SURGERY AND GEOMETRY

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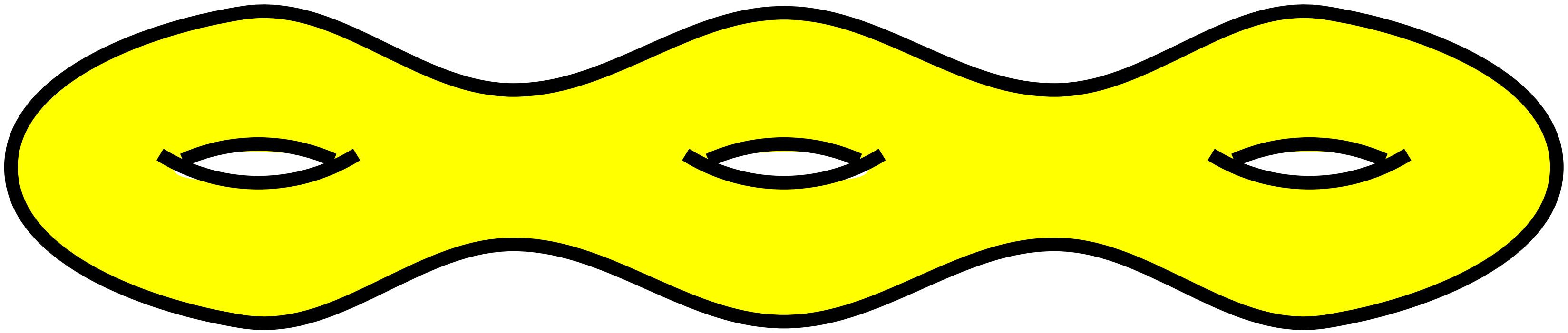
an n-dim (topological) manifold

