RIMS Kôkyûroku Bessatsu
B39 (2013), 197–214

On Controlled Assembly Maps

By

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Abstract

Theorem 3.9 of [9] says that the $L^{-\infty}$ homology theory and the controlled $L^{-\infty}$ theory of a simplicially stratified control map $p : E \to X$ are equivalent. Unfortunately the proof given there contains serious errors. In this paper I give a correct statement and a correct proof.

§ 1. Introduction

For a covariant functor $\mathbb{J} = \{\mathbb{J}_n\}$ from spaces to spectra and a map $p : E \to X$, Quinn defined a homology spectrum $\mathbb{H}(X; \mathbb{J}(p))$ [4].

$\mathbb{H}(-; \mathbb{J}(-))$ defines a covariant functor which sends a pair $(X, p : E \to X)$ to a spectrum $\mathbb{H}(X; \mathbb{J}(p))$. Suppose we are given a covariant functor $\mathbb{J}(-;-)$ which sends a pair (X, p) to a spectrum $\mathbb{J}(X; p)$. Then we can define a covariant functor, also denoted $\mathbb{J}$, from spaces to spectra by $\mathbb{J}(E) = \mathbb{J}(*; E \to *)$. And then we obtain a homology spectrum $\mathbb{H}(X; \mathbb{J}(p))$ for a map $p : E \to X$. Quinn showed that, if the original functor $\mathbb{J}(-;-)$ satisfies three axioms (the restriction, continuity, and inverse limit axioms) and p is nice (*i.e.* it is a stratified system of fibrations [4]), then there is a homotopy equivalence

$$A : \mathbb{H}(X; \mathbb{J}(p)) \longrightarrow \mathbb{J}(X; p)$$

when X is compact [4, Characterization Theorem, p.421]. If X is non-compact, we need to consider a locally-finite homology theory.

In [9], I considered an $L^{-\infty}$-theory functor $\mathbb{L}(-;-)$. To avoid confusion with the ordinary L^h-theory functor, $\mathbb{L}(X; p)$ will be denoted $\mathbb{L}^{-\infty}(X; p)$ with decoration $-\infty$ in this paper. Note that $\mathbb{L}_j^{-\infty}(X; p)$'s are not only spaces but Δ-sets, and that the

Received September 30, 2011. Revised Dcember 15, 2012.
2000 Mathematics Subject Classification(s): 18F25
Key Words: assembly map, controlled assembly map
This work was partially supported by JSPS KAKENHI Grant Numbers 20540100 and 22540212.
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index j is the negative of the usual index for spectra. $\mathbb{L}^{-\infty}$ is functorial; there is a "forget-control" map

$$F : \mathbb{L}^{-\infty}(X; p : E \to X) \to \mathbb{L}^{-\infty}(*; E \to *) = \mathbb{L}^{-\infty}(E).$$

If E has the homotopy type of a connected CW-complex, then the homotopy groups of $\mathbb{L}^{-\infty}(E)$ are isomorphic to the groups $L^{-\infty}(\pi_1(E))$ of Wall and Ranicki.

We only consider the case when X is a finite polyhedron, and also assume that p is a *simplicially stratified fibration*, *i.e.* X is a geometric realization of a finite ordered simplicial complex K and that p has an iterated mapping cylinder decomposition with respect to K in the sense of Hatcher [1, p.105] [2, p.457]. See §3 for details about simplicially stratified fibrations.

The following is the key technical result of [9, Theorem 3.9]:

Theorem 1.1 (Characterization Theorem). *Suppose $p : E \to |K|$ is a simplicially stratified fibration on a finite polyhedron K. Then there is a homotopy equivalence ("controlled assembly map")*

$$A_j : \mathbb{H}_j(K; \mathbb{L}^{-\infty}(p)) \longrightarrow \mathbb{L}^{-\infty}_{-j}(|K|; p)$$

such that its composition with the forget-control map F is the ordinary assembly map $a_j : \mathbb{H}_j(K; \mathbb{L}^{-\infty}(p)) \to \mathbb{L}^{-\infty}_{-j}(E)$.

Unfortunately there were erroros in the construction of the controlled assembly map A_j. An analysis of the errors will be given in a separate paper, since any quick repair does not seem to be possible.

In this paper, I employ homology theory re-defined by Quinn in his 1995 paper [5]. There he showed that homology for bordism-type theories can be represented by "cycles". This notion of 'cycles' was independently developed about the same time by Ranicki [6]. The key is the use of dual cones. If the control map $p : E \to |K|$ is simplicially stratified, then cycles are collections of pieces on dual cones of the vertices of K, which can be glued to define the assembly map. These pieces can be shrunk to the center of the cones, and we can inductively use such shrinking operations to define Alexander tricks on the pieces. And the Alexander tricks will produce the desired controlled assembly maps.

One may wonder why we do not appeal to Quinn's Characterization Theorem mentioned above. One reason is that it seems very difficult to check that $\mathbb{L}^{-\infty}$ satisfies the restriction axiom. Another reason is that I feel safer working with the theory developed in [5].

In §2, we review the definition of various $\mathbb{L}^h/\mathbb{L}^{-\infty}$-spaces. In §3, we describe homology and the bordism spectra of cycles. In §4, we adapt the squeezing technique of [3] and use it to construct A. In §5, we give a proof of Theorem 1.1.

I would like to thank Frank Quinn for advising me to work with dual cones instead of simplices, when we discussed the difficulty in the argument of [9]. And I would like to thank the referee for pointing out many errors and improper arguments and making various invaluable suggestions. He also emphasized the benefits of dual cones to me in his comment to the first revision. I close the introduction by quoting a part of his comment. The map $p : E \to |K|$ is as above and $D(\sigma)$ denotes the dual cone of a simplex σ of K.

> The first benefit of the set of dual cones is that it is closed under intersections. The second is that the star of each $\partial D(\sigma)$ in the second barycentric subdivision is a *tubular* neighborhood of $\partial D(\sigma)$. Then, when E is a triangulated manifold, any simplicial map $p : E \to K$ is automatically transverse to each $\partial D(\sigma)$, upon taking second barycentric subdivisions and without needing a homotopy.

§ 2. L-specta and controlled L-spectra

Constructions of various L-theoretic spaces (Δ-sets) will be given in terms of geometric modules and geometric morphisms. Roughly speaking, a geometric module on a space E is a free module generated by "points" in E, and a geometric morphism between two geometric modules G and G' on E is a locally finite sum of paths in E which connect generators of G to generators of G' with integer coefficients. If $f : G \to G'$ is a geometric morphism between geometric modules on E and if $\widetilde{E}$ is a covering space of E with the deck transformation groups π, then f lifts to a $\mathbb{Z}\pi$-equivariant geometric morphism $\tilde{f} : \widetilde{G} \to \widetilde{G}'$ between the corresponding covers $\widetilde{G}$ and $\widetilde{G}'$ of G and G', because paths constituting f have lifts to $\widetilde{E}$. There are also notions of homotopies between geometric morphisms which are actually homotopies of the paths. See [9, §2] for details.

