WHITNEY'S TRICK FOR THREE 2-DIMENSIONAL HOMOLOGY CLASSES OF 4-MANIFOLDS

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ABSTRACT. In his recent paper, Y. Matsumoto has defined a triple product of 2-homology classes of simply-connected oriented 4-manifolds, when the intersection numbers are zero. In the present paper, the author establishes that three 2-homology classes can be homotopically separated if the intersection numbers and the triple product vanish.

1. Introduction. Let $M$ be a simply-connected oriented 4-manifold possibly with boundary. We shall say that homology classes $x_i \in H_2(M; \mathbb{Z})$, $i = 1, \ldots, n$, can be separated, if there exist continuous maps $f_i: S^2 \to M$ representing $x_i$, $i = 1, \ldots, n$, such that $f_i(S^2) \cap f_j(S^2) = \emptyset$ for $i \neq j$.

In [2], establishing a homotopy version of Whitney's trick in dimension 4, Kobayashi proved that homology classes $x_1$ and $x_2 \in H_2(M; \mathbb{Z})$ can be separated if and only if the intersection number $x_1 \cdot x_2 = 0$.

On the other hand, Matsumoto [3] defined a "secondary intersection triple", which we shall call Matsumoto triple, $\langle x_1, x_2, x_3 \rangle \in \mathbb{Z}/I$, where $x_1, x_2, x_3$ are 2-dimensional homology classes of $M$ such that $x_i \cdot x_j = 0$ for $i \neq j$ and $I$ is an ideal of $\mathbb{Z}$, $\{ x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 \in H_3(M; \mathbb{Z}) \}$. See §4 for the definition. He showed that $x_1, x_2, x_3$ cannot be separated if the triple $\langle x_1, x_2, x_3 \rangle \neq 0$.

In this paper we shall prove the following:

THEOREM. Let $M$ be a simply-connected oriented 4-manifold, then three homology classes $x_1, x_2, x_3 \in H_2(M; \mathbb{Z})$ can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$ and the Matsumoto triple $\langle x_1, x_2, x_3 \rangle = 0$.

COROLLARY. When $M$ is closed, homology classes $x_1, x_2, x_3$ can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$.

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2. Some fundamental devices. Throughout this paper, we shall denote by $M$ a simply-connected oriented 4-manifold. Let $f_1, f_2: S^2 \to M$ be smooth generic immersions in the sense that all the self- and mutual-intersections of

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$S_1 (= f_1(S^2))$ and $S_2 (= f_2(S^2))$ are transversal double points. Suppose that $S_1 \cap S_2$ consists of positive double points, say $p_1, \ldots, p_m$, and negative ones, say $q_1, \ldots, q_n$, where $m, n \neq 0$. Draw smooth arcs $\gamma_1, \gamma_2$ connecting $p_1$ and $q_1$ on $S_1, S_2$, respectively. We may assume that $\gamma_1$ and $\gamma_2$ are generic in the sense that they have no self-intersection points and pass through neither the self-intersection points nor the mutual intersection points of $S_1$ and $S_2$. As $M$ is simply-connected, $\gamma_1 \cup \gamma_2$ bounds a smoothly immersed 2-disk $\Delta$, called a Whitney disk, which is generic with respect to $S_1$ and $S_2$. Let $\phi$ be a nonzero vector field on $\gamma_1 \cup \gamma_2$ such that when restricted to $\gamma_2$ it gives a cross-section of a normal 1-vector bundle $\nu(\gamma_2 \hookrightarrow S_2)$ and when restricted to $\gamma_1$ the unique extension of $\phi_{|\gamma_1}$ over $\gamma_1$, which is normal to both $S_1$ and $\Delta$. Let $\theta(\Delta) \in Z = \pi_1(SO_2)$ be the obstruction to extending $\phi$ over $\Delta$. We shall say that the Whitney disk $\Delta$ is good, if $\theta(\Delta) = 0$.

Device 1 (Making the Whitney disk $\Delta$ good). When the Whitney disk $\Delta$ is not good, one can obtain a good Whitney disk spanning $\gamma_1 \cup \gamma_2$ by spinning $\Delta$ around $\gamma_1$ or $\gamma_2$ (see [1]).