$=$ a subset of $\mathbb{R}^N$ locally homeomorphic to $\mathbb{R}^n$

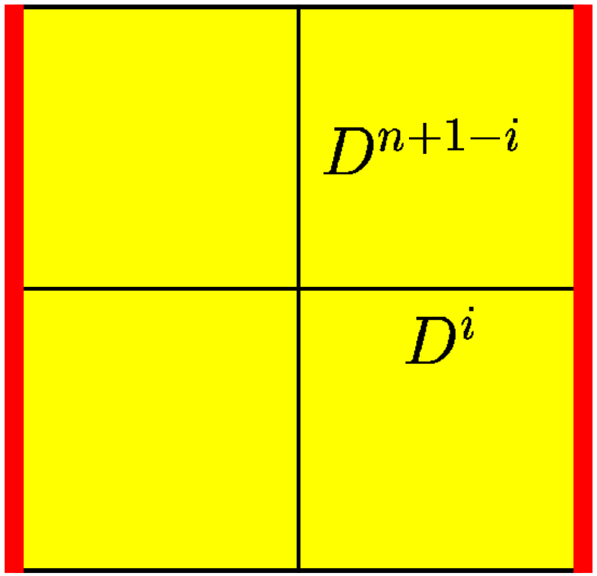


will only consider oriented manifolds in this talk