Recall that a Δ-set is a semi-simplicial set *without degeneracies* [7]. When X and Y are Δ-sets, their geometric product $X \otimes Y$ is defined as follows. First construct css-sets (semi-simplicial sets *with degeneracies*) $F(X)$ and $F(Y)$ by adding degenerate simplices to X and Y. Their product $F(X) \times F(Y)$ is a css-set whose n-simplices are the pairs (μ, ν) of n-simplices $\mu \in F(X)$ and $\nu \in F(Y)$. Face maps and degeneracy maps are induced by those of $F(X)$ and $F(Y)$ in the obvious manner. Then $X \otimes Y$ is the Δ-set made up of non-degenerate simplices of $F(X) \times F(Y)$. Its realization $|X \otimes Y|$ is homeomorphic to the product CW-complex $|X| \times |Y|$ of the realizations $|X|$ and $|Y|$. The notation $X \times Y$ is used to represent the "non-geometric product" whose n-simplices are the pairs (μ, ν) of n-simplices $\mu \in X$ and $\nu \in Y$. For example, let us regard the two ordered simplicial complexes Δ^k and $[0, 1]$ as Δ-sets. Then the product $\Delta^k \times [0, 1]$ has only 0-simplices and 1-simplices; on the other hand, the pair

$$(\langle 0,1,\ldots,j,j+1,\ldots,k-1,k\rangle, \langle \underbrace{0,0,\ldots,0}_{j+1},\underbrace{1,\ldots,1,1}_{k-j+1}\rangle)$$

of two degenerate $(k+1)$-simplices is non-degenerate in $F(\Delta^k)\times F([0,1])$, and hence defines a $(k+1)$-simplex in the geometric product $\Delta^k\otimes[0,1]$. We will denote this simplex by listing the vertices as follows:

$$\langle (0,0),(1,0),\ldots,(j,0),(j,1),\ldots,(k-1,1),(k,1)\rangle\ .$$

Δ-sets are usually defined using certain $(k+2)$-ads as k-simplices; see *e.g.* [8, §0]. For these we use d_i's to denote the face operators. In this article we heavily use materials from Quinn [5, §§3,4]; there he uses "$[k]$-ads" instead of $(k+2)$-ads, where $[k]$ denotes the set $\{0,1,2,\ldots,k\}$. For a finite set A, an *A-ad x of dimension n* is an object with codimension 1 faces $\{\partial_a x | a\in A\}$ such that

(1) $\partial_a x$ is an $(A-\{a\})$-ad of dimension $n-1$;

(2) for each n and A, there is an 'empty' A-ad of dimension n;

(3) there is a reindexing operation which changes the label set A to another set B via a bijection $A\to B$;

(4) there is an expansion operation of the label set A by adding empty faces;

(5) there is an orientation reversing operation, $x\mapsto -x$, on A-ads; and

(6) if $a\neq b\in A$, then $\partial_a\partial_b x = -\partial_b\partial_a x$.

A $[k]$-ad $x=(x;\partial_0 x,\partial_1 x,\ldots,\partial_k x)$ looks like a $(k+2)$-ad, but it is not one, because the indexing schemes and orientation conventions are different. If we want to construct a Δ-set using $[k]$-ads, then we need to change the faces in the following manner: if x is a $[k]$-ad, then $\partial_i x$ is not a $[k-1]$-ad, but is an $([k]-\{i\})$-ad; so define $d_i x$ by reindexing the $([k]-\{i\})$-ad $(-1)^i\partial_i x$ via the order-preserving bijection $[k-1]\to[k]-\{i\}$, and continue changing the faces of these and so on. Then we have the usual identity $d_i d_j = d_{j-1}d_i$ for $i<j$.

If A is an ordered set of $k+1$ elements, then one can use the order preserving bijection from A to $[k]$ to identify an A-ad with a $[k]$-ad, and also with a $(k+2)$-ad by the construction above.

A $[k]$-ad x is said to be *special* if $\partial_0\partial_1\ldots\partial_k x=\emptyset$.

Let $A=\{a_0,\ldots,a_i\}$ and $B=\{b_0,\ldots,b_j\}$ be sets. Then their *disjoint union* $A\sqcup B$ is the union $A\cup B'$ of A and a copy B' of B which is disjoint from A. When A and B

are ordered, B' is given the order coming from the identification with B and elements of B' are defined to be larger than elements of A. When there is no ambiguity, we pretend A and B are disjoint and omit mentioning a copy of B. An $A \sqcup B$-ad x is said to be *special in* A (*resp. in* B) if $\partial_{a_0} \dots \partial_{a_i} x = \emptyset$ (*resp.* $\partial_{b_0} \dots \partial_{b_j} x = \emptyset$).

A class of A-ads with arbitrary A is called a *bordism-type theory* if it satisfies a certain Kan condition [5, §3]. We introduce two 'homotopy invariant' functors $\mathcal{L}^h$ and $\mathcal{L}^{-\infty}$ from spaces to bordism-type theories.

Let E be a space. An *A-ad of dimension n in* $\mathcal{L}^h(E)$ is a strictly n-dimensional finitely generated geometric module quadratic Poincaré A-ad on E. $\mathcal{L}^h(E)$ satisfies Kan condition and, hence, it is a bordism-type theory. The definition of these A-ads is given inductively by a "low-tech" way as explained in [5, 6.3B]. To define an $[n]$-ad, we need the notion of $(A - \{a\})$-ads and glueing of these. Glueing of more than two A-ads are performed inductively, and we may encounter a difficulty in general as was pointed out in [9], but if the pieces are arranged just like the faces of a simplex, then there is no problem for glueing.

The definition of $\mathcal{L}^{-\infty}$ is slightly more complicated. First a *primitive A-ad of dimension n in* $\mathcal{L}^{-\infty}(E)$ is, for some integer $l \geq 0$, a strictly $(n+l)$-dimensional geometric module quadratic Poincaré A-ad x on $\mathbb{R}^l \times E$ such that x is locally finitely generated and has bounded radius with respect to the projection $\mathbb{R}^l \times E \to \mathbb{R}^l$. The *external suspension* of such an x is the pullback of the tensor product [1] $\sigma^*(S^1) \otimes x$ of the geometric module symmetric Poincaré complex of S^1 and x via the covering map $\mathbb{R}^{l+1} \times E = \mathbb{R} \times \mathbb{R}^l \times E \to S^1 \times \mathbb{R}^l \times E$. Iterated external suspensions are also called the external suspensions. The integer $n+l$ will be called the *real dimension*. Now an A-ad of dimension n in $\mathcal{L}^{-\infty}(E)$ is an equivalence class of primitive A-ads of dimension n in $\mathcal{L}^{-\infty}(E)$, where the equivalence relation is generated by identifying x with its external suspensions. Such an A-ad of dimension n in $\mathcal{L}^{-\infty}(E)$ is said to have real dimension $\leq k$ if there is a representative with real dimension $\leq k$. $\mathcal{L}^{-\infty}(E)$ satisfies the Kan condition and it is a bordism-type theory.

A bordism-type theory determines an Ω-spectrum called the *bordism spectrum*. In the case of $\mathcal{L}^h(E)$, it is denoted $\mathbb{L}^h(E)$. $\mathbb{L}^h_n(E)$ is a Δ-set whose j-simplices are the special $[j]$-ads x of dimension $n+j$ in $\mathcal{L}(E)$. The basepoint $\emptyset$ is the trivial complex 0. The bordism spectrum $\mathbb{L}^{-\infty}(E)$ of $\mathcal{L}^{-\infty}(E)$ is defined in the same manner.

The functors $\mathcal{L}^h$ and $\mathcal{L}^{-\infty}$ are 'homotopy invariant', *i.e.* a homotopy equivalence $E \to E'$ of spaces induces homotopy equivalences $\mathbb{L}^h(E) \to \mathbb{L}^h(E')$ and $\mathbb{L}^{-\infty}(E) \to \mathbb{L}^{-\infty}(E')$ of spectra. If E has the homotopy type of a connected CW-complex, then the homotopy groups $\pi_n(\mathbb{L}^h(E))$ and $\pi_n(\mathbb{L}^{-\infty}(E))$ are isomorphic to the L^h-groups $L^h_n(\mathbb{Z}\pi)$

[1] It is unfortunate that the same symbol $\otimes$ is used to represent the tensor product of a quadratic complex and a symmetric complex and to represent the geometric product of two Δ-sets. I hope there will be no confusions.

and the $L^{-\infty}$-groups $L_n^{-\infty}(\mathbb{Z}\pi)$ of the fundamental group $\pi = \pi_1(E)$ of E.

