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WHITNEY'S TRICK FOR THREE 2-DIMENSIONAL HOMOLOGY CLASSES OF 4-MANIFOLDS

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ABSTRACT. In his recent paper, Y. Matsumoto has defined a triple product of 2-homology classes of simply-connected oriented 4-manifolds, when the intersection numbers are zero. In the present paper, the author establishes that three 2-homology classes can be homotopically separated if the intersection numbers and the triple product vanish.

1. Introduction. Let M be a simply-connected oriented 4-manifold possibly with boundary. We shall say that homology classes $x_i \in H_2(M; Z)$, $i = 1, \ldots, n$, *can be separated*, if there exist continuous maps f_i: $S^2 \to M$ representing x_i, $i = 1, \ldots, n$, such that $f_i(S^2) \cap f_j(S^2) = \varnothing$ for $i \neq j$.

In [2], establishing a homotopy version of Whitney's trick in dimension 4, Kobayashi proved that homology classes x_1 and $x_2 \in H_2(M; Z)$ can be separated if and only if the intersection number $x_1 \cdot x_2 = 0$.

On the other hand, Matsumoto [3] defined a "secondary intersection triple", which we shall call Matsumoto triple, $\langle x_1, x_2, x_3 \rangle \in Z/I$, where x_1, x_2, x_3 are 2-dimensional homology classes of M such that $x_i \cdot x_j = 0$ for $i \neq j$ and I is an ideal of Z, $\{x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 \in Z; y_1, y_2, y_3 \in H_2(M, Z)\}$. See §4 for the definition. He showed that x_1, x_2, x_3 cannot be separated if the triple $\langle x_1, x_2, x_3 \rangle \neq 0$.

In this paper we shall prove the following:

THEOREM. *Let M be a simply-connected oriented 4-manifold, then three homology classes $x_1, x_2, x_3 \in H_2(M; Z)$ can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$ and the Matsumoto triple $\langle x_1, x_2, x_3 \rangle = 0$.*

COROLLARY. *When M is closed, homology classes x_1, x_2, x_3 can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$.*

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2. Some fundamental devices. Throughout this paper, we shall denote by M a simply-connected oriented 4-manifold. Let f_1, f_2: $S^2 \to M$ be smooth generic immersions in the sense that all the self- and mutual-intersections of

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$S_1(=f_1(S^2))$ and S_2 $(=f_2(S^2))$ are transversal double points. Suppose that $S_1 \cap S_2$ consists of positive double points, say $p_1, \ldots, p_m$, and negative ones, say $q_1, \ldots, q_n$, where $m, n \neq 0$. Draw smooth arcs γ_1, γ_2 connecting p_1 and q_1 on S_1, S_2, respectively. We may assume that γ_1 and γ_2 are generic in the sense that they have no self-intersection points and pass through neither the self-intersection points nor the mutual intersection points of S_1 and S_2. As M is simply-connected, $\gamma_1 \cup \gamma_2$ bounds a smoothly immersed 2-disk Δ, called a Whitney disk, which is generic with respect to S_1 and S_2. Let ϕ be a nonzero vector field on $\gamma_1 \cup \gamma_2$ such that when restricted to γ_2 it gives a cross-section of a normal 1-vector bundle $\nu(\gamma_2 \hookrightarrow S_2)$ and when restricted to γ_1 the unique extension of $\phi|_{\{p_1, q_1\}}$ over γ_1 which is normal to both S_1 and Δ. Let $\mathcal{O}(\Delta) \in Z = \pi_1(SO_2)$ be the obstruction to extending ϕ over Δ. We shall say that the Whitney disk Δ is good, if $\mathcal{O}(\Delta) = 0$.

Device 1 (*Making the Whitney disk* Δ *good*). When the Whitney disk Δ is not good, one can obtain a good Whitney disk spanning $\gamma_1 \cup \gamma_2$ by spinning Δ around γ_1 or γ_2 (see **[1]**).

Device 2 (*Making* S_1 *escape from the intersection with* int Δ *across* γ_1). If $S_1 \cap \text{int}\,\Delta \neq \varnothing$, for a point $p \in S_1 \cap \text{int}\,\Delta$ we take a point $p' \in \text{int}\,\gamma_1$ and a simple arc γ connecting p and p' on Δ such that $\gamma \cap S_1 = \{p, p'\}$ and $\gamma \cap S_2 = \varnothing$. Pushing a neighborhood of the intersecting point p in S_1 along the arc γ off Δ as in Figure 1, we can make S_1 escape from the intersection point p with int Δ across γ_1, by adding two self-intersection points with opposite sign for S_1.

