On the existence and classification of isovariant maps

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Today’s talk

(I) Existence or nonexistence results on isovariant maps, in particular, Borsuk-Ulam type results.

(II) Classification results on isovariant maps, in particular, Hopf type results.
Isovariant maps

Let $G$ be a compact Lie group. All maps are assumed to be continuous.

**Definition.** A $G$-map $f : X \to Y$ between $G$-spaces is called $G$-iso-variant if $G_{f(x)} = G_x$ for all $x \in X$.

**Example.**

(1) If both $X$ and $Y$ are free $G$-spaces, then an arbitrary $G$-map is a $G$-iso-variant map.

(2) Suppose $X$ is a $G$-space with nontrivial action and $Y$ is a $G$-space with $Y^G \neq \emptyset$. In this case, a map $f : X \to Y^G \subset Y$ is equivariant, but not isovariant.
**Isovariant homotopy classes**

**Definition.** A $G$-homotopy $F : X \times I \to Y$ is called a $G$-isovariant homotopy if $F$ is $G$-isovariant.

Let $[X, Y]^{\text{isov}}_G$ denote the set of isovariant homotopy classes of $G$-isovariant maps from $X$ to $Y$.

As usual, $[X, Y]_G$ denotes the set of $G$-homotopy classes of $G$-maps from $X$ to $Y$. 
Notation

(1) $C_n$: a cyclic group of order $n$.

(2) $U_k (= \mathbb{C}), k \in \mathbb{Z}$: the unitary 1-dimensional representation of $C_n$ on which a generator $c \in C_n$ acts by $c \cdot z = \xi_n^k z$, where $z \in U_k$ and $\xi_n = \exp(2\pi \sqrt{-1}/n)$.

(3) $SV$: the unit sphere of a representation $V$ of $G$, which is called a representation sphere or linear $G$-sphere.
An existence problem

Let $G = C_{pq}$, where $p$, $q$ are distinct primes.

Set

$$U_1^r = U_1 \oplus \cdots \oplus U_1 \text{ (r times)}$$

and

$$W = U_p \oplus U_q.$$ 

Note that $G$ acts freely on $SU_1^r$, but not freely on $SW$.

In fact, the singular set (nonfree part) of $SW$:

$$SW^{>1} = SW^{C_p} \cup SW^{C_q} = SU_p \bigsqcup SU_q.$$
An existence problem

In equivariant case, as an application of equivariant obstruction theory, one can see that, for any $r \geq 1$, there exists a $C_{pq}$-map

$$g : SU^r_1 \rightarrow S(U_p \oplus U_q).$$

Question. What about a $C_{pq}$-isovariant map?

Does there exist a $C_{pq}$-isovariant map from $SU^r_1$ to $SW$?
The answer

If \( r = 1 \), then there is an isovariant map. For example, one can define an isovariant map \( f_{0,0} : SU_1 \to SW \) by

\[
f_{0,0}(z) = (z^p, z^q)/\sqrt{2}.
\]

(In fact, \( f_{0,0} \) is a \( G \)-embedding.)

When \( r \geq 2 \), the answer is “No.”

This is shown by a Borsuk-Ulam type theorem.
In transformation group theory, the classical Borsuk-Ulam theorem is stated as follows.

**Theorem 1.** Let $S^m$ and $S^n$ be spheres with antipodal $C_2$-action. If there is a $C_2$-map $f : S^m \to S^n$, then the inequality $m \leq n$ holds.

Thus the Borsuk-Ulam theorem provides the nonexistence of a $C_2$-map. In fact, if $m > n$, then there is no $C_2$-map from $S^m$ to $S^n$. 
A generalization of the Borsuk-Ulam theorem

Many generalizations of the Borsuk-Ulam theorem are known. The following is one of them.