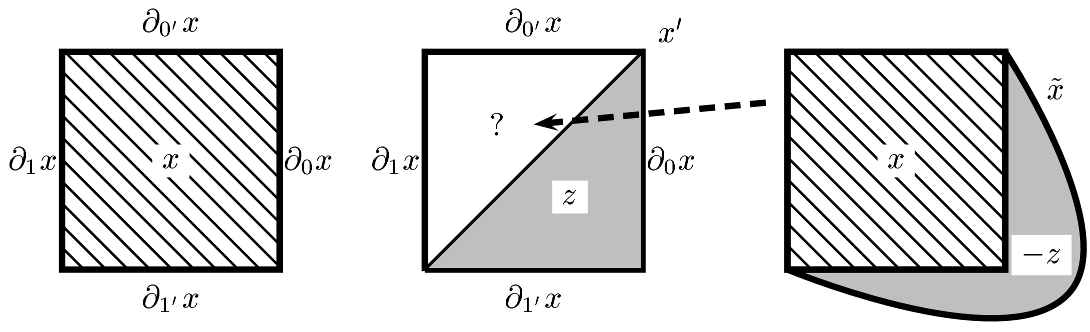
Let $\mathcal{J}$ represent both $\mathcal{L}^h$ and $\mathcal{L}^{-\infty}$, and $\mathbb{J}(E)$ denote its bordism spectrum. Suppose we are given a $[j]\sqcup[1]$-ad x of dimension $n+j+1$ in $\mathcal{J}(E)$ which is special both in $[j]$ and in $[1]$. It is a 'bordism' between $\partial_{0'}x$ and $\partial_{1'}x$, where $\{0', 1'\}$ is a copy of $[1]$. Although the two ends $\partial_{0'}x$ and $\partial_{1'}x$ define j-simplices of $\mathbb{J}_n(E)$, x itself defines a $(j+2)$-simplex of $\mathbb{J}_{n-1}(E)$ and cannot be used to relate the two ends in $\mathbb{J}_n(E)$ directly. The next lemma shows that we can "subdivide" x into mutually adjacent $j+1$ $(j+1)$-simplices of $\mathbb{J}_n(E)$ connecting $\partial_{0'}x$ and $\partial_{1'}x$, just like the $(j+1)$-simplices of the geometric product $\Delta^j \otimes [0,1]$ of the two Δ-sets Δ^j and $[0,1]$.

Lemma 2.1 (Triangulation of $[j] \sqcup [1]$-ads). *Suppose x is a $[j]\sqcup[1]$-ad of dimension $n+1+j$ in $\mathcal{J}(E)$ which is special both in $[j]$ and in $[1]$. Then there is a Δ-map φ from the geometric product $\Delta^j \otimes [0,1]$ to $\mathbb{J}_n(E)$ such that x is bordant to the union of the images of the $(j+1)$-simplices by φ, fixing the two ends $\partial_{0'}x$ and $\partial_{1'}x$.*

Proof. If $j = 0$, then x represents a 1-simplex of $\mathbb{J}_n(E)$, so there is nothing to prove. Assume inductively that we have constructed such a Δ-map for $j < k$, and consider a $[k] \sqcup [1]$-ad x of dimension$(n+1+k)$ in $\mathcal{J}(E)$. By assumption we have a Δ-map $\Delta^{k-1} \otimes [0,1]$ to $\mathbb{J}_n(E)$ for each face $\partial_i x$. Since the construction is by induction, we may assume that these match up along the common faces. The geometric product $\Delta^k \otimes [0,1]$ contains the following $k+1$ $(k+1)$-simplices:

$$\Delta_0^{k+1} = \langle \underline{0}, \underline{1}, \ldots, \underline{k-1}, \underline{k}, \overline{k}\rangle,\ \Delta_1^{k+1} = \langle \underline{0}, \underline{1}, \ldots, \underline{k-1}, \overline{k-1}, \overline{k}\rangle,$$
$$\ldots, \Delta_{k-1}^{k+1} = \langle \underline{0}, \underline{1}, \overline{1}, \ldots, \overline{k}\rangle, \Delta_k^{k+1} = \langle \underline{0}, \overline{0}, \overline{1}, \ldots, \overline{k}\rangle,$$

where $\underline{i}$ and $\overline{i}$ denote $(i,0)$ and $(i,1)$, respectively. Starting from the bottom $(k+1)$-simplex Δ_0^{k+1}, we keep extending the already constructed Δ-map over the $(k+1)$-simplices except for the last one, using the Kan condition. Now on the boundary of the top $(k+1)$-simplex Δ_k^{k+1} of $\Delta^k \otimes [0,1]$, a Δ-map is defined. We need to define a $(k+1)$-simplex of $\mathbb{J}_n(E)$ with the given faces using x.



Let z be the union of the images of the first k $(k+1)$-simplices. Glue $-z$ to x along the images of the faces whose realization are subsets of $\partial(|\Delta^k| \times |[0,1]|)$, then we obtain a $(k+1)$-simplex $\tilde{x}$ of $\mathbb{J}_n(E)$ which fits into the already constructed map. Since z is actually obtained by glueing trivial cobordisms, this Δ-map has the required property. □

Remark. If compatible Δ-maps on some of the prisms $(d_{i_0} \dots d_{i_l} \Delta^j) \otimes [0,1]$ are already given, one can use them because the proof is by induction.

Lemma 2.2 (Triangulation of $[j] \sqcup [1] \sqcup [1]$-ads). *Suppose y is a $[j] \sqcup [1] \sqcup [1]$-ad of dimension $n+j+2$ in $\mathcal{J}(E)$ which is special in $[j]$ and also in the two $[1]$'s. Then there is a Δ-map $\varphi : \Delta^j \otimes [0,1] \otimes [0,1] \to \mathbb{J}_n(E)$ such that y and the union of the images of the $(j+2)$-simplices by φ are bordant fixing the two ends $\partial_{0''} y$ and $\partial_{1''} y$.*

Proof. Let $\{\Delta_i^{k+1}\}$ be the $(k+1)$-simplices used in the proof of Lemma 2.1. Then $\{|\Delta_i^{k+1}| \times |[0,1]|\}$ gives a decomposition of $|\Delta^k| \times |[0,1]| \times |[0,1]|$ into $k+1$ prisms. An inductive process using Kan condition (*i.e.* glueing) gives $k+1$ $[k+1] \sqcup [1]$-ads where the union is bordant to y, realizing the model above. Now apply Lemma 2.1 inductively to each prism to get a desired Δ-map. □

Next we also fix a control map $p : E \to X$ to a metric space X. Then we can talk about the radius of a simplex of $\mathbb{L}_n^h(E)$ by looking at the radii of the paths and homotopies involved. For a positive number ε, $\mathbb{L}_n^h(X, p, \varepsilon)$ denotes the Δ-subset of $\mathbb{L}_n^h(E)$ made up of all the simplices with radius $\leq \varepsilon$.