Device 2 (Making $S_1$ escape from the intersection with $\text{int} \Delta$ across $\gamma_i$). If $S_1 \cap \text{int} \Delta \neq \emptyset$, for a point $p \in S_1 \cap \text{int} \Delta$ we take a point $p' \in \text{int} \gamma_1$ and a simple arc $\gamma$ connecting $p$ and $p'$ on $\Delta$ such that $\gamma \cap S_1 = \{p, p'\}$ and $\gamma \cap S_2 = \emptyset$. Pushing a neighborhood of the intersecting point $p$ in $S_1$ along the arc $\gamma$ off $\Delta$ as in Figure 1, we can make $S_1$ escape from the intersection point $p$ with $\text{int} \Delta$ across $\gamma_1$, by adding two self-intersection points with opposite sign for $S_1$.

Device 3 (Whitney’s trick for $S_1$ and $S_2$ across $\gamma_1$). Let $\Delta$ be a good immersed Whitney disk such that $S_1 \cap \text{int} \Delta = \emptyset$. We set

$$D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1, y > 0\},$$

$$D' = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < (6/5)^2, y > 0\},$$

$$C_i = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1, y > 0\},$$

$$C'_i = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = (6/5)^2, y > 0\}.$$

![Figure 1](image-url)
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\[ C_2 = \{(x, 0) \in \mathbb{R}^2; -1 < x < 1\}, \]

\[ C'_2 = \{(x, 0) \in \mathbb{R}^2; -6/5 < x < 6/5\}. \]

Let \( f: D \to M \) be an immersion such that \( f(D) = \Delta, f(C_2) = \gamma_1, f(C'_2) = \gamma_2 \). Adding a collar along \( \gamma_1 \), we have an extension \( f': D' \to M \) of \( f \), so that \( f'(C'_2) \cap S_1 = \emptyset, f'(C'_2) \subset S_2 \). Since \( \Delta \) is good, we have an immersion \( \tilde{f}: D' \times [-1, 1] \to M \) using the vector field \( \tilde{\phi} \) (the extension of \( \phi \) over \( \Delta \)) such that \( \tilde{f} \) restricted to \( D' = D' \times 0 \) coincides with \( f' \). Now the immersed 2-sphere \( S_2 \) shall be modified as follows.

\[ S'_2 = \left( S_2 - \tilde{f}(C'_2 \times [-1, 1]) \right) \cup \tilde{f}(C'_2 \times [-1, 1]) \cup \tilde{f}(D' \times [-1, 1]). \]

Rounding the corners, one can assume \( S'_2 \) is a generic immersed 2-sphere. Now \( S_1 \cap S'_2 = \{ p_2, \ldots, p_m, q_2, \ldots, q_n \} \). It is easy to construct a generic immersion \( f': S^2 \to M \) with \( f'(S^2) = S'_2 \) which is regularly homotopic to \( f_2 \). This process will be referred to as Whitney’s trick for \( S_1 \) and \( S_2 \) across \( \gamma_1 \) (see [4, Theorem 6.6]).

Remark. (1) Let \( X \) be a compact subset of \( M \). If \( \Delta \cap X = \emptyset \), then \( S_1 \cap X = S_2 \cap X \) in Device 3.

(2) These can be applied to generic intersections of immersed disks and spheres not only of immersed spheres.

Using the Devices 1, 2 and 3 repeatedly, we obtain

**Proposition (Kobayashi [2]).** Let \( x_1, x_2 \in H_q(M; Z) \) be homology classes such that \( x_1 \cdot x_2 = r \). Then \( x_1 \) and \( x_2 \) can be represented by continuous maps of \( S^2 \) whose images have \( |r| \) points in common. In particular, if \( x_1 \cdot x_2 = 0 \), \( x_1 \) and \( x_2 \) can be separated.

**3. The key lemma.** Let \( S_1, S_2, S_3 \) be smoothly immersed generic 2-spheres in \( M \) such that their mutual algebraic intersection numbers are all zero. We denote by \( p_1^{(i,j)} \) (or \( q_1^{(i,j)} \)) the \( \lambda \)-th positive (or negative) intersection point of \( S_i \) and \( S_j \). Draw a smoothly imbedded arc \( \gamma_1^{(i,j)} \) (or \( \gamma_2^{(i,j)} \)) connecting \( p_1^{(i,j)} \) and \( q_1^{(i,j)} \) on the immersed sphere \( S_i \) (or \( S_j \)). We assume that \( \gamma_1^{(i,j)} \cap \gamma_1^{(k,l)} = \emptyset \) (\( \lambda \neq \mu \) or \( j \neq k \)). Let \( \Delta_1^{(i,j)} \) be a smoothly immersed generic 2-disk bounding the circle \( \gamma_1^{(i,j)} \cup \gamma_1^{(k,l)} \).