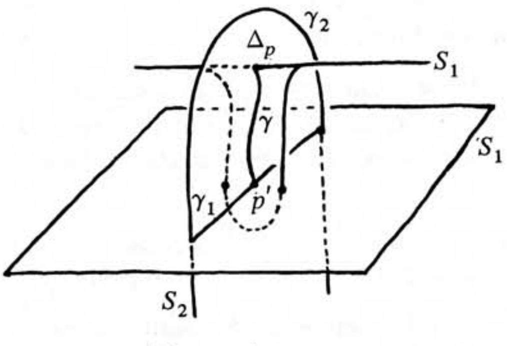


FIGURE 1

Device 3 (*Whitney's trick for* S_1 *and* S_2 *across* γ_1). Let Δ be a good immersed Whitney disk such that $S_1 \cap \text{int}\,\Delta = \varnothing$. We set

$$D = \{(x, y) \in R^2;\ x^2 + y^2 < 1, y > 0\},$$
$$D' = \{(x, y) \in R^2;\ x^2 + y^2 < (6/5)^2, y > 0\},$$
$$C_1 = \{(x, y) \in R^2;\ x^2 + y^2 = 1, y > 0\},$$
$$C_1' = \{(x, y) \in R^2;\ x^2 + y^2 = (6/5)^2, y > 0\},$$

$$C_2 = \{(x, 0) \in R^2;\ -1 \leqslant x \leqslant 1\},$$
$$C_2' = \{(x, 0) \in R^2;\ -6/5 \leqslant x \leqslant 6/5\}.$$

Let $f\colon D \to M$ be an immersion such that $f(D) = \Delta$, $f(C_1) = \gamma_1$, $f(C_2) = \gamma_2$. Adding a collar along γ_1, we have an extension $f'\colon D' \to M$ of f, so that $f'(C_1') \cap S_1 = \varnothing$, $f'(C_2') \subset S_2$. Since Δ is good, we have an immersion $\tilde{f}\colon D' \times [-1, 1] \to M$ using the vector field $\tilde{\phi}$ (the extension of ϕ over Δ) such that $\tilde{f}$ restricted to $D' = D' \times 0$ coincides with f'. Now the immersed 2-sphere S_2 shall be modified as follows.

$$S_2' = \big(S_2 - \tilde{f}(C_2' \times [-1, 1])\big) \cup \tilde{f}(C_1' \times [-1, 1]) \cup \tilde{f}(D' \times \{-1, 1\}).$$

Rounding the corners, one can assume S_2' is a generic immersed 2-sphere. Now $S_1 \cap S_2' = \{p_2, \ldots, p_m, q_2, \ldots, q_n\}$. It is easy to construct a generic immersion $f_2'\colon S^2 \to M$ with $f_2'(S^2) = S_2'$ which is regularly homotopic to f_2. This process will be referred to as Whitney's trick for S_1 and S_2 across γ_1 (see [**4**, Theorem 6.6]).

REMARK. (1) Let X be a compact subset of M. If $\Delta \cap X = \varnothing$, then $S_2' \cap X = S_2 \cap X$ in Device 3.

(2) These can be applied to generic intersections of immersed disks and spheres not only of immersed spheres.

Using the Devices 1, 2 and 3 repeatedly, we obtain

PROPOSITION (KOBAYASHI [**2**]). *Let $x_1, x_2 \in H_2(M; Z)$ be homology classes such that $x_1 \cdot x_2 = r$. Then x_1 and x_2 can be represented by continuous maps of S^2 whose images have $|r|$ points in common. In particular, if $x_1 \cdot x_2 = 0$, x_1 and x_2 can be separated.*

3. The key lemma. Let S_1, S_2, S_3 be smoothly immersed generic 2-spheres in M such that their mutual algebraic intersection numbers are all zero. We denote by $p_\lambda^{\{i,j\}}$ (or $q_\lambda^{\{i,j\}}$) the λth positive (or negative) intersection point of S_i and S_j. Draw a smoothly imbedded arc $\gamma_{\lambda,i}^{\{i,j\}}$ (or $\gamma_{\lambda,j}^{\{i,j\}}$) connecting $p_\lambda^{\{i,j\}}$ and $q_\lambda^{\{i,j\}}$ on the immersed sphere S_i (or S_j). We assume that $\gamma_{\lambda,i}^{\{i,j\}} \cap \gamma_{\mu,i}^{\{i,k\}} = \varnothing$ ($\lambda \neq \mu$ or $j \neq k$). Let $\Delta_\lambda^{\{i,j\}}$ be a smoothly immersed generic 2-disk bounding the circle $\gamma_{\lambda,i}^{\{i,j\}} \cup \gamma_{\lambda,j}^{\{i,j\}}$.