The Δ-set $\mathbb{L}_n^h(X; p)$ is a sort of homotopy inverse limit of $\mathbb{L}_n^h(X, p, \varepsilon)$'s as $\varepsilon \to 0$: a j-simplex is a Δ-map from the geometric product $\Delta^j \otimes [0, \infty)$ of the standard j-simplex Δ^j and the interval $[0, \infty)$ with the standard triangulation to $\mathbb{L}_n^h(E)$ satisfying the following condition: there is a sequence ε_i monotone decreasing to 0 such that the image of $\Delta^j \otimes [i, \infty)$ lies in $\mathbb{L}_n^h(X, p, \varepsilon_i)$. Then there is a natural homotopy equivalence $T : \Omega\mathbb{L}_n^h(X; p) \to \mathbb{L}_{n+1}^h(X; p)$ and these spaces form an Ω-spectrum $\mathbb{L}^h(X; p)$, which will be called the *controlled L^h-spectrum* of the pair $(X, p : E \to X)$. Suppose we have a k-simplex σ of $\Omega\mathbb{L}_n^h(X; p)$. It is actually a Δ-map $\sigma : \Delta^k \otimes [0, \infty) \otimes [0,1] \to \mathbb{L}_n^h(E)$. There is a sequence ε_i monotone decreasing to 0 such that the image of $\Delta^j \otimes [i, \infty) \otimes [0,1]$ lies in $\mathbb{L}_n^h(X, p, \varepsilon_i)$, and σ sends the simplices of $\Delta^k \otimes [0, \infty) \otimes \{0, 1\}$ to 0. We define $T\sigma : \Delta^k \otimes [0, \infty) \to \mathbb{L}_{n+1}^h(E)$ as follows. Let τ be an m-simplex of $\Delta^k \otimes [0, \infty)$. The images of m-simplices of the geometric product $\tau \otimes [0,1]$ by σ can be glued together to form an m-simplex $(T\sigma)(\tau)$ of $\mathbb{L}_{n+1}^h(E)$, because $\sigma(\tau \otimes 0) = \sigma(\tau \otimes 1) = 0$. Furthermore the glueing processes are the same for all $\Delta^k \otimes [i, i+1] \otimes [0,1]$ $(i = 0, 1, 2, \dots)$, there is a positive constant C such that the image of $\Delta^k \otimes [i, \infty)$ by $T\sigma$ lies in $\mathbb{L}_{n+1}^h(X, p, C\varepsilon_i)$. So $T\sigma$ is a k-simplex of $\mathbb{L}_{n+1}^h(X; p)$. The construction of a homotopy inverse of T is essentially the same as the construction in the $L^{-\infty}$ case described in [9, Theorem 3.4].

There is a "restriction" Δ-map

$$R : \mathbb{L}_n^h(X;p) \to \mathbb{L}_n^h(E) \ ; \ \varphi \mapsto \varphi(0) \ .$$

When X is a single point, R is a homotopy equivalence $\mathbb{L}_n^h(*; E \to *) \simeq \mathbb{L}_n^h(E)$ and we can identify these, because everything has radius 0 and $\mathbb{L}_n^h(*; E \to *)$ is just the path space of $\mathbb{L}_n^h(E)$.

Therefore we can identify the restriction Δ-map $R : \mathbb{L}_n^h(X;p) \to \mathbb{L}_n^h(E)$ with the "forget-control" Δ-map

$$F : \mathbb{L}_n^h(X;p) \to \mathbb{L}_n^h(*; E \to *)$$

which is induced by the pair $(1 : E \to E, X \to *)$.

The controlled $L^{-\infty}$-spectrum $\mathbb{L}^{-\infty}(X;p)$ is defined in the following manner. We measure the radius of a simplex using the map $\mathbb{R}^l \times E \xrightarrow{\text{projection}} E \xrightarrow{p} X$. (Recall that we required that the radius measured in $\mathbb{R}^l$ be bounded.) If k is a non-negative integer and ε is a positive number, then we define $\mathbb{L}_n^{-\infty}(X,p,\varepsilon)^{(k)}$ to be the Δ-subset of $\mathbb{L}_n^{-\infty}(E)$ made up of all the simplices whose radius and real dimension are less than or equal to ε and k, respectively.

A j-simplex of the Δ-set $\mathbb{L}_n^{-\infty}(X;p)$ is a Δ-map from $\Delta^j \otimes [0,\infty)$ to $\mathbb{L}_n^{-\infty}(E)$ satisfying the following condition: there exist a non-negative integer k and a sequence ε_i monotone decreasing to 0 such that the image of $\Delta^j \otimes [i,\infty)$ lies in $\mathbb{L}_n^{-\infty}(X,p,\varepsilon_i)^{(k)}$. This is the controlled $L^{-\infty}$-spectrum of (X,p). As in the case of $\mathbb{L}^h$, the spectrum $\mathbb{L}^{-\infty}(*; E \to *)$ is homotopy equivalent to Ranicki's spectrum $\mathbb{L}^{-\infty}(E)$.

§ 3. L-homology and cycles

This section is a review of Quinn's description of homology and cycles [5, §4] in a very special case. Ranicki's description [6, §12] is also used at several places. Although we do not use the homology spectrum directly in this paper, we first review its construction. And then we review the notion of cycles which are used as substitutes of homology classes.

We first review the notion of itereated mapping cylinders ([2, p.457]). Suppose we have a sequence of maps $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n$. The *iterated mapping cylinder* $M(f_1,\dots,f_n)$ is defined to be the usual mapping cylinder $(X_0 \times [0,1] \cup X_1)/(x,1) \sim f_0(x)$ for $n = 1$. There is a canonical (strong deformation) retraction $r_1 : M(f_1) \to X_1$. For $n > 1$, assume inductively that we defined $M(f_1,\dots,f_{n-1})$ and constructed a (strong deformation) retraction $r_{n-1} : M(f_1,\dots,f_{n-1}) \to M_{n-1}$. Now we define $M(f_1,\dots,f_n)$ to be the mapping cylinder of the composition

$$M(f_1,\dots,f_n) \xrightarrow{r_{n-1}} X_{n-1} \xrightarrow{f_n} X_n,$$

and define $r_n : M(f_1, \dots, f_n) \to X_n$ to be the obvious (strong deformation) retraction. There is a natural projection $M(f_1, \dots, f_n) \to \sigma = \langle v_0, v_1, \dots, v_n \rangle$, since σ is the iterated mapping cylinder of the following sequence of trivial maps:

$$\{v_0\} \longrightarrow \{v_1\} \longrightarrow \dots \longrightarrow \{v_n\} \ .$$

In this section, we only consider the case when $p : E \to X$ is a simplicially stratified fibration (*i.e.* p has an iterated mapping cylinder decomposition [1][2] in the following sense).

Definition 3.1. Let K be a finite ordered simplicial complex and let $p : E \to X = |K|$ be a map. An *iterated mapping cylinder decomposition* of p is a collection of maps

$$\{f_{uv} : p^{-1}(u) \to p^{-1}(v) |\ u \text{ and } v \text{ are vertices of the same simplex of } K \text{ and } u < v\}$$

which satisfy the following conditions.

- f_{uv}'s satisfy $f_{uw} = f_{vw} \circ f_{uv}$.
- For each simplex $\sigma = \langle v_0, \dots, v_n \rangle \in K$, $p^{-1}(\sigma)$ is the iterated mapping cylinder $M(f_{v_0v_1}, \dots, f_{v_{n-1}v_n})$, and $p|p^{-1}(\sigma)$ is the natural projection $M(f_{v_0v_1}, \dots, f_{v_{n-1}v_n}) \to \sigma$ coming from the iterated mapping cylinder structure.