**Theorem 2** (N-Hara-Kawakami-Ushitaki, Biasi-de Mattos). Let $X$ be a free $C_n$-space and $Y$ a Hausdorff free $C_n$-space. Suppose that there exists $m \geq 1$ such that

$$\tilde{H}_q(X; \mathbb{Z}/n) = 0 \quad \text{for} \quad 0 \leq q \leq m,$$

and

$$H_{m+1}(Y/C_n; \mathbb{Z}/n) = 0.$$

Then there is no $C_n$-map from $X$ to $Y$.

Here the homology is the singular homology.
A generalization of the Borsuk-Ulam theorem

This theorem deduces a well-known result below.

**Corollary 3** (mod $p$ Borsuk-Ulam theorem). Assume that $C_p$ ($p$: prime) acts freely on $X$ with $H_*(X; \mathbb{Z}/p) \cong H_*(S^m; \mathbb{Z}/p)$ and on (Hausdorff) $Y$ with $H_*(Y; \mathbb{Z}/p) \cong H_*(S^n; \mathbb{Z}/p)$. If there is a $C_p$-map $f : X \to Y$, then $m \leq n$.

In other words, if $m > n$, then there is no $C_p$-map from $X$ to $Y$. 
Proof of the nonexistence

It suffices to show this when $r = 2$. Suppose

$$f : SU_1^2 = S(U_1 \oplus U_1) \to S(U_p \oplus U_q) = SW$$

is an isovariant map.

By restricting the action, we get a $C_p$-map $f : SU_1^2 \to SW \setminus SW^{C_p}$

Since $SW \setminus SW^{C_p} \cong S^1$, $SW \setminus SW^{C_p}$ is a free $C_p$-homology sphere of (homological) dimension 1.

By the mod $p$ Borsuk-Ulam theorem, we have $\dim SU_1^2 \leq 1$, however this is a contradiction. \qed
Homologically linear actions

The above example is generalized.

Set

\[ R_G = \begin{cases} \mathbb{Z}/|G| & \text{if } \dim G = 0, \\ \mathbb{Z} & \text{if } \dim G > 0. \end{cases} \]

**Definition.** A smooth closed \( G \)-manifolds \( \Sigma \) is called an \( R_G \)-homologically linear \( G \)-sphere if for every (closed) subgroup \( H \), the \( H \)-fixed point set \( \Sigma^H \) is an \( R_G \)-homology sphere or the empty set; namely,

\[ H_\ast(\Sigma^H; R_G) \cong H_\ast(S^{m(H)}; R_G), \quad m(H) = \dim \Sigma^H. \]

For convenience, we set \( \dim \Sigma^H = -1 \) if \( \Sigma^H \) is empty.
More general results

Then we have

**Theorem 4** (Isovariant Borsuk-Ulam theorem). *Let $G$ be a solvable compact Lie group. Let $\Sigma_1$ and $\Sigma_2$ be $R_G$-homologically linear $G$-spheres. If there is a $G$-isovariant map $f : \Sigma_1 \to \Sigma_2$, then the inequality*

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

*holds.*

**Remark.** Wasserman first proved this result for representation spheres.
Nonsolvable case

Using a result of Oliver, we have

**Proposition 5.** If $G$ is nonsolvable, then there exists a sequence

$$
\cdots \xrightarrow{h_n} \Sigma_n \xrightarrow{h_{n-1}} \Sigma_{n-1} \xrightarrow{h_{n-2}} \cdots \xrightarrow{h_1} \Sigma_1
$$

such that

- each $\Sigma_n$ is a homologically linear $G$-sphere
  (in fact $\Sigma_n$ can be taken to be a semilinear $G$-sphere),
- each $h_n$ is a $G$-isovariant map,
- $\Sigma_n^G = \emptyset$ and $\lim_{n \to \infty} \dim \Sigma_n = \infty$.  

Nonsolvable case

Take a $G$-embedding $i : \Sigma_1 \subset SW$ for some representation $W$. Then an isovariant map $f_n : \Sigma_n \to SW$ is defined by composition.

Thus there is an integer $n_0$ such that

$$\dim \Sigma_n + 1 > \dim SW - \dim SW^G$$

for any $n > n_0$.