LEMMA. *Suppose that $\Delta_1^{\{i,j\}} \cap S_k = \{a_1, \ldots, a_m\}$ and $\Delta_2^{\{i,j\}} \cap S_k = \{b_1, \ldots, b_n\}$ where $\{i, j, k\} = \{1, 2, 3\}$. Then one can regularly homotope S_1, S_2, S_3 to obtain S_1', S_2', S_3' and Whitney disks $\{\Delta_\lambda'^{\{i,j\}}\}$ such that:*

(1) $S_i' \cap S_j' = S_i \cap S_j$ ($\forall i, j$),

(2) $\Delta_\lambda'^{\{i',j'\}} \cap S_{k'}' = \Delta_\lambda^{\{i',j'\}} \cap S_k$, for $\lambda > 2$, and

(3) $\Delta_1'^{\{i,j\}} \cap S_k' = \{a_2, \ldots, a_m\}$, $\Delta_2'^{\{i,j\}} \cap S_k' = \{b_0, b_1, \ldots, b_n\}$ where b_0 and a_1 have the same sign.

PROOF. (See Figure 2.) Make S_k escape from the intersection point a_1 with $\Delta_1^{\{i,j\}}$ across $\gamma_{1,i}^{\{i,j\}}$, adding new intersections of S_k and S_i, $p_0^{\{k,i\}}$ and $q_0^{\{k,i\}}$; then we obtain a small Whitney disk Δ'. Choose an imbedding $g\colon B = [-1, 1] \times [0, 1] \to S_i$ such that

$$g([-1, 1] \times \{0\}) = \partial\Delta' \cap S_i, \qquad g(\{1\} \times \{0\}) = p_0^{\{k,i\}},$$
$$g(\{-1\} \times \{0\}) = q_0^{\{k,i\}}, \qquad g(B) \cap \gamma_{\lambda,i}^{\{i,l\}} = \varnothing$$

except for $l = j$ and $\lambda = 1$ or 2, and

$$g(B) \cap \gamma_{1,i}^{\{i,j\}} = \gamma_{1,i}^{\{i,j\}}\left(\left[0, \tfrac{1}{2}\right]\right) = g_1, \qquad g(B) \cap \gamma_{2,i}^{\{i,j\}} = \gamma_{2,i}^{\{i,j\}}\left(\left[\tfrac{1}{2}, 1\right]\right) = g_2,$$

where

$$\gamma_{1,i}^{\{i,j\}}(0) = p_1^{\{i,j\}}, \qquad \gamma_{2,i}^{\{i,j\}}(1) = q_2^{\{i,j\}}.$$

Let $\gamma_{0,k}^{\{k,i\}} = \partial\Delta' \cap S_k$,

$$\gamma_{0,i}^{\{k,i\}} = g(\{1, -1\} \times [0, 1] \cup [-1, 1] \times \{1\}),$$

and $\tilde{\tilde{\Delta}}_0^{\{k,i\}} = g(B) \cup \Delta'$. Then $\tilde{\tilde{\Delta}}_0^{\{k,i\}} \cap S_j = \{p_1^{\{i,j\}}, q_2^{\{i,j\}}\}$. Let ψ be a vector field on the arc $g(\{0\} \times [0, 1])$ which does not lie in $T(S_i)$, $T(S_i)|g_l + T(\Delta_l^{\{i,j\}})|g_l$ $(l = 1, 2)$. Push $\tilde{\tilde{\Delta}}_0^{\{k,i\}}$ off S_i along ψ keeping $\gamma_{0,i}^{\{k,i\}}$ fixed; then we obtain an imbedded disk $\tilde{\Delta}_0^{\{k,i\}}$ bounded by $\gamma_0^{\{k,i\}} \cup \gamma_{0,k}^{\{k,i\}}$ such that it meets S_i normally along $\gamma_{0,i}^{\{k,i\}}$, S_k normally along $\gamma_{0,k}^{\{k,i\}}$, and