If we apply the functor $\mathbb{L}^h$ to the sequence of maps above, consider the iterated mapping cylinder of spectra, and glue all the pieces corresponding to the simplices of K, then we should get a desired spectral sheaf $\mathbb{L}^h(p)$. This spectral sheaf is not suitable for directly defining the assembly map, and we need to fatten the fibers without changing the fiber homotopy type of the sheaf. Again let σ be a simplex of K, and let $D(\sigma)$ denote the dual cone of σ in K, *i.e.* the union of all simplices of the barycentric subdivision of K which intersect σ in exactly the barycenter of σ [5, p.256]. Then $\mathcal{U} = \{D(v) | v \in K^{(0)}\}$ is a covering of X and its nerve is K itself. Let us define the *boundary* $\partial D(\sigma)$ of $D(\sigma)$ by $\partial D(\sigma) = \bigcup_{\tau > \sigma} D(\tau)$. Then $D(\sigma)$ is the cone on $\partial D(\sigma)$.

Now define the Δ-set version of the spectral sheaf $\mathbb{L}^h(p)$ by:

$$\left\{ \mathbb{L}^h_n(p) = \left(\coprod_{\sigma \in K} \mathbb{L}^h_n(p^{-1}(D(\sigma))) \otimes \sigma \right) / \sim \right\} \ ,$$

where a simplex $\sigma \in K$ is given the obvious Δ-set structure, $\otimes$ denotes the geometric product of two Δ-sets, and the equivalence relation $\sim$ is generated by: a simplex in $\mathbb{L}^h_n(p^{-1}(D(d_j\sigma))) \otimes d_j\sigma$ is identified with its image in $\mathbb{L}^h_n(p^{-1}(D(\sigma))) \otimes \sigma$.

Definition 3.2. The homology spectrum $\mathbb{H}(K;\mathbb{L}^h(p))$ is an Ω-spectrum of Δ-sets defined by

$$\mathbb{H}(K;\mathbb{L}^h(p)) = \operatorname*{holim}_{n\to\infty} \Omega^n(|\mathbb{L}^h_{-n}(p)|/s_{-n}(|K|)) ,$$

where $s_{-n} : |K| \to |\mathbb{L}^h_{-n}(p)|$ is the 0-section.

Instead of directly using the homology defined above, we use 'cycles' to define the controlled assembly maps. For this purpose, we fix an embedding of K into the boundary $\partial\Delta^{m+1}$ of the standard $(m+1)$-simplex. Here $m+1$ is the number of vertices of K; since the vertices are ordered, we identify $K^{(0)}$ with the vertices of the standard m-simplex $\Delta^m = \langle 0, 1, \dots, m\rangle$ which is a face of Δ^{m+1}. Geometrically we identify these vertices with the points $(0, \dots, 0, 1, 0, \dots, 0)$ in $\mathbb{R}^{m+2}$. This fixes the metric of $X = |K|$. If σ is a simplex in K, then we use $[m+1]-\sigma$ to denote the complement of the vertices of σ in $[m+1] = \{0, 1, \dots, m+1\}$.

Definition 3.3. An $\mathcal{L}^h$-n-cycle in (K, p) is a function $N : K \to \mathcal{L}^h$ such that

(1) if $\sigma = \langle v_0, \dots, v_k\rangle$ $(v_0 < \cdots < v_k)$ is a k-simplex of K, then $N(\sigma)$ is an $(n-k)$-dimensional $([m+1]-\sigma)$-ad in $\mathcal{L}^h(p^{-1}(D(\sigma)))$, and

(2) these satisfy the compatibility condition; *i.e.* the functorial image of $N(\sigma)$ in $\mathcal{L}^h(p^{-1}(D(d_j\sigma)))$ is equal to $(-1)^j\partial_{v_j}N(d_j\sigma)$ for $j = 0, \dots, k$.

The pieces $N(v)$'s $(v \in K^{(0)})$ in an $\mathcal{L}^h$-n-cycle N in (K, p) can be glued together to produce an $\emptyset$-ad (= a special [0]-ad) of dimension n in $\mathcal{L}^h(E)$. This process is called the assembly.

The next step is to introduce a bordism-type theory $\mathrm{Cycles}^{\mathcal{L}^h}(K, p)$ of cycles.

Definition 3.4. An A-ad of dimension n in $\mathrm{Cycles}^{\mathcal{L}^h}(K, p)$ is a function $N : K \to \mathcal{L}^h$ such that

(1) if $\sigma = \langle v_0, \dots, v_k\rangle$ $(v_0 < \cdots < v_k)$ is a k-simplex of K, then $N(\sigma)$ is an $(n-k)$-dimensional $([m+1]-\sigma) \sqcup A$-ad in $\mathcal{L}^h(p^{-1}(D(\sigma)))$, and

(2) these satisfy the compatibility condition; *i.e.* the functorial image of $N(\sigma)$ in $\mathcal{L}^h(p^{-1}(D(d_j\sigma)))$ is equal to $(-1)^j\partial_{v_j}N(d_j\sigma)$ for $j = 0, \dots, k$

Quinn verified that this is a bordism-type theory. Thus we can take its bordism spectrum, which we denote by $\mathbb{H}(\mathrm{Cycles}^{\mathcal{L}^h}(K, p))$. It is a Δ-set whose j-simplex is a $[j]$-ad N of dimension $-n+j$ in $\mathrm{Cycles}^{\mathcal{L}^h}(K, p)$ such that the $([m+1]-\sigma)\sqcup[j]$-ad $N(\sigma)$ is special in $[j]$, *i.e.* $\partial_0 \dots \partial_j N(\sigma) = 0$. See [5, p.223] for an explanation of the $-$ sign of n.

The $\mathcal{L}^{-\infty}$-versions of these are defined in the same way.

Quinn uses Ω instead of $\mathbb{H}$ in the notation; but our notation is not too inappropriate, because he showed that it is actually a homology:

Theorem 3.5 (Representation Theorem [5]). *There are homotopy equivalences of spectra*

$$\mathbb{H}(\mathrm{Cycles}^{\mathcal{L}^h}(K,p)) \longrightarrow \mathbb{H}(K;\mathbb{L}^h(p)) \ ,$$
$$\mathbb{H}(\mathrm{Cycles}^{\mathcal{L}^{-\infty}}(K,p)) \longrightarrow \mathbb{H}(K;\mathbb{L}^{-\infty}(p)) \ .$$

We will use this cycle description of homology to define the controlled assembly maps. In [9], I tried to define the (controlled) assembly maps by glueing pieces on simplices of a triangulation of PL manifolds. Unfortunately the proof of the glueing lemma over manifolds [9, Theorem 2.10] is incorrect, because the induction argument using the dual cones fails. If we use the cycle description, this local problem does not occur because each piece already lies over the dual cones and there are no difficulties in glueing these pieces. Furthermore there is no need to assume that the space is a manifold; we can directly handle polyhedrons. When we consider the metric control using the map p, we should note that glueing increases radii. But there is a universal constant C_1 such that an object obtained from two objects with radius $\le \delta$ by a single glueing operation has radius $\le C_1\delta$. This constant is universal in the sense that it does not depend on the dimension. This can be observed using the explicit formulas concerning the union operation given in the note I wrote with the help of A. Ranicki during my stay in Edingburgh [10]. If we repeat glueing then the radius may become large, but if we fix the simplicial complex K, then we have an estimate of the radius of the union. A j-simplex N of $\mathbb{H}_n(\mathrm{Cycles}^{\mathcal{J}}(K,p))$ is said to *have radius* $\le \varepsilon$ if $N(\sigma)$ has radius $\le \varepsilon$ for each $\sigma \in K$. The *assembly* $a(N)$ of such an N is the union of all $N(v)$'s along faces.

