Z, $f^{-1}(C)$ is definably compact.

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  Let $X \subset R^n, Z \subset R^m$ be definable open sets and $f : X \to Z$ a definable map. We say that f is a definable C^r map if f is of class C^r.
  A definable C^r map is a definable C^r diffeomorphism if f is a C^r diffeomorphism.

Definition

(1) Let r be a non-negative integer or ∞. A Hausdorff space X is an *n-dimensional definable C^r manifold*
if there exist a **finite** open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of X, **finite** open sets $\{V_\lambda\}_{\lambda \in \Lambda}$ of R^n, and **finite** homeomorphisms $\{\phi_\lambda : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ such that for any λ, ν with $U_\lambda \cap U_\nu \neq \emptyset$, $\phi_\lambda(U_\lambda \cap U_\nu)$ is definable and $\phi_\nu \circ \phi_\lambda^{-1} : \phi_\lambda(U_\lambda \cap U_\nu) \to \phi_\nu(U_\lambda \cap U_\nu)$ is a definable C^r diffeomorphism.

- This pair $(U_\lambda, \phi_\lambda)$ of sets and homeomorphisms is called a *definable C^r coordinate system*.

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Theorem

(1) For any positive integer r, a definable group G admits a unique definable C^r group structure up to definable C^r group isomorphism.
(2) If $\mathcal{N}$ is an o-minimal expansion of the standard structure of $\mathbb{R}$ and it admits the C^∞ cell decomposition, then we can take $r = \infty$ in (1).

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Corollary

If $R = \mathbb{R}$ and G is a compact Lie group of positive dimension, then $\chi(G) = 0$.

Facts on definable sets and definable maps

Theorem

(1) (Definable triangulation). Let $S \subset R^n$ *be a definable set and* $S_1, \dots, S_k$ *definable subsets of* S*. Then there exist a finite simplicial complex* K *in* R^n *and a definable map* $\phi : S \to R^n$ *such that* ϕ *maps* S *and each* S_i *definably homeomorphically onto a union of open simplexes of* K*. If* S *is definably compact, then we can take* $K = \phi(S)$*.*
(2) (Piecewise definable trivialization). Let X *and* Z *be definable sets and* $f : X \to Z$ *a definable map. Then there exist a finite partition* $\{T_i\}_{i=1}^k$ *of* Z *into definable sets and definable homeomorphisms* $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(z_i)$ *such that* $f|f^{-1}(T_i) = p_i \circ \phi_i$*,* $(1 \leq i \leq k)$*, where* $z_i \in T_i$ *and* $p_i : T_i \times f^{-1}(z_i) \to T_i$ *denotes the projection.*
(3) (Existence of definable quotient). Let G *be a definably compact definable group and* X *a definable* G *set. Then the orbit space* X/G *exists as a definable set and the orbit map* $\pi : X \to X/G$ *is surjective, definable and definably proper.*

Definable subgroups and definable G maps

Definition

A subgroup of a definable group is a definable subgroup of it if it is a definable subset of it.

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Definition

A definable map (resp. A definable homeomorphism) between definable G sets is a definable G map (resp. a definable G homeomorphism) if it is a G map.

Definable set with a definable G actions

Definition

Let G be a definable group. A definable set with a definable G action is a pair (X, ϕ) consisting of a definable set X and a group action $\phi : G \times X \longrightarrow X$ such that ϕ is a definable map.

- This action is not necessarily linear (orthogonal). Similarly, we can define definable G maps and definable G homeomorphisms between them.

- Using Theorem, if H is a definable subgroup of a definably compact definable group G, then G/H is a definable set, and the standard action $G \times G/H \longrightarrow G/H$ defined by $(g, g'H) \mapsto gg'H$ of G on G/H makes G/H a definable set with a definable G action.

Definable G CW complexes

Definition

Let G be a definably compact definable group.

(1) A definable G CW complex is a finite G CW complex $(X, \{c_i | i \in I\})$ satisfying the following three conditions.

(a) The underlying space $|X|$ of X is a definable G set.

(b) The characteristic map $f_{c_i} : G/H_{c_i} \times \Delta \to \overline{c_i}$ of each open G cell c_i is a definable G map and $f_{c_i}|G/H_{c_i} \times \text{Int}\,\Delta : G/H_{c_i} \times \text{Int}\,\Delta \to c_i$ is a definable G homeomorphism, where H_{c_i} is a definable subgroup of G, Δ denotes a standard closed simplex, $\overline{c_i}$ is the closure of c_i in X, and Int Δ means the interior of Δ.

(c) For each c_i, $\overline{c_i} - c_i$ is a finite union of open G cells.

(2) Let X and Z be definable G CW complexes. A cellular G map $f : X \to Z$ is definable if $f : |X| \to |Z|$ is definable.