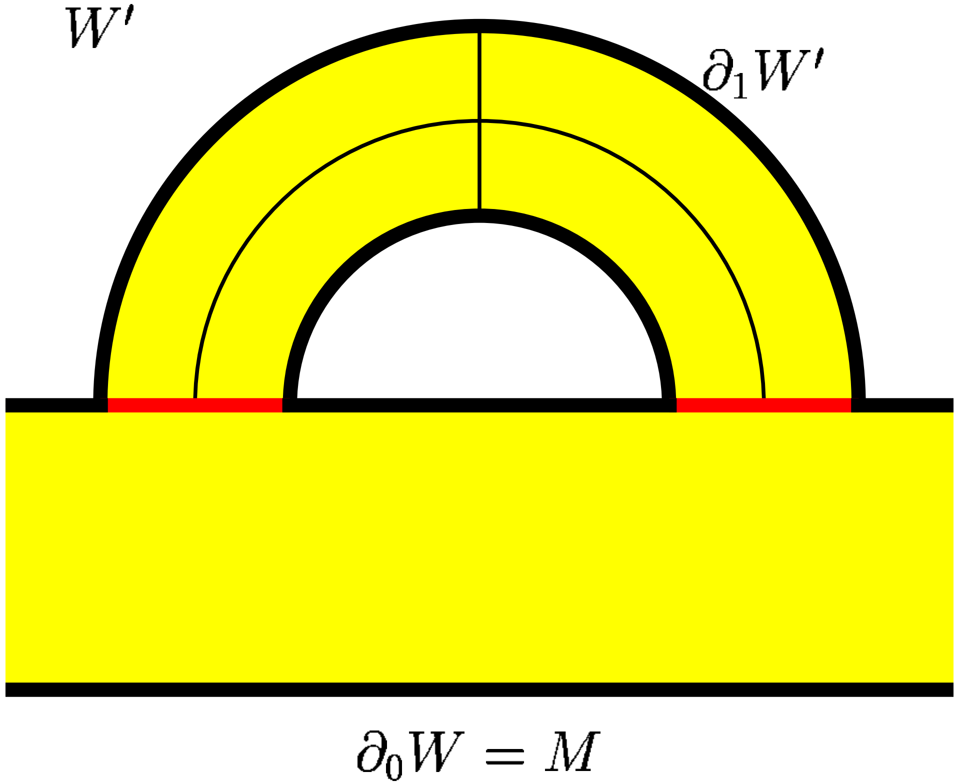
SURGERY

M : n-dim manifold

$H = D^i \times D^{n+1-i}$: $(n+1)$-dim i-handle

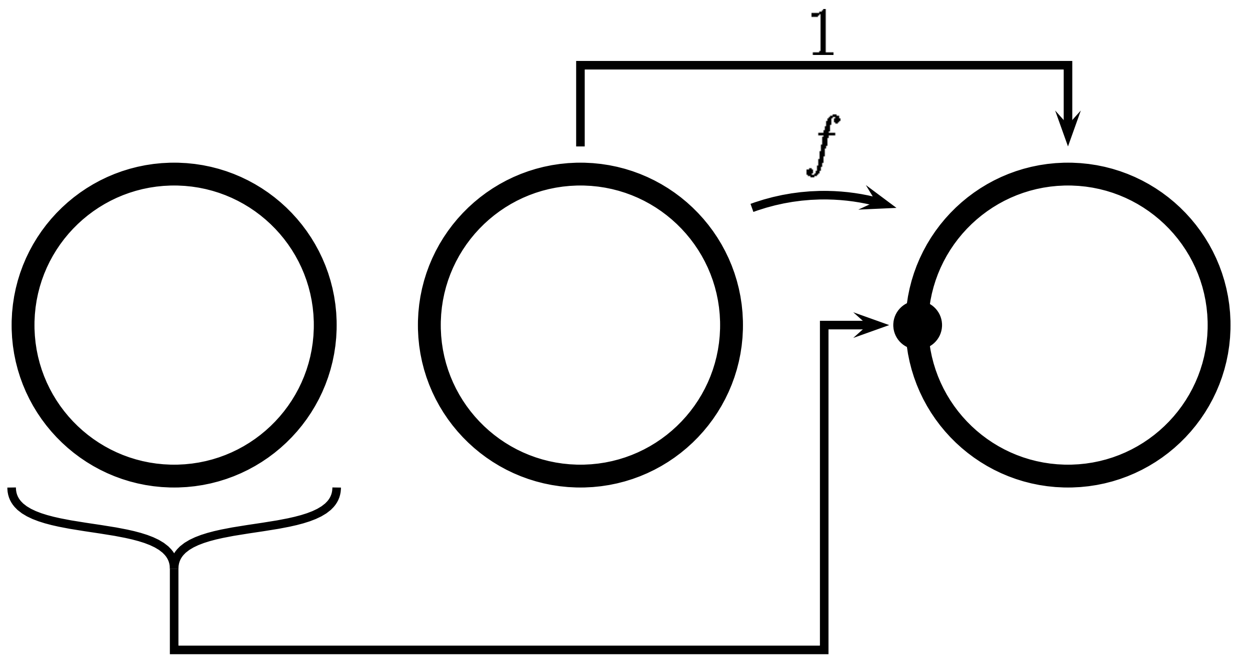


$W = M \times [0, 1]$