This shows that the isovariant Borsuk-Ulam theorem does not hold for a nonsolvable compact Lie group $G$. 

Remark

Hence, for $R_G$-homologically linear actions, the isovariant Borsuk-Ulam theorem holds if and only if $G$ is solvable.

Remark. The problem whether the above $\Sigma_n$ can be taken to be a linear $G$-sphere is still open.

In equivariant case, the following is known.

**Proposition 6 (Bartsch).** Let $G$ be a finite group. the Borsuk-Ulam theorem (in a weak sense) holds if and only if $G$ is of prime power order.
Corollary

Another result is obtained from the isovariant Borsuk-Ulam theorem.

**Corollary 7 (N-Ushitaki).** Let $G$ be a finite group and $\Sigma$ an $R_G$-homology sphere with free $G$-action. Let $SW$ be the representation sphere of a representation $W$ of $G$. If there is a $G$-isovariant map $f : \Sigma \to SW$, then the inequality

$$\dim \Sigma + 1 \leq \dim SW - \dim SW^{>1},$$

where $SW^{>1} = \bigcup_{1 \neq H \leq G} SW^H$ (the singular set).

**Remark.** This result still holds when $G = S^1$, $\text{Pin}(2) \cong N_{S^3}(S^1)$. However, in the case of $G = S^3$, it is unknown.
Existence results

The isovariant Borsuk-Ulam theorem provides nonexistence results of isovariant maps.

we here discuss an existence problem under some conditions.

As is seen before, there is a $C_{pq}$-isovariant map

$$f_{0,0} : SU_1 \rightarrow S(U_p \oplus U_q).$$

This is generalized as follows.
Existence results

**Proposition 8.** Let $M$ be a free $G$-manifold. Let $W$ be a representation of $G$. Suppose

$$\dim M/G + 1 \leq \dim SW - \dim SW^{>1}.$$ 

*Then there exists a $G$-isovariant map from $M$ to $SW$.*

Outline of Proof.

Set $SW_{\text{free}} = SW \setminus SW^{>1}$ and $d = \dim SW - \dim SW^{>1}$.

**Fact.** $SW_{\text{free}}$ is $(d - 2)$-connected.
Existence results

It suffices to construct a $G$-map from $M$ to $SW_{\text{free}}$.

Fix a $G$-CW complex structure of $M$. One can inductively construct a $G$-map as follows.

Suppose that a $G$-map $f_k : X_k \to SW_{\text{free}}$ is constructed on the $k$-skeleton $X_k$ of $M$.

Let $X_{k+1} = X_k \cup \phi G \times D^{k+1} \cup \cdots$. Then

$$f_k \circ \phi|_{\partial D^{k+1}} : 1 \times \partial D^{k+1} \to SW_{\text{free}}$$

is extended to $f_{k+1} : D^{k+1} \to SW_{\text{free}}$, since $k \leq d - 2$ and $SW_{\text{free}}$ is $(d - 2)$-connected. Hence $f_k$ equivariantly extends to a $G$-map $f_{k+1} : X_{k+1} \to SW_{\text{free}}$. \qed
Next we discuss a classification problem. Let $G = C_{pq}$. Recall the isovariant map

$$f_{0,0} : SU_1 \to SW = S(U_p \oplus U_q)$$

$$f_{0,0}(z) = (z^p, z^q)/\sqrt{2}.$$

One can find other isovariant maps. Indeed, a map $f_{\alpha,\beta} : SU_1 \to SW$ defined by

$$f_{\alpha,\beta}(z) = (z^{p(1+\alpha q)}, z^{q(1+\beta p)})/\sqrt{2},$$

$f_{\alpha,\beta}$ is $G$-isovariant for $(\alpha, \beta) \in \mathbb{Z}^2$.

**Question.** Do these maps represent different isovariant homotopy classes?
Classification problem — An example

The answer is “Yes.” In fact,

**Proposition 9.** If \( f_{\alpha, \beta} \) and \( f_{\alpha', \beta'} \) are isovariantly homotopic, then \( (\alpha, \beta) = (\alpha', \beta') \).