We have the following:

Lemma 3.6 (Glueing of $\mathcal{L}^h$-cycles / $\mathcal{L}^{-\infty}$-cycles). *Let $\mathcal{J}$ be either $\mathcal{L}^h$ or $\mathcal{L}^{-\infty}$, and $\mathbb{J}$ be $\mathbb{L}^h$ or $\mathbb{L}^{-\infty}$, accordingly. Fix a simplicially stratified fibration $p : E \to |K|$ as above. Assembly defines a Δ-map $a : \mathbb{H}_n(\mathrm{Cycles}^{\mathcal{J}}(K,p)) \to \mathbb{J}_{-n}(E)$. Furthermore, there exists a positive constant C which depends only on K such that if N has radius $\le \varepsilon$, then $a(N)$ has radius $\le C\varepsilon$.*

Thus, the ordinary assembly maps are defined using this assembly process:

$$a : \mathbb{H}(K;\mathbb{L}^h(p)) \simeq \mathbb{H}(\mathrm{Cycles}^{\mathcal{L}^h}(K,p)) \xrightarrow{a} \mathbb{L}^h(E)$$
$$a : \mathbb{H}(K;\mathbb{L}^{-\infty}(p)) \simeq \mathbb{H}(\mathrm{Cycles}^{\mathcal{L}^{-\infty}}(K,p)) \xrightarrow{a} \mathbb{L}^{-\infty}(E)$$

To define the controlled assembly maps, we need to make the radii of the pieces arbitrarily small. This squeezing problem will be discussed in the next section.

To prove the characterization theorem, we need the stable splitting lemma. The proof given for "the stable splitting lemma over manifolds" [9, Theorem 2.11] uses the same inductive argument on dual cones as above, so it is incorrect. But the following stable splitting into a cycle is correct.

Lemma 3.7 (Stable splitting lemma). *Fix an integer k and a finite ordered simplicial complex K. Then there exist positive constants δ and C which depend only on k and K such that the following holds: If*

(i) $p : E \to |K|$ *is a simplicially stratified fibration,*

(ii) x *is a* $[j]$*-ad of dimension* n *in* $\mathcal{L}^{-\infty}(|K|, p)$ *which has real dimension* $\leq k$ *and radius* $\leq \varepsilon \leq \delta$*, and*

(iii) N *is a* $[j-1]$*-ad of dimension* $n-1$ *in* $\mathrm{Cycles}^{\mathcal{L}^{-\infty}}(K, p)$ *such that* $\partial_j x = a(N)$*,*

then there exists a $[j]$*-ad* M *of dimension* n *in* $\mathrm{Cycles}^{\mathcal{L}^{-\infty}}(K, p)$ *and a* $[j] \sqcup [1]$*-ad* y *of dimension* $n+1$ *in* $\mathcal{L}^{-\infty}(E)$ *satisfying:*

(1) $\partial_j M = N$,

(2) y *is special in* $[1] = \{0', 1'\}$,

(3) y *has real dimension* $\leq Ck$ *and radius* $\leq C\varepsilon$*, and*

(4) $\partial_{0'} y = x$ *and* $\partial_{1'} y = a(M)$.

Proof. In the case of $\mathcal{L}^{-\infty}$, we can use "the stable splitting lemma for geometric quadratic Poincaré pairs" [9, Lemma 2.5] to split a sufficiently controlled object into pieces, each lying over a neighbourhood of $D(v)$, since each boundary $\partial D(v)$ is bicollared in $|K|$. Furthermore, since the control map p is simplicially stratified, each $p^{-1}(\partial D(v))$ is also bicollared, so we can retract each piece into the corresponding subset $p^{-1}(D(v))$. If a part of the boundary is already split, then we can use the given one there [9, Remark 2.12]. □

Remarks. (1) In the statement of [9, Lemma 2.5] mentioned above and in the rest of the paper [9], there are expressions like Y^{ε} and $Y^{-\varepsilon}$ for a subset Y of a metric space X and a positive number ε without any explanations. These are Quinn's notations for the ε-neighborhood of Y in X and the subset $X - (X - Y)^{\varepsilon}$, respectively, used in [4] and also in its sequels.
(2) Since we need to use stabilizations, this does not hold in the $\mathcal{L}^h$ case.
(3) Note that the cobordism y between the original object x and the assembled object $a(M)$ is less controlled than the original x, but it is still "under control".

§ 4. Squeezing and controlled assembly maps

In this section, we describe the squeezing technique on cycles, and use it to define controlled assembly maps. The argument in the $\mathcal{L}^{-\infty}$ case is exactly the same as the $\mathcal{L}^h$ case; so we discuss only the $\mathcal{L}^h$ case. Everything in this section holds true when we replace $\mathcal{L}^h$ and $\mathbb{L}^h$ by $\mathcal{L}^{-\infty}$ and $\mathbb{L}^{-\infty}$, respectively.

Theorem 4.1. *Fix a simplicially stratified fibration $p : E \to |K|$. Then there exists a positive number C which depends only on K such that the following holds. For any $\mathcal{L}$-n-cycle N of radius $\leq \delta$ in (K,p) and for any $\varepsilon > 0$, there is a special* [1]*-ad H of dimension $n+1$ in* $\mathrm{Cycles}^{\mathcal{L}}(K,p)$ *satisfying:*

(1) $\partial_0 H = N$,

(2) $\partial_1 H$ *has radius $\leq \varepsilon$, and*

(3) H *has radius $\leq C\delta$.*

Proof. We construct H (and $M = \partial_1 H$) inductively. Let $m+1$ be the number of vertices of K as before.

First consider the top dimensional simplices of K, and let σ be one of them. The dual cone $D(\sigma)$ of σ in K is a single point $\hat{\sigma}$ (the barycenter of σ), and hence $N(\sigma)$ has radius 0. So we set $M(\sigma) = N(\sigma)$ and set $H(\sigma)$ to be the trivial $([m+1]-\sigma) \sqcup [1]$-ad in $\mathcal{L}(p^{-1}(\hat{\sigma}))$ such that $\partial_0 H(\sigma) = \partial_1 H(\sigma) = N(\sigma)$. This is the first step of a finite induction.

Assume inductively that we have constructed $M(\tau)$'s and $H(\tau)$'s for simplices τ of dimension $> k$, where

(1) $M(\tau)$ is an $(n - \dim\tau)$-dimensional $([m+1]-\tau)$-ad in $\mathcal{L}^h(p^{-1}(D(\tau)))$ of radius $\leq \varepsilon$,

(2) $H(\tau)$ is an $(n - \dim\tau + 1)$-dimensional $([m+1]-\tau) \sqcup [1]$-ad in $\mathcal{L}^h(p^{-1}(D(\tau)))$ which is special in [1] and has radius $\leq C\delta$,

(3) $\partial_0 H(\tau) = N(\tau)$ and $\partial_1 H(\tau) = M(\tau)$, and

(4) these satisfy the compatibility condition required for cycles and special [1]-ads of cycles.

The radius assumptions given in (1) and (2) above is actually not sufficient for the next step. We will modify these later.