- Since G and every standard closed simplex are definably compact and by definition, every definable G CW complex X is definably compact. Note that a G CW subcomplex of a definable G CW complex is a definable G CW complex itself.

Definable strong deformation G retract

Definition

Let X be a definable G set and Y a definable G subset of X.
(1) We say that a definable G map $l : X \to Y$ is a definable G retraction from X to Y if $l|Y = id_Y$.
(2) A definable strong G deformation retraction from X to Y is a definable G map $L : X \times [0,1]_R \to X$ such that $L(x,0) = x$ for all $x \in X$, $L(y,t) = y$ for all $y \in Y, t \in [0,1]_R$ and $L(X,1) = Y$, where the action on $[0,1]_R = \{x \in R | 0 \leq x \leq 1\}$ is trivial.
Note that $L(\cdot,1) : X \to Y$ is a definable G retraction from X to Y.

Our result

Theorem (2010)

Let G be a definably compact definable group and X a definable G set. Then there exists a definable strong G deformation retraction L from X to a definably compact definable G subset Y of X.

Definably G homotopy classes

Definition

Let G be a definable group, X, Y definable G sets and $f, h : X \to Y$ definable G maps.
(1) We say that f is definably G homotopic to h if there exists a definable G map $F : X \times [0,1]_R \to Y$ such that $F(x,0) = f(x)$ for all $x \in X$ and $F(x,1) = h(x)$ for all $x \in X$, where the action on $[0,1]_R = \{x \in R | 0 \leq x \leq 1\}$ is trivial.
(2) We denote $[X,Y]^G_{def}$ (resp. $[X,Y]^G$) by the set of definable G (resp. G) homotopy classes of definable G (resp. continuous G) maps from X to Y. For a definable G map (resp. continuous G) map f, $[f]^G_{def}$ (resp. $[f]^G$) means the definable G (resp. G) homotopy class of f.

The case where $\mathcal{N}$ is an o-minimal expansion of $\mathbb{R}$

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Theorem (2004)

If $\mathcal{N}$ is an o-minimal expansion of $\mathbb{R}$, G is a compact definable group and X, Y are definable G sets, then the map $[X, Y]^G_{def} \to [X, Y]^G$ defined by $[f]^G_{def} \mapsto [f]^G$ is bijective.

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Remark

Even a non-equivariant version of the above theorem is not true in general $\mathcal{N}$.

Definable G CW complex structure of a definable G set

Theorem

Let G be a definably compact definable group. Let X be a definable G set and Y a definable closed G subset of X. Then there exist a definable G CW complex Z in a representation space Ξ of G, a G CW subcomplex W of Z, and a definable G map $f : X \longrightarrow Z$ such that:

1. *f maps X and Y definably G homeomorphically onto G invariant definable subsets Z_1 and W_1 of Z and W obtained by removing some open G cells from Z and W, respectively.*
2. *The orbit map $\pi : Z \longrightarrow Z/G$ is a definable cellular map.*
3. *The orbit space Z/G is a finite simplicial complex compatible with $\pi(Z_1)$ and $\pi(W_1)$.*
4. *For each open G cell c of Z, $\pi|\overline{c} : \overline{c} \longrightarrow \pi(\overline{c})$ has a definable section $s : \pi(\overline{c}) \longrightarrow \overline{c}$, where $\overline{c}$ denotes the closure of c in Z.*

Moreover, if X is definably compact, then $Z = f(X)$ and $W = f(Y)$.

Key results for the proof of definable G CW complex structure theorem

Lemma

Let G be a definably compact definable group, K, H definable subgroups of G with $K < H$ and X is a definable K set. Then the map $G \times_K X \to G \times_H (H \times_K X), [g, x] \mapsto [g, [e, x]]$ is a definable G homeomorphism, where e denotes the unit element of G.

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Theorem

Let G be a definably compact definable group. Then every definable G set has only finitely many orbit types.

Key results for the proof of definable G CW complex structure theorem

Theorem (Equivariant piecewise definable trivialization)

Let G be a definably compact definable group, X a definable G set, Z a definable set and $f : X \to Z$ a G invariant definable map. Then there exist a finite decomposition $\{T_i\}_{i=1}^k$ of Z into definable sets and definable G homeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(z_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \leq i \leq k)$, where p_i denotes the projection $T_i \times f^{-1}(y_i) \to T_i$ and $z_i \in T_i$.

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  Let Y denote the maximum definably compact G CW subcomplex of X. In other words, Y is the union of all open G cells c of X such that $\overline{c} \subset X$, where $\overline{c}$ denotes the closure of c in C.

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  To complete the proof, we find a definable strong G deformation retraction L and prove that Y is a definable strong G deformation retract of X.

Thank you very much.