In order to show this, we introduce the *multidegree* as an isovariant homotopy invariant.

If \( f \) is an isovariant map, then we obtain a \( G \)-map \( f : SU_1 \to SW_{\text{free}} \).

Consider the induced homomorphism

\[
f_* : H_1(SU_1) \to H_1(SW_{\text{free}}).
\]
Lemma 10.

\[ \pi_1(SW_{\text{free}}) \cong H_1(SW_{\text{free}}) \cong \mathbb{Z} \oplus \mathbb{Z}. \]

Proof. \( SW_{\text{free}} = SW \setminus (SU_p \cup SU_q) \cong (U_p^\perp - 0) \times (U_q^\perp - 0) \cong SU_q \times SU_p. \)

We define the multidegree \( \text{mDeg}(f) \) of \( f \) by

\[ \text{mDeg}(f) = f_*([SU_1]) \in \mathbb{Z} \oplus \mathbb{Z}. \]
The multidegree of $f_{\alpha,\beta}$ is

$$\text{mDeg } f_{\alpha,\beta} = (q(1 + \beta p), p(1 + \alpha q)),$$

This shows that if $(\alpha, \beta) \neq (\alpha', \beta')$, then $\text{mDeg } f_{\alpha,\beta} \neq \text{mDeg } f_{\alpha',\beta'}$.

Hence the isovariant maps $f_{\alpha,\beta}$ represent different isovariant homotopy classes.
Classification problem — An example

In this case, the converse is true; in fact,

**Proposition 11.** Let \( f, g : SU_1 \to SW \) be isovariant maps. If \( m\text{Deg} f = m\text{Deg} g \), then \( f \) and \( g \) are isovariantly homotopic.

**Outline of Proof.**

Set \( G = C_{pq} \).

It suffices to construct a \( G \)-homotopy \( F : SU_1 \times I \to SW_{\text{free}} \) between \( f \) and \( g \).
Classification problem — An example

Consider the commutative diagram:

\[
[SU_1, SW_{\text{free}}]_G \xrightarrow{\gamma G} H^1(SU_1/G, \pi_1) = \mathbb{Z}^2
\]

\[
\varepsilon \downarrow \quad \quad \quad \quad \downarrow \pi^*
\]

\[
[SU_1, SW_{\text{free}}] \xrightarrow{\gamma} H^1(SU_1, \pi_1) = \mathbb{Z}^2,
\]

where \( \pi_1 = \pi_1(SW_{\text{free}}) = \mathbb{Z}^2 \). The vertical map \( \varepsilon \) is the forgetful map and \( \pi : SU_1 \to SU_1/G \) is the orbit map.
Classification problem — An example

\[
\begin{align*}
[\text{SU}_1, \text{SW}_{\text{free}}]_G & \xrightarrow{\gamma_G} H^1(\text{SU}_1/G, \pi_1) = \mathbb{Z}^2 \\
\varepsilon & \downarrow \\
[\text{SU}_1, \text{SW}_{\text{free}}] & \xrightarrow{\gamma} H^1(\text{SU}_1, \pi_1) = \mathbb{Z}^2,
\end{align*}
\]

Fix a \( G \)-map \( g : \text{SU}_1 \to \text{SW}_{\text{free}} \). The horizontal maps are defined by

\[
\gamma_G([f]) = \circ_G(f, g) \quad \text{and} \quad \gamma([f]) = \circ(f, g),
\]

which are bijections as a consequence of the equivariant obstruction theory.
Classification problem — An example

\[
[SU_1, SW_{\text{free}}]_G \xrightarrow{\gamma G} H^1(SU_1/G, \pi_1) = \mathbb{Z}^2
\]

One can see that

\[\pi^* \text{ is multiplication by } pq\]

and

\[\pi^*(\sigma_G(f, g)) = \sigma(f, g).\]
Classification problem — An example

\[ [SU_1, SW_{\text{free}}]_G \xrightarrow{\gamma_G} \mathbb{H}^1(SU_1/G, \pi) = \mathbb{Z}^2 \]

\[ \varepsilon \downarrow \quad \downarrow \pi^* \]

\[ [SU_1, SW_{\text{free}}] \xrightarrow{\gamma} \mathbb{H}^1(SU_1, \pi) = \mathbb{Z}^2, \]

Hence \( \pi^* \) is injective, and the forgetful map \( \varepsilon \) is injective.