Let σ be a k-simplex of K. The union $\bigcup_{\tau>\sigma} H(\tau)$ is a cobordism between $\bigcup_{\tau>\sigma} N(\tau)$ and $\bigcup_{\tau>\sigma} M(\tau)$. Define $N'(\sigma)$ by glueing $\bigcup_{\tau>\sigma} H(\tau)$ to $N(\sigma)$:

$$N'(\sigma) = N(\sigma) \cup \bigcup_{\tau>\sigma} H(\tau).$$

Its radius is $\leq C'C\delta$ for some $C' \geq 1$ which depends only on K, and its boundary $\bigcup_{\tau>\sigma} M(\tau)$ has radius $\leq C'\varepsilon$. Let L be a large integer, and triangulate the interval $[0, L]$ using integer points. This triangulation defines a 1-dimensional geometric module symmetric Poincaré [2]-ad on $[0, L]$, which we denote by $\sigma^*[0, L]$. [2] We tensor it with the [1]-ad $(N'(\sigma); \bigcup M(\tau))$ to obtain a geometric module quadratic Poincaré [2]-ad on $p^{-1}(D(\sigma)) \times [0, L]$:

$$(N'(\sigma) \otimes \sigma^*[0, L]; N'(\sigma) \otimes 0, N''(\sigma)),$$

where $N''(\sigma) = (N'(\sigma) \otimes L) \cup (\bigcup M(\tau) \otimes \sigma^*[0, L])$. This can be also viewed as a [3]-ad

$$y = (N'(\sigma) \otimes \sigma^*[0, L]; N(\sigma), \bigcup H(\tau), N''(\sigma)) \ .$$

Observe that the radial deformation retraction $\{r_t\}$ of $D(\sigma)$ to the barycenter $\hat{\sigma}$ is covered by a canonical strong deformation retraction $\{\tilde{r}_t\}$ of $p^{-1}(D(\sigma))$ to $p^{-1}(\hat{\sigma})$, because p is simplicially stratified. We elongate $\tilde{r}$ into a map

$$f : p^{-1}(D(\sigma)) \times [0, L] \to p^{-1}(D(\sigma)) \times [0, L]; \quad (e, t) \mapsto (\tilde{r}_{t/L}(e), t) \ .$$

Now set

$$(H(\sigma); N(\sigma), \bigcup H(\tau), M(\sigma))$$

to be the functorial image of y by the composite map

$$p^{-1}(D(\sigma)) \times [0, L] \xrightarrow{f} p^{-1}(D(\sigma)) \times [0, L] \xrightarrow{\text{projection}} p^{-1}(D(\sigma)) \subset E \ .$$

$H(\sigma)$ has radius $\leq C'C\delta$, and, if L is sufficiently large, then $M(\sigma)$ has radius at most that of $\bigcup_{\tau>\sigma} M(\tau)$, which is $\leq C'\varepsilon$. This will be called the *Alexander trick of level L*.

Thus in order to accomplish the desired radius condition for M, we needed to start by squeezing into a much smaller object at each stage. For the radius condition for H, we need to replace C by a bigger number. $\square$

Proposition 4.2. *There exists a constant $C > 0$ which depends only on K such that the following holds. If N is a j-simplex of $\mathbb{H}_n(\mathrm{Cycles}^{\mathcal{L}^h}(K, p))$ whose radius is $\leq \delta$, then, for any $\varepsilon > 0$, there exist a j-simplex M of $\mathbb{H}_n(\mathrm{Cycles}^{\mathcal{L}^h}(K, p))$ and a Δ-map $f : \Delta^j \otimes [0, 1] \to \mathbb{L}^h_{-n}(|K|, p, C\delta)$ satisfying:*

(1) *$a(M)$ is a j-simplex of $\mathbb{L}^h_{-n}(|K|, p, \varepsilon)$*

(2) *$f(\Delta^j \otimes 0) = a(N)$, and*

(3) *$f(\Delta^j \otimes 1) = a(M)$.*

Furthermore, we can arrange so that M and f are natural with respect to taking faces.

[2]This σ^* is not related to the simplex σ of K; it is Ranicki's notation for a symmetric object.

Proof. We use an inductive application of "Alexander tricks" similar to those used in the proof of Theorem 4.1. It will produce a sufficiently small j-simplex M of $\mathbb{H}_n(\mathrm{Cycles}^{\mathcal{L}^h}(K,p))$ such that $a(M)$ has radius $\leq \varepsilon$, and $a(N)$ and $a(M)$ are cobordant, *i.e.* there is a $[j]\sqcup[1]$-ad connecting $a(N)$ and $a(M)$. By using Lemma 2.1, we can replace this with a Δ-map, which is essentially unique by Lemma 2.2. The extra requirement on naturality can be established by using the same level for all the simplices at each inductive step. $\square$

Now we may define the L^h-version of the controlled assembly map of Theorem 1.1:

$$A_n : \mathbb{H}_n(K; \mathbb{L}^h(p)) \longrightarrow \mathbb{L}^h_{-n}(|K|; p),$$

where $p : E \to |K|$ is a simplicially stratified fibration and K is finite.

Define $\bar{\varepsilon}_0$ to be the diameter of K, and choose a sequence $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, $\bar{\varepsilon}_3$, ... monotone decreasing to 0 such that $\bar{\varepsilon}_i > C\bar{\varepsilon}_{i+1}$, where C is the constant given in Proposition 4.2.

Pick a j-simplex $N = N_0$ of $\mathbb{H}_n(\mathrm{Cycles}^{\mathcal{L}}(K,p))$. Its radius is certainly smaller than or equal to $\bar{\varepsilon}_0$. Apply Proposition 4.2 to get a new j-simplex N_1 whose assembly $a(N_1)$ has radius $\leq \bar{\varepsilon}_2$ (this is not a typo). Continue applying Proposition 4.2 to the new j-simplices to get a sequence N_0, N_1, N_2, ...and Δ-maps $f_i : \Delta \otimes [i, i+1] \to \mathbb{L}_{-n}(|K|, p, C\bar{\varepsilon}_{i+1})$ connecting $a(N_i)$ and $a(N_{i+1})$ such that the radius of $a(N_i)$ is $\leq \bar{\varepsilon}_{i+1}$ for $i \geq 1$. Since $C\bar{\varepsilon}_{i+1} < \bar{\varepsilon}_i$, we get a Δ-map from $\Delta^j \otimes [0,\infty) \to \mathbb{L}^h_{-n}(E)$ such that the image of $\Delta^j \otimes [i,\infty)$ lies in $\mathbb{L}^h_{-n}(|K|, p, \bar{\varepsilon}_i)$, *i.e.* a j-simplex of $\mathbb{L}^h_{-n}(|K|; p)$. We may assume that these match up to define a Δ-map

$$A_n : \mathbb{H}_n(\mathrm{Cycles}^{\mathcal{L}^h}(K,p)) \longrightarrow \mathbb{L}^h_{-n}(|K|; p)\ .$$

The composition with the homotopy equivalence of Theorem 3.5 is the desired map A. Obviously, the standard assembly map a factors through it.

As mentioned at the beginning of the section, the arguments work equally well in the $\mathcal{L}^{-\infty}$ case.

§ 5. Proof of the characterization theorem

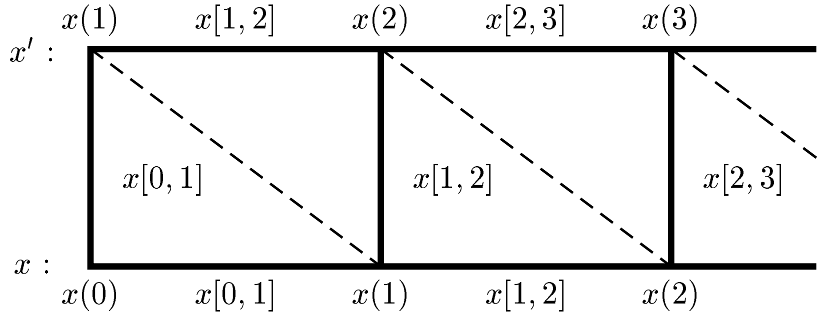
By Theorem 3.5, it suffices to show that the controlled assembly map below is a homotopy equivalence:

$$A : \mathbb{H}_{-n}(\mathrm{Cycles}^{\mathcal{L}^{-\infty}}(K,p)) \longrightarrow \mathbb{L}^{-\infty}_n(|K|; p)\ .$$

For convenience, we changed the sign for n.