**Lemma.** Suppose that \( \Delta_1^{(i,j)} \cap S_k = \{ a_1, \ldots, a_m \} \) and \( \Delta_1^{(i,j)} \cap S_k = \{ b_1, \ldots, b_n \} \) where \( \{ i, j, k \} = \{ 1, 2, 3 \} \). Then one can regularly homotope \( S_1, S_2, S_3 \) to obtain \( S'_1, S'_2, S'_3 \) and Whitney disks \( \Delta_1^{(i,j)} \) such that:

1. \( S'_1 \cap S'_2 = S'_1 \cap S'_3 \) (for \( i, j, k \)),
2. \( \Delta_1^{(i,j)} \cap S'_k = \Delta_1^{(i,j)} \cap S'_k \) for \( \lambda > 2 \), and
3. \( \Delta_1^{(i,j)} \cap S'_k = \{ a_1, \ldots, a_m \}, \Delta_1^{(i,j)} \cap S'_k = \{ b_0, b_1, \ldots, b_n \} \) where \( b_0 \) and \( a_j \) have the same sign.

**Proof.** (See Figure 2). Make \( S_k \) escape from the intersection point \( a_1 \) with \( \Delta_1^{(i,j)} \) across \( \gamma_1^{(i,j)} \), adding new intersections of \( S_k \) and \( S_i \) \( p_1^{(k,i,j)} \) and \( q_1^{(k,i,j)} \); then we obtain a small Whitney disk \( \Delta' \). Choose an imbedding \( g: B = [-1, 1] \times [0, 1] \to S_i \) such that
\[ g([0, 1]) = \gamma(0) \times \{0\}, \quad g([1, 0]) = \gamma(1) \times \{0\}, \quad g([-1, 1] \times \{0\}) = \gamma([0, 1]) \times \{0\}, \quad g(-1) \times \{0\}) = \gamma([-1, 1]) \times \{0\}, \quad g(0) = \{0\}
\]

except for \(l = j\) and \(\lambda = 1\) or 2, and

\[ g(B) \cap \gamma^{(j)} = \gamma^{(j)}([0, 1]) = g, \quad g(B) \cap \gamma^{(j)} = \gamma^{(j)}([1, 0]) = g^2. \]

where

\[ \gamma^{(j)}(0) = p^{(j)}, \quad \gamma^{(j)}(1) = q^{(j)}. \]

Let \(\gamma^{(j)} = \partial \Delta \cap S_k\),

\[ \gamma^{(j)} = \gamma([0, 1]) \cup \gamma([-1, 1]). \]

and \(\tilde{\Delta}^{(j)} = g(B) \cup \Delta. \) Then \(\tilde{\Delta}^{(j)} \cap S_j = \{p^{(j)}, q^{(j)}\}. \) Let \(\psi\) be a vector field on the arc \(g([0, 1])\) which does not lie in \(T(S), T(S) \cap S_j, T(S) \cap S_j (l = 1, 2)\). Push \(\tilde{\Delta}^{(j)}\) off \(S_i\) along \(\psi\) keeping \(\gamma^{(j)}\) fixed; then we obtain an imbedded disk \(\Delta^{(j)}\) bounded by \(\gamma^{(j)} \cup \gamma^{(k)}\) such that it meets \(S_i\) normally along \(\gamma^{(k)}\), \(S_k\) normally along \(\gamma^{(k)}\), and

\( S_i \cap \text{Int} \tilde{\Delta}^{(j)} = \emptyset, \quad S_k \cap \text{Int} \tilde{\Delta}^{(k)} = \emptyset, \)