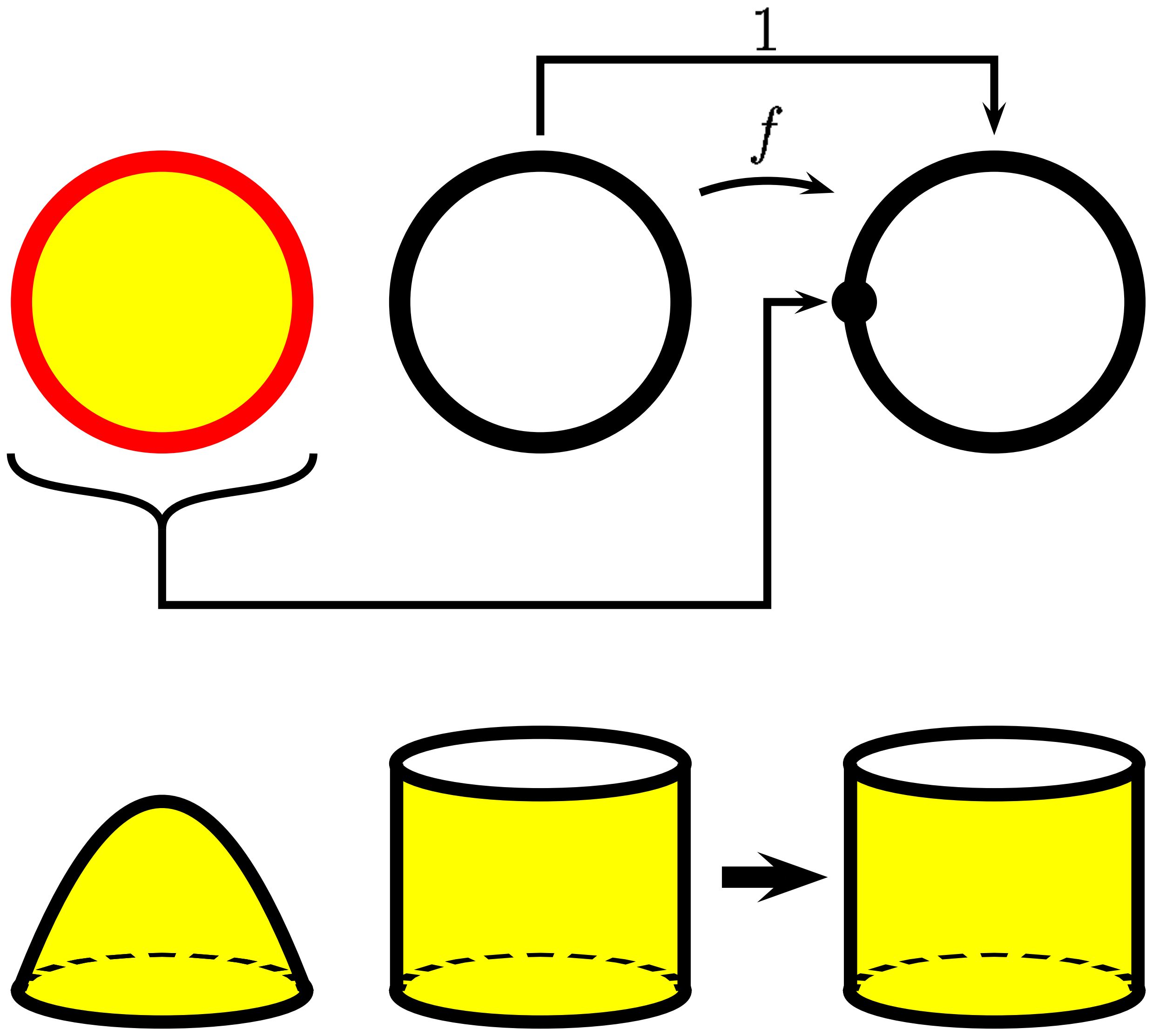


Can change a given "map" $f : M^n \longrightarrow X^n$ into a htpy equiv. using surgery?

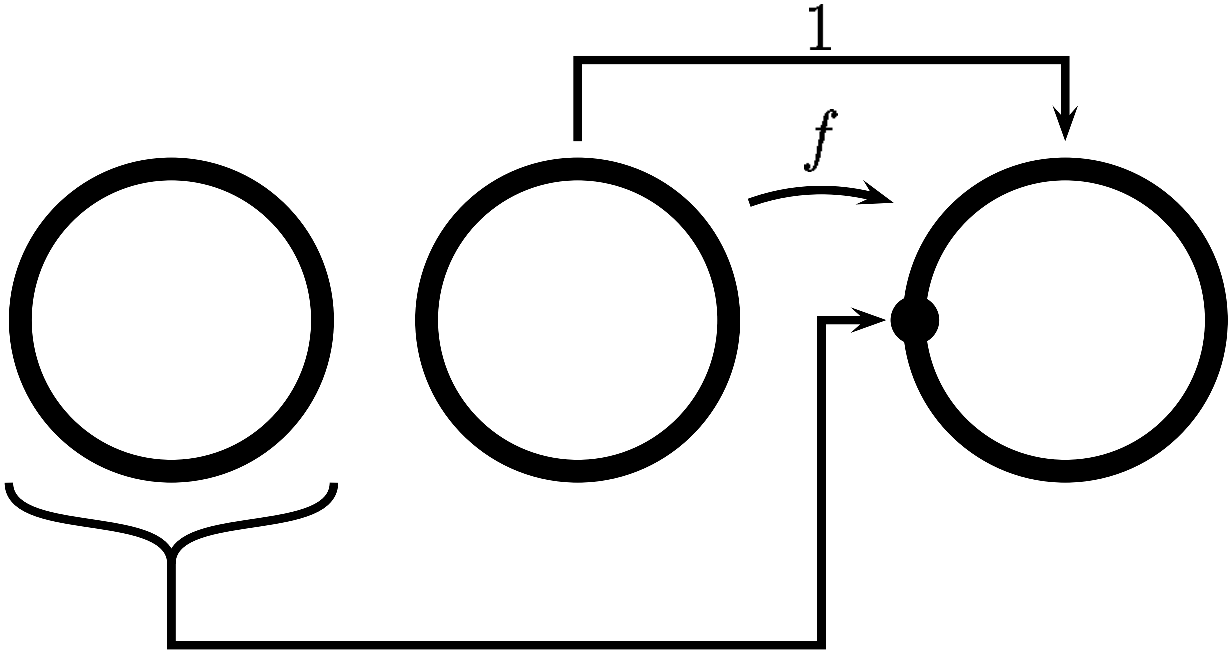
Given Map :