First we show that A maps into every component. Let x be a 0-simplex of $\mathbb{L}^{-\infty}_n(|K|; p)$; it is a map from $[0,\infty)$ to $\mathbb{L}^{-\infty}_n(E)$ such that $x[i,\infty) \subset \mathbb{L}^{-\infty}_n(X, p, \varepsilon_i)^{(k)}$

for some integer k and a sequence ε_0, ε_1, ... monotone decreasing to 0. Note that x and another map x' defined by "$x'(t) = x(t+1)$" is in the same connected component. This can be observed using Lemma 2.1 or by directly constructing a 1-simplex connecting them.

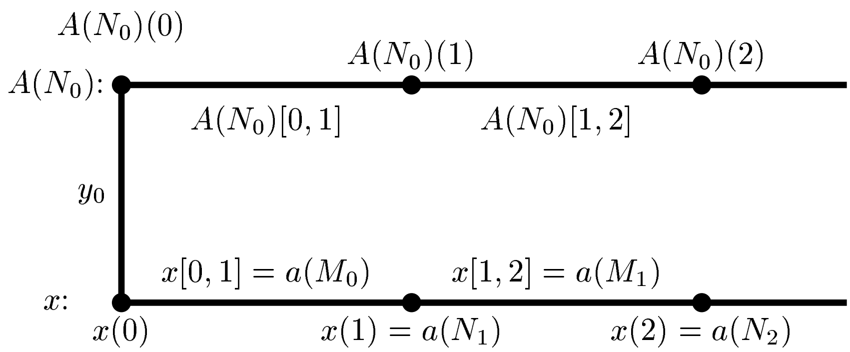


Choose i large so that ε is quite small compared with the δ posited in Lemma 3.7. Another map $x^{(i)}$ defined by "$x'(t) = x(t+i)$" is in the same connected component, repeating the argument above. So we may assume from the beginning that $\varepsilon_0 \ll \delta$. Then, by applying the absolute version of Lemma 3.7 to the $x(i)$'s, we get 0-simplices N_i of $\mathbb{H}_{-n}(\text{Cycles}^{\mathcal{L}^{-\infty}}(K,p))$ and 1-simplices y_i of $\mathbb{L}_n^{-\infty}(E)$ connecting $x(i)$ and $a(N_i)$. We may assume that the unions $y_i \cup x[i,i+1] \cup y_{j+1}$ have radius $\leq \delta$, so we can apply the relative version of Lemma 3.7 to these to obtain special [1]-ads M_i of cycles connecting N_i and N_j. We can use the triangulation argument that the sequence

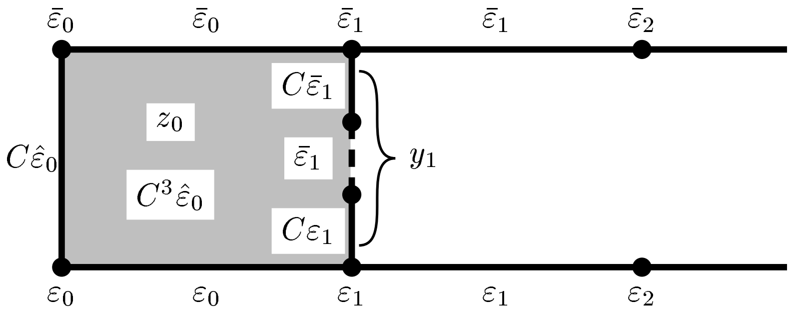
$$a(N_0), a(M_0), a(N_1), a(M_1), a(N_2), \ldots$$

is in the same connected component as x. Thus we may assume from the very beginning that x is equipped with such a splitting into cycles. Note that we still do not know that x is in the image of the controlled assembly map A, because the sequence is not obtained by successive application of Alexander tricks.

Now we would like to show that x and $A(N_0)$ are in the same connected component.

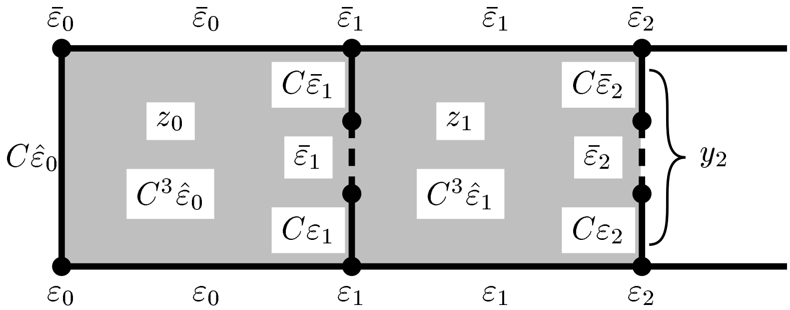


Let us define $\hat{\varepsilon}_i$ to be $\max\{\varepsilon_i, \bar{\varepsilon}_i\}$, where $\bar{\varepsilon}_i$'s are the numbers that were used during the definition of the controlled assembly map A. Since $x(0)$ and $A(N_0)(0)$ are both $a(N_0)$, there is a trivial cobordism y_0 between them. Take the union of y_0, $x[0,1]$, and $A(N_0)[0,1]$, and apply the Alexander trick used to define $A(N_0)(1)$ to it. Then we obtain a 1-simplex y_1 connecting $x(1)$ and $A(N_0)(1)$ and a "2-cell" z_0 of radius $\leq C^3\hat{\varepsilon}_0$ which fills in the rectangle as suggested by the picture below.



The dashed line indicates the squeezed piece.

The second step is to apply the squeezing to the union of y_1, $x[1,2]$, and $A(N_0)[1,2]$, as shown in the picture below:



Continuing this process, we can find 1-simplices y_i's connecting $x(i)$ and $A(N_0)(i)$ and find "2-cells" z_i's of radius $\leq C^2\hat{\varepsilon}_i$ which fill in the rectangles

$$y_i \cup x[i,i+1] \cup y_{i+1} \cup A(N_0)[i,i+1].$$

Now use Lemma 2.1 to show that x and $A(N_0)$ are in the same connected component.

Next we show that the relative homotopy groups $\pi_j(A)$ of A vanish for all j. The proof is quite the same as the previous part. The only difference is that we need to use Lemma 2.2, instead of Lemma 2.1, to construct a relative homotopy. An element of $\pi_j(A)$ is represented by a Δ-map

$$x : \Delta^j \otimes [0,\infty) \longrightarrow \mathbb{L}_n^{-\infty}(E)$$

such that $x|d_i\Delta^k \otimes [0,\infty) = 0$ for $i < j$, $x|d_j\Delta^j = A(N)$ for some $[j-1]$-ad N of cycles, and the image of $x(\Delta^j \otimes [i,\infty))$ is contained in $\mathbb{L}_n^{-\infty}(X,p,\varepsilon_i)^{(k)}$ for some k and a sequence $\{\varepsilon_i\}$. As before, we may assume that $\varepsilon_0 \ll \delta$ by changing x using a relative homotopy. Then we can use the relative splitting lemma and we may assume that x is split. Now we can use the Alexander tricks and the Lemma 2.2 to show that x represents the trivial element in the relative homotopy group. This completes the proof of Theorem 1.1.

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