$$S_i \cap \operatorname{int} \tilde{\Delta}_0^{\{k,i\}} = \varnothing, \qquad S_k \cap \operatorname{int} \tilde{\Delta}_0^{\{k,i\}} = \varnothing,$$
$$S_j \cap \tilde{\Delta}_0^{\{k,i\}} = \{p, q\}, \qquad \tilde{\Delta}_0^{\{k,i\}} \cap \Delta_\lambda^{\{i,j\}} = \varnothing$$

for $\lambda = 1, 2$. We can cancel these intersection points p and q as follows. Let γ (or γ') be a generic arc connecting p and q on $\tilde{\Delta}_0^{\{k,i\}}$ (or S_j), and let Δ be a good generic immersed disk bounded by $\gamma \cup \gamma'$. We can make S_j escape from the intersection with int Δ across γ'. Doing Whitney's trick for $\tilde{\Delta}_0^{\{k,i\}}$ and S_j across γ', we obtain a new immersed disk $\Delta_0^{\{k,i\}}$ such that $S_j \cap \Delta_0^{\{k,i\}} = \varnothing$. We may assume that $\Delta_0^{\{k,i\}}$ is good, and

$$\Delta_0^{\{k,i\}} \cap \Delta_\lambda^{\{j,k\}} = \varnothing, \qquad \operatorname{int} \Delta_0^{\{k,i\}} \cap \operatorname{int} \Delta_\mu^{\{i,j\}} = \varnothing.$$

For example, if $\Delta_0^{\{k,i\}} \cap \Delta_\lambda^{\{j,k\}} \neq \varnothing$, we can make $\Delta_0^{\{k,i\}}$ escape from the intersection with $\Delta_\lambda^{\{j,k\}}$ across $\gamma_{\lambda,k}^{\{j,k\}}$ by adding two intersection points of $\Delta_0^{\{k,i\}}$ and S_k. Possibly int $\Delta_0^{\{k,i\}} \cap S_k \neq \varnothing$, int $\Delta_0^{\{k,i\}} \cap S_i \neq \varnothing$. Make $\Delta_0^{\{k,i\}}$ escape from this intersection with S_k across $\gamma_{0,k}^{\{k,i\}}$, if necessary.

Using $\Delta_0^{\{k,i\}}$, we can do Whitney's trick for S_k and S_i across $\gamma_{0,k}^{\{k,i\}}$ and we obtain a new immersed 2-sphere S_i' such that

$$S_i' \cap S_k = S_i \cap S_k - \{p_0^{\{k,i\}}, q_0^{\{k,i\}}\} \quad \text{and} \quad \Delta_\lambda^{\{j,k\}} \cap S_i' = \Delta_\lambda^{\{j,k\}} \cap S_i.$$

Let $\tilde{f}$ denote the immersion: $D' \times [-1, 1] \to M$ in Device 3 such that $\tilde{f}(C_2 \times \{0\}) = \gamma_{0,i}^{\{k,i\}}$. We may assume that

$$\tilde{f}(\{(0, 0)\} \times [-1, 1]) = \gamma_{2,i}^{\{i,j\}} \cap \tilde{f}(D' \times [-1, 1]).$$

We shall modify the disk $\Delta_2^{\{i,j\}}$ as follows:

$$\Delta_2'^{\{i,j\}} = \Delta_2^{\{i,j\}} \cup \tilde{f}(\{(x, 0); 0 \leqslant x \leqslant 6/5\} \times [-1, 1]).$$

Then we obtain a new intersection point b_0. Q.E.D.