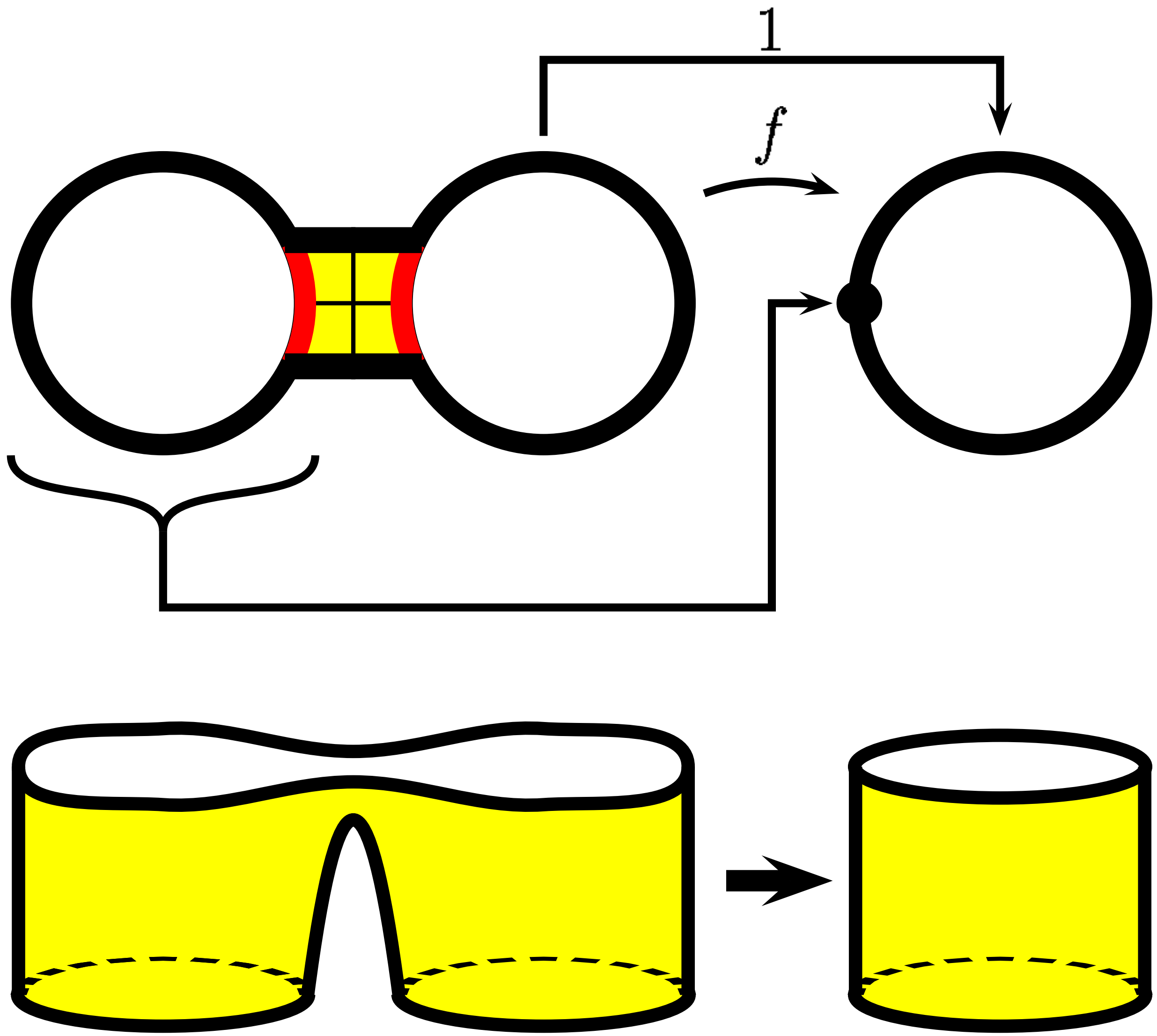
Method 1 : attach a 2-handle



Given Map :



Method 2 : attach a 1-handle



Surgery Obstruction

a degree 1 normal map $f : M^n \to X^n$ $\Longrightarrow \sigma(f) \in L_n(\mathbb{Z}\pi_1 X)$

A **normal map** is a map $f : M \to X$ together with a stable TOP bundle map $b : \nu_M \to \eta$ to some TOP bundle η over X covering f.