By calculation of the obstruction class, we have

\[ \gamma([f]) = \sigma(f, g) = \text{mDeg } f - \text{mDeg } g. \]

Hence if \( \text{mDeg } f = \text{mDeg } g \), then we have \( \sigma_G(f, g) = 0 \) and so a \( G \)-map \( f \amalg g \) extends to a \( G \)-homotopy \( F \).
**A classification result**

Furthermore it is seen that

\[ \text{mDeg } f - \text{mDeg } g \in pq\mathbb{Z}^2. \]

Taking \( g = f_{0,0} \), we can define an injective map

\[ D : [SU_1, SW_{\text{free}}]_G \to \mathbb{Z} \oplus \mathbb{Z} \]

by \( D[f] = (\text{mDeg } f - \text{mDeg } f_{0,0})/pq \).

Since \( D([f_\alpha, \beta]) = (\beta, \alpha) \), it follows that \( D \) is surjective. Hence \( D \) is a bijection.
A classification result

Thus we have the following classification result.

**Proposition 12.** There is a one-to-one correspondence

\[ D : [SU_1, SW]_{iso}^G \rightarrow \mathbb{Z} \oplus \mathbb{Z}. \]

In particular, the maps \( f_{\alpha,\beta} \) represent all isovariant homotopy classes.

Using the notion of degree, H. Hopf showed that

\[ \text{deg} : [M, S^n] \rightarrow \mathbb{Z} \]

is a bijection for an orientable closed \( n \)-manifold \( M \). We call this sort of result a **Hopf type theorem**.
A Hopf type theorem

The above example is generalized as follows.

We assume the following.

• $G$ is a finite group.
• $M$ is a connected, closed free $G$-manifold.
• $SW$ is a unitary representation sphere of $G$.
• $\dim M + 1 = \dim SW - \dim SW^1$.

Notation

• $\mathcal{A} = \{ H \in \text{Iso } W \mid \dim SW^H = \dim SW^1 \}$.
• $\mathcal{A}/G = \{ (H) \mid H \in \mathcal{A} \}$.
**A Hopf type theorem**

**Theorem 13** *(Isovariant Hopf theorem).*  *With the above assumption*

(1) *If $M$ is orientable and the $G$-action on $M$ is orientation-preserving, then there is a one-to-one correspondence* \[
[M, SW]_{G}^{iso} \cong \bigoplus_{(H) \in A/G} \mathbb{Z}.
\]

*Every isovariant homotopy class is determined by the multidegree.*
A Hopf type theorem

(2) If $M$ is non-orientable, then there is a one-to-one correspondence

$$[M, SW]^{isov}_G \cong \bigoplus_{(H) \in \mathcal{A}/G} \mathbb{Z}/2.$$

If $G$ is of odd order, then every isovariant homotopy class is determined by the mod 2 multidegree.
Further results

(1) In the case where $M$ is orientable, if the $G$-action is \textit{not} orientation-preserving, then some $\mathbb{Z}/2$ components appear in $[M, SW]_{G}^{i\text{so}v}$, and the multidegree does not determine the isovariant homotopy classes.

$$[M, SW]_{G}^{i\text{so}v} \cong \bigoplus \mathbb{Z} \oplus \bigoplus \mathbb{Z}/2.$$ 

(2) In the case where $M$ is non-orientable, if $G$ is not of odd order, then the mod 2 multidegree does not always determine the isovariant homotopy classes.