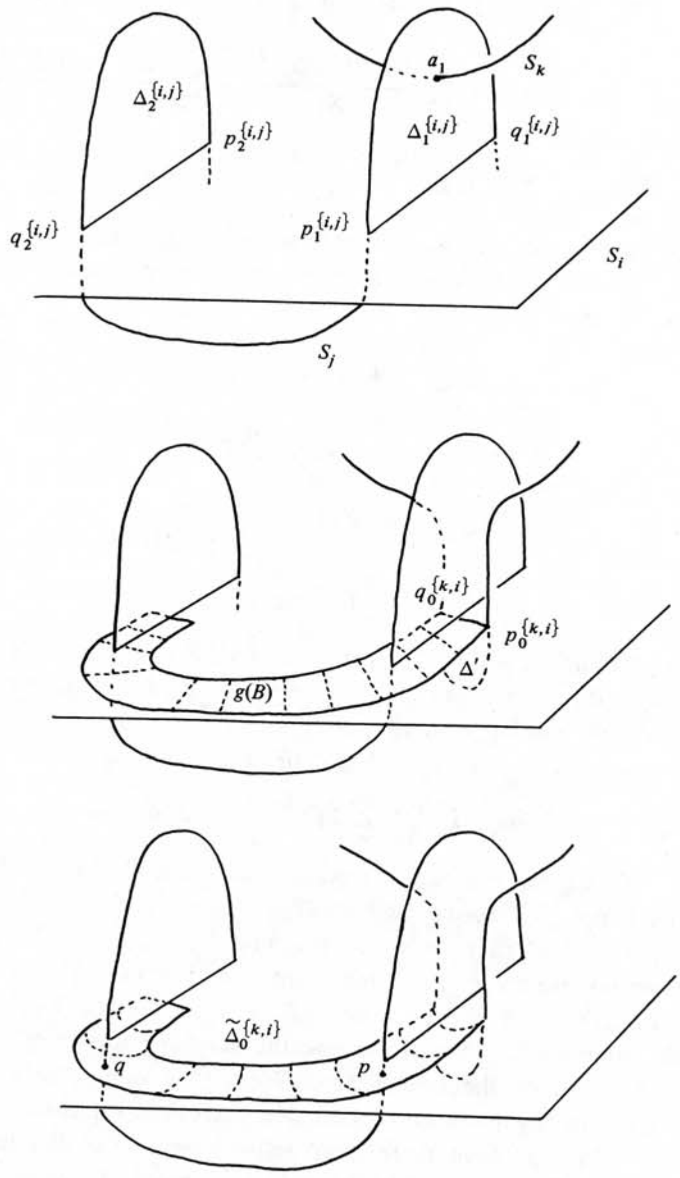
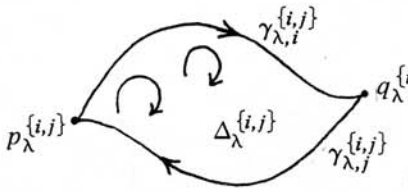


FIGURE 2

4. Proof of theorem and corollary. Let $x_1, x_2, x_3 \in H_2(M; Z)$ be homology classes such that $x_i \cdot x_j = 0$ for $i \neq j$. Represent x_1, x_2, x_3 by smoothly immersed generic 2-spheres S_1, S_2, S_3, and let $p_\lambda^{\{i,j\}}, \gamma_{\lambda,i}^{\{i,j\}}, \Delta_\lambda^{\{i,j\}}$ be as in §3, but we do not require the condition $\gamma_{\lambda,i}^{\{i,j\}} \cap \gamma_{\mu,i}^{\{i,k\}} = \varnothing$ ($\lambda \neq \mu$ or $j \neq k$). The Whitney disk $\Delta_\lambda^{\{i,j\}}$ is oriented as in Figure 3. Now the Matsumoto triple $\langle x_1, x_2, x_3\rangle$ is defined as follows:

$$\langle x_1, x_2, x_3\rangle = \sum_\lambda S_1 \cdot \Delta_\lambda^{\{2,3\}} + \sum_\mu S_2 \cdot \Delta_\mu^{\{3,1\}} + \sum_\nu S_3 \cdot \Delta_\nu^{\{1,2\}} + \sum_{\mu,\nu} \frac{\partial\Delta_\mu^{\{3,1\}} \cdot \partial\Delta_\nu^{\{1,2\}}}{S_1} + \sum_{\nu,\lambda} \frac{\partial\Delta_\nu^{\{1,2\}} \cdot \partial\Delta_\lambda^{\{2,3\}}}{S_2} + \sum_{\lambda,\mu} \frac{\partial\Delta_\lambda^{\{2,3\}} \cdot \partial\Delta_\mu^{\{3,1\}}}{S_3} \mod I,$$

where $S_1 \cdot \Delta_\lambda^{\{2,3\}}$, etc., denote the intersection number of S_1 and $\Delta_\lambda^{\{2,3\}}$, etc., and $(\partial\Delta_\mu^{\{3,1\}} \cdot \partial\Delta_\nu^{\{1,2\}})/S_1$, etc. denote the intersection number of $\partial\Delta_\mu^{\{3,1\}}$ and $\partial\Delta_\nu^{\{1,2\}}$ on S_1, etc.