Notation for later use

$\mathrm{TOP}_n = \{h : \mathbb{R}^n \to \mathbb{R}^n \ ; \ \text{homeo.}, h(O) = O\}$

$B\mathrm{TOP}$: the classifying sp. of stable TOP bundles

$G_n = \{h : S^{n-1} \to S^{n-1} \ ; \ \text{htpy equiv.}\}$

BG : the classifying sp. of stable spherical fibrations

G/TOP : htpy fiber of $B\mathrm{TOP} \to BG$

Assume $n \geqq 5$.

$\sigma(f) = 0 \quad \Longrightarrow$
can change f into a htpy equiv. by surgery

Similarly for manifolds w/ boundary.

$$\left.\begin{array}{c} f : (M^n, \partial M) \to (X^n, \partial X) \\ \partial f : \partial M \xrightarrow{\simeq} \partial X \end{array}\right\} \Rightarrow \sigma(f) \in L_{n+1}(\mathbb{Z}\pi)$$

$\sigma(f) = 0 \quad \Longrightarrow \quad$ surgery *rel* ∂ is possible

Summary : **Surgery Exact Sequence**

$$\to \mathcal{S}(X \times I, \partial) \to \mathcal{N}(X \times I, \partial) \to L_{n+1}(\mathbb{Z}\pi)$$

$$\longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \longrightarrow L_n(\mathbb{Z}\pi)$$

$\mathcal{S}(X) = \{f : M^n \xrightarrow{\simeq} X\}/\sim$

$\mathcal{N}(X) = \{f : M^n \to X \text{ degree 1 normal map}\}/\sim$

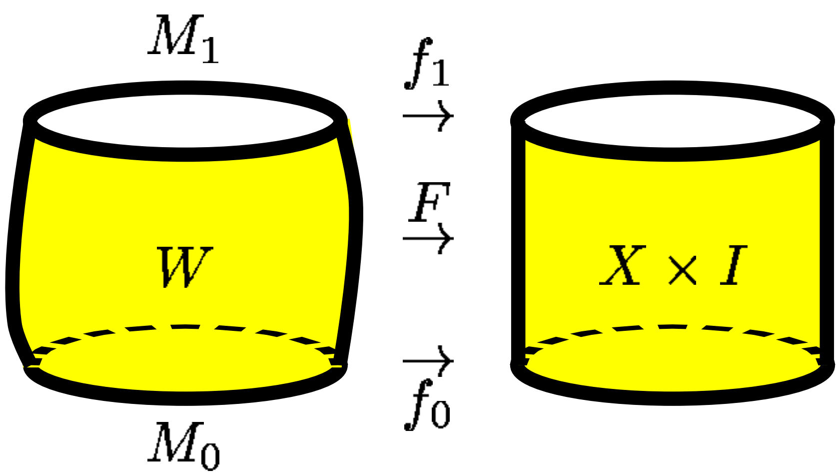
Require that f be a homeo. on the boundary.

Equiv. Rel. $\sim$ in the def. of $\mathcal{S}(X)$:

$$f_0 : M_0 \xrightarrow{\simeq} X \quad \sim \quad f_1 : M_1 \xrightarrow{\simeq} X \qquad \Longleftrightarrow$$

$$\exists W; \partial W = M_0 \cup M_1, \quad \exists F : W \xrightarrow{\simeq} X \times [0,1]$$

$$F|M_0 = f_0, \quad F|M_1 = f_1$$



$$f_0 : M_0 \xrightarrow{\simeq} X \quad \sim \quad f_1 : M_1 \xrightarrow{\simeq} X$$

$\Longrightarrow \quad (W; M_0)$ is an h-cobordism

If $Wh(\pi_1(X)) = 0$ and $n \geq 5$,

$\Longrightarrow \quad W \cong M_0 \times I$ (s-cobordism theorem)