\( S_j \cap \text{Int} \tilde{\Delta}^{(j)} = \{p, q\}, \quad \tilde{\Delta}^{(j)} \cap \Delta^{(j)} = \emptyset. \)

for \(\lambda = 1, 2\). We can cancel these intersection points \(p\) and \(q\) as follows. Let \(\gamma\) (or \(\psi\)) be a generic arc connecting \(p\) and \(q\) on \(\Delta^{(j)}\) (or \(S_j\)), and let \(\Delta\) be a good generic immersed disk bounded by \(\gamma \cup \gamma\). We can make \(S_j\) escape from the intersection with \(\Delta\) across \(\gamma\). Doing Whitney's trick for \(\Delta^{(j)}\) and \(S_j\) across \(\gamma\), we obtain a new immersed disk \(\Delta^{(j)}\) such that \(S_j \cap \Delta^{(j)} = \emptyset\). We may assume that \(\Delta^{(j)}\) is good, and

\( \Delta^{(j)} \cap \Delta^{(k)} = \emptyset, \quad \text{Int} \Delta^{(j)} \cap \text{Int} \Delta^{(k)} = \emptyset. \)

For example, if \(\Delta^{(j)} \cap \Delta^{(k)} \neq \emptyset\), we can make \(\Delta^{(j)}\) escape from the intersection with \(\Delta^{(k)}\) across \(\gamma^{(j)}\) by adding two intersection points of \(\Delta^{(j)}\) and \(S_k\). Possibly \(\text{Int} \Delta^{(j)} \cap S_k \neq \emptyset, \text{Int} \Delta^{(k)} \cap S_j \neq \emptyset\). Make \(\Delta^{(j)}\) escape from this intersection with \(S_k\) across \(\gamma^{(k)}\), if necessary.

Using \(\Delta^{(j)}\), we can do Whitney's trick for \(S_k\) and \(S_j\) across \(\gamma^{(j)}\) and we obtain a new immersed 2-sphere \(S_i\) such that

\( S_i \cap S_k = S_i \cap S_j = \{p^{(j)}, q^{(i)}\} \text{ and } \Delta^{(j)} \cap S_i = \Delta^{(j)} \cap S_k. \)

Let \(\tilde{f}\) denote the immersion: \(D' \times [-1, 1] \to M\) in Device 3 such that \(\tilde{f}(C_2 \times (0)) = \gamma^{(j)}\). We may assume that

\(\tilde{f}((0, 0) \times [-1, 1]) = \gamma^{(j)} \cap \tilde{f}(D' \times [-1, 1]).\)

We shall modify the disk \(\Delta^{(j)}\) as follows:

\(\Delta^{(j)} = \Delta^{(j)} \cup \tilde{f}((x, 0); 0 < x < 6/5) \times [-1, 1]).\)

Then we obtain a new intersection point \(b_0\). Q.E.D.
4. Proof of theorem and corollary. Let $x_1, x_2, x_3 \in H_2(M; \mathbb{Z})$ be homology classes such that $x_i \cdot x_j = 0$ for $i \neq j$. Represent $x_1, x_2, x_3$ by smoothly immersed generic 2-spheres $S_1, S_2, S_3$, and let $p^{(i)}_\lambda, \gamma^{(i)}_\lambda, \Delta^{(i)}_\lambda$ be as in §3, but we do not require the condition $\gamma^{(i)}_\lambda \cap \gamma^{(k)}_\mu = \emptyset$ ($\lambda \neq \mu$ or $i \neq k$). The Whitney disk $\Delta^{(i)}_\lambda$ is oriented as in Figure 3. Now the Matsumoto triple $\langle x_1, x_2, x_3 \rangle$ is defined as follows:
\[ \langle x_1, x_2, x_3 \rangle = \sum_{\lambda} S_1 \cdot \Delta^{(2,3)} + \sum_{\mu} S_2 \cdot \Delta^{(3,1)} + \sum_{\nu} S_3 \cdot \Delta^{(1,2)} \]

\[ + \sum_{\alpha} \frac{\partial \Delta^{(2,3)}_{\alpha}}{S_1} + \sum_{\beta} \frac{\partial \Delta^{(3,1)}_{\beta}}{S_2} + \sum_{\gamma} \frac{\partial \Delta^{(1,2)}_{\gamma}}{S_3} \mod I, \]

where \( S_1 \cdot \Delta^{(2,3)} \), etc., denote the intersection number of \( S_1 \) and \( \Delta^{(2,3)} \), etc., and \( (\partial \Delta^{(3,1)}_{\beta} \cdot \partial \Delta^{(1,2)}_{\gamma}) / S_1 \), etc. denote the intersection number of \( \partial \Delta^{(3,1)} \) and \( \partial \Delta^{(1,2)} \) on \( S_1 \), etc.