$(i, j) = (1, 2), (2, 3), (3,1)$

FIGURE 3

PROOF OF THEOREM. Let $x_1, x_2, x_3 \in H_2(M; Z)$ be homology classes such that $x_i \cdot x_j = 0$ for $i \neq j$ and $\langle x_1, x_2, x_3\rangle = 0 \in Z/I$. Let S_i, $\gamma_{\lambda,i}^{\{i,j\}}$, $\Delta_\lambda^{\{i,j\}}$ be as above. Now we assume as in §3 that $\gamma_{\lambda,i}^{\{i,j\}} \cap \gamma_{\mu,i}^{\{i,k\}} = \varnothing$ ($\lambda \neq \mu$ or $j \neq k$); then the Matsumoto triple $\langle x_1, x_2, x_3\rangle$ is defined as

$$\sum_\lambda S_1 \cdot \Delta_\lambda^{\{2,3\}} + \sum_\mu S_2 \cdot \Delta_\mu^{\{3,1\}} + \sum_\nu S_3 \cdot \Delta_\nu^{\{1,2\}}.$$

We may assume that this sum is zero. In fact, if the ideal I is $\{0\}$, it is always zero. If I is not $\{0\}$, there exist homology classes $y_1, y_2, y_3 \in H_2(M; Z)$ such that $\langle x_1, x_2, x_3\rangle = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$. Let F_1, F_2, F_3 be immersed 2-spheres representing y_1, y_2, y_3. Make connected-sums of $\Delta_1^{\{2,3\}}$ and $-F_1$, $\Delta_1^{\{3,1\}}$ and $-F_2$, $\Delta_1^{\{1,2\}}$ and $-F_3$, where $-F$ is an immersed 2-sphere with the reversed orientation. Then if we use the resulting immersed disks instead of$\Delta_1^{\{2,3\}}$, $\Delta_1^{\{3,1\}}$, $\Delta_1^{\{1,2\}}$, the sum is zero. We may assume that every Whitney disk is good in the sense of §2 and that there is no mutual-intersection of Whitney disks (and even there is no self-intersection of Whitney disks, i.e. every Whitney disk is an imbedded disk). (See proof of lemma.) As $x_1 \cdot x_3 = 0$, we may assume $S_1 \cap S_3 = \varnothing$ by the proposition. Escaping the intersection $S_1 \cap \operatorname{int} \Delta_\lambda^{\{1,2\}}$ across $\gamma_{\lambda,1}^{\{1,2\}}$, we obtain a new immersed 2-sphere S_1', so that $S_1' \cap \operatorname{int} \Delta_\lambda^{\{1,2\}} = \varnothing$ and $S_1' \cap S_3 = \varnothing$. Using $\Delta_\lambda^{\{1,2\}}$, do Whitney's trick for S_1' and S_2 across $\gamma_{\lambda,1}^{\{1,2\}}$, and we shall obtain a new immersed 2-sphere S_2' such that $S_1' \cap S_2' = \varnothing$. By Lemma, we may assume that $S_1' \cap \Delta_\lambda^{\{2,3\}} = \varnothing$ for $\lambda \neq 1$. Then $S_1' \cdot \Delta_1^{\{2,3\}} = 0$. Using Devices 1, 2 and 3, we obtain $S_1' \cap \Delta^{\{2,3\}} = \varnothing$. Now we can do Whitney's trick for S_2' and S_3 (using Device 2), and we obtain the required maps. Q.E.D.

PROOF OF COROLLARY. If one of x_1, x_2, x_3 is 0, then this follows immediately from the proposition. If one of x_1, x_2, x_3, say x_1, is a primitive element, i.e. there is no homology class $x \in H_2(M; Z)$ such that $x_1 = mx$ ($m \in Z, \neq 1, -1$), then there exists a homology class $y \in H_2(M; Z)$ such that $x_1 \cdot y = 1$. Therefore $\langle x_1, x_2, x_3 \rangle = 0 \bmod I = (1)$ and they can be separated by the theorem. If x_1, x_2, x_3 can be separated, also mx_1, x_2, x_3 can be separated by using the "self-connected-sum" of the immersed 2-sphere representing x_i ($m \in Z$). Q.E.D.

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