$\Longrightarrow \quad M_0 \cong M_1$

$Wh(G)$: the Whitehead group of G

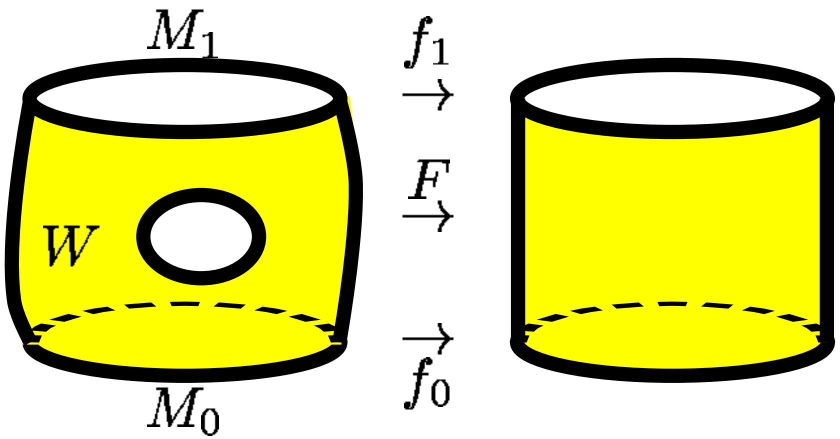
$Wh(\{1\}) = 0$, $Wh(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) = 0$

Equiv. Rel. $\sim$ in the def. of $\mathcal{N}(X)$:

$$f_0 : M_0 \to X \quad \sim \quad f_1 : M_1 \to X \qquad \Longleftrightarrow$$

$$\exists W ; \partial W = M_0 \cup M_1, \quad \exists F : W \to X \times [0,1]$$

$$F|M_0 = f_0, \quad F|M_1 = f_1$$



$\Longleftrightarrow$ f_0 can be changed into f_1 by surgery $(n \neq 4)$

Typical Examples ($n \geqq 5$, $n = 4$: Freedman)

(1) $X = S^n \Longrightarrow \mathcal{S}(X^n) = \{0\}$ (Smale, ...)

(2) $X = T^n \Longrightarrow \mathcal{S}(X^n) = \{0\}$ (Hsiang-Wall)

Any htpy equiv. $M^n \to X^n$ is htpic to a homeo.

Generalization?

(1)$\Longrightarrow$ simply-connected manifolds ... No.

(2)$\Longrightarrow$ aspherical manifolds ... Borel Conjecture.

Borel Conjecture

X^n:an aspherical manifold $[\pi_i(X) = 0\ (i > 1)]$

$\Longrightarrow$

Any htpy equiv. $M^n \xrightarrow{\simeq} X$ is htpic to a homeo.

Suffices to show $Wh(\pi_1 X) = 0$, $\mathcal{S}(X) = \{0\}$.

Conj. G : torsion-free $\Longrightarrow Wh(G) = 0$

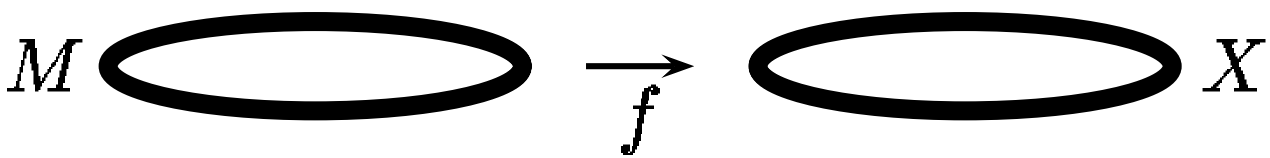
Rem. Asperical manifolds have torsion-free π_1.

Criterion for $\mathcal{S}(X) = 0$:

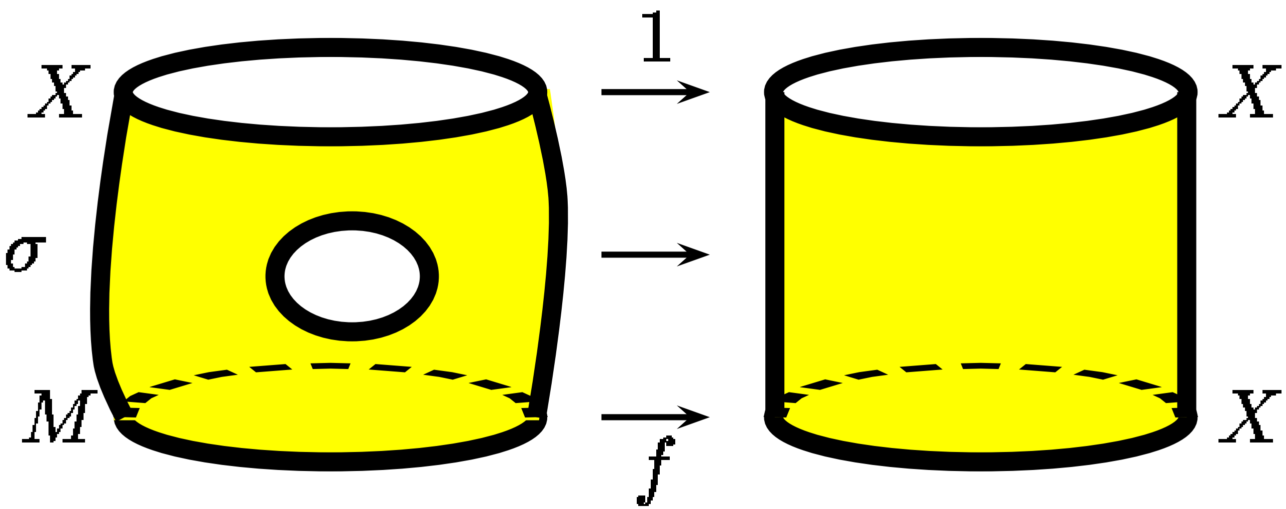
$$\mathcal{N}(X \times I, \partial) \xrightarrow{\text{surjective}} L_{n+1}(\mathbb{Z}\pi) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \xrightarrow{\text{injective}} L_n(\mathbb{Z}\pi)$$