**Figure 3**

**Proof of Theorem.** Let \( x_1, x_2, x_3 \in H_2(M; \mathbb{Z}) \) be homology classes such that \( x_i \cdot x_j = 0 \) for \( i \neq j \) and \( \langle x_1, x_2, x_3 \rangle = 0 \in \mathbb{Z}/I \). Let \( S_1 \), \( S_2 \), \( S_3 \) be as above. Now we assume as in §3 that \( \gamma^{(i,j)}_{\lambda,k} \cap \gamma^{(l,k)}_{\mu} = \emptyset \) (\( \lambda \neq \mu \) or \( j \neq k \)); then the Matsumoto triple \( \langle x_1, x_2, x_3 \rangle \) is defined as

\[ \sum_{\lambda} S_1 \cdot \Delta^{(2,3)} + \sum_{\mu} S_2 \cdot \Delta^{(3,1)} + \sum_{\nu} S_3 \cdot \Delta^{(1,2)}. \]

We may assume that this sum is zero. In fact, if the ideal \( I \) is \( \{0\} \), it is always zero. If \( I \) is not \( \{0\} \), there exist homology classes \( y_1, y_2, y_3 \in H_2(M; \mathbb{Z}) \) such that \( \langle x_1, x_2, x_3 \rangle = x_1 \cdot y_2 + x_2 \cdot y_3 \). Let \( F_1, F_2, F_3 \) be immersed 2-spheres representing \( y_1, y_2, y_3 \). Make connected-sums of \( \Delta^{(2,3)} \) and \( -F_1, \Delta^{(1,2)} \) and \( -F_2, \Delta^{(1,2)} \) and \( -F_3, \Delta^{(1,2)} \), where \( -F \) is an immersed 2-sphere with the reversed orientation. Then if we use the resulting immersed disks instead of \( \Delta^{(2,3)}, \Delta^{(3,1)}, \Delta^{(1,2)} \), the sum is zero. We may assume that every Whitney disk is good in the sense of §2 and that there is no mutual-intersection of Whitney disks (and even there is no self-intersection of Whitney disks, i.e., every Whitney disk is an imbedded disk). (See proof of lemma.) As \( x_1 \cdot x_3 = 0 \), we may assume \( S_1 \cap S_3 = \emptyset \) by the proposition. Escaping the intersection \( S_1 \cap \text{int} \Delta^{(1,2)} \), we obtain a new immersed 2-sphere \( S'_1 \) such that \( S'_1 \cap \text{int} \Delta^{(1,2)} = \emptyset \) and \( S'_1 \cap S_1 = \emptyset \). Using \( \Delta^{(1,2)} \), do Whitney's trick for \( S'_1 \) and \( S_1 \) across \( \gamma^{(1,2)}_{\mu} \), and we shall obtain a new immersed 2-sphere \( S'_2 \) such that \( S'_1 \cap S'_2 = \emptyset \). By Lemma, we may assume that \( S'_1 \cap \Delta^{(2,3)} = \emptyset \) for \( \lambda \neq 1 \). Then \( S'_1 \cdot \Delta^{(2,3)} = 0 \). Using Devices 1, 2 and 3, we obtain \( S'_1 \cap \Delta^{(2,3)} = \emptyset \). Now we can do Whitney's trick for \( S'_1 \) and \( S_1 \) (using Device 2), and we obtain the required maps. Q.E.D.
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Proof of Corollary. If one of \( x_1, x_2, x_3 \) is 0, then this follows immediately from the proposition. If one of \( x_1, x_2, x_3 \), say \( x_1 \), is a primitive element, i.e. there is no homology class \( x \in H_2(M; \mathbb{Z}) \) such that \( x_1 = mx \) \((m \in \mathbb{Z}, \neq 1, -1)\), then there exists a homology class \( y \in H_2(M; \mathbb{Z}) \) such that \( x_1 \cdot y = 1 \). Therefore \( \langle x_1, x_2, x_3 \rangle = 0 \mod l = (1) \) and they can be separated by the theorem. If \( x_1, x_2, x_3 \) can be separated, also \( mx_1, x_2, x_3 \) can be separated by using the "self-connected-sum" of the immersed 2-sphere representing \( x_i \) \((m \in \mathbb{Z})\). Q.E.D.

References


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