Will sketch the mechanism . . .

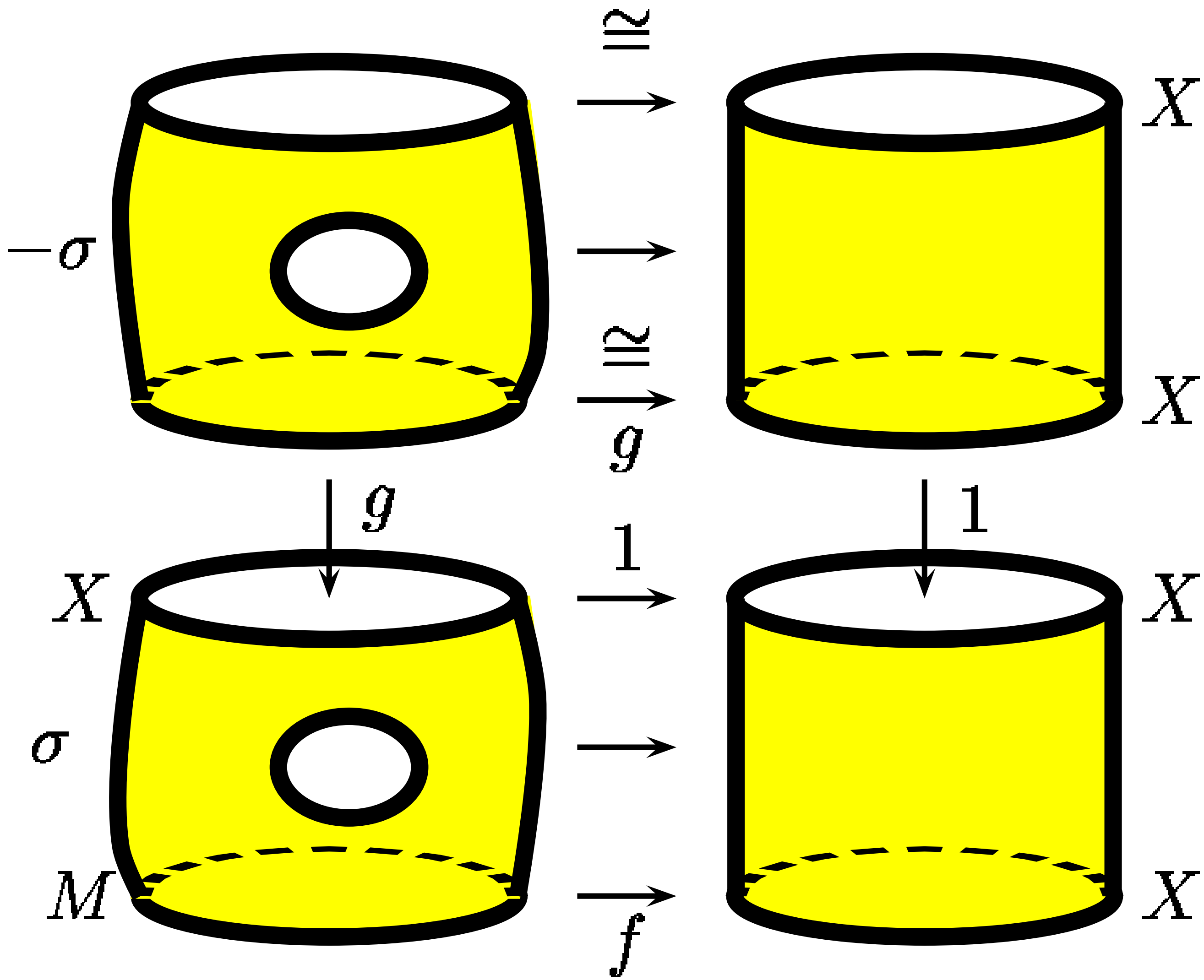
Given an $f \in \mathcal{S}(X)$,



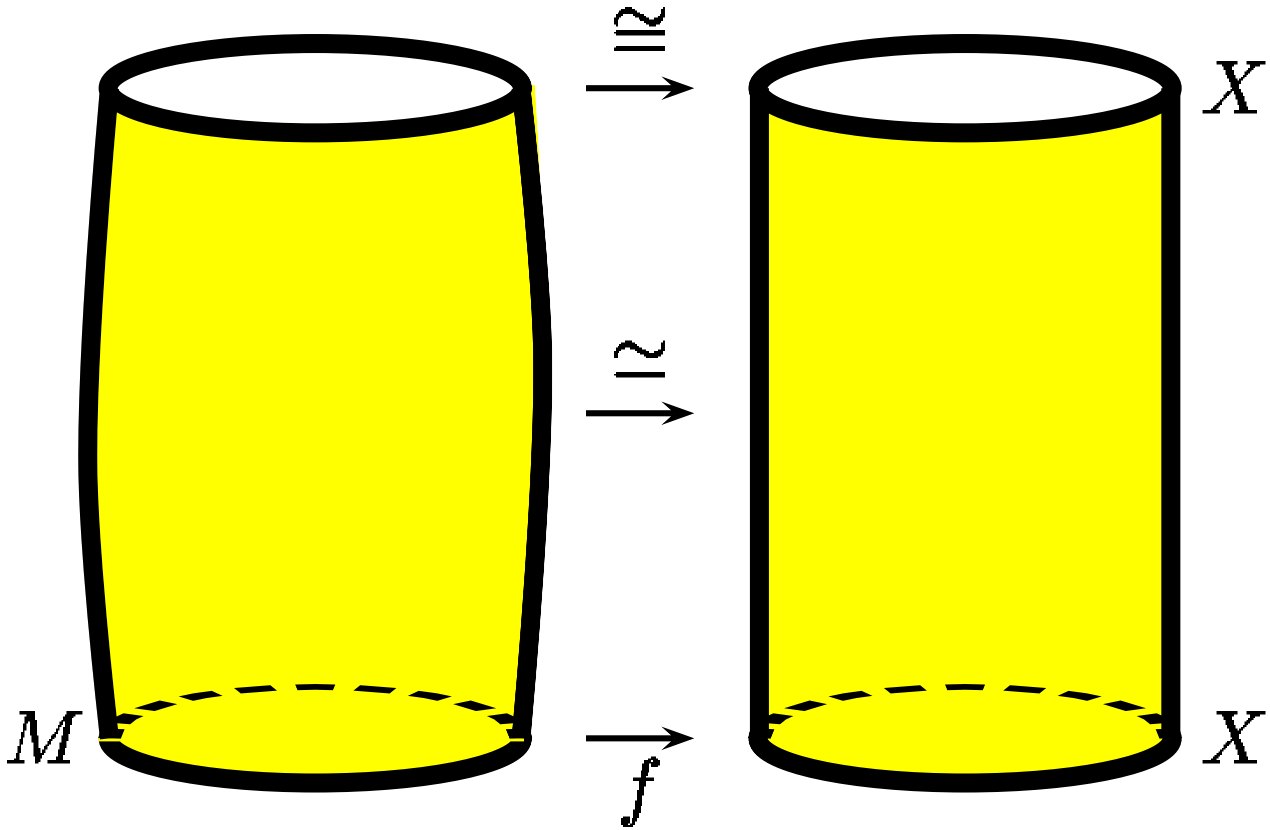
$$\mathcal{N}(X) \to L_n(\mathbb{Z}\pi) : \text{ injective} \Rightarrow [f] = 0 \in \mathcal{N}(X)$$



$$\mathcal{N}(X \times I, \partial) \longrightarrow\!\!\!\!\!\rightarrow L_{n+1}(\mathbb{Z}\pi) \;\Rightarrow \text{can realize } -\sigma$$



Now perform surgery to get a htpy equiv.



Ranicki's Quadratic Complexes (QC)

$A = \mathbb{Z}\pi$ or a ring with involution $(x \mapsto \overline{x})$

$L_j(A) = \{j$-dim QC over $A\}/$cobordism

$\mathbb{L}(A)_0 =$ the space of 0-dim QC's over A

$$\implies \pi_j(\mathbb{L}(A)_0) = L_j(A)$$

a 0-dim QC $=$ a quadratic form

$=$ (a free A-module $K, \psi_0 : K^* \to K$)

s.t. $\psi_0 + \psi_0^* : K^* \to K$ an iso.

Delooping of $\mathbb{L}(A)_0$

a j-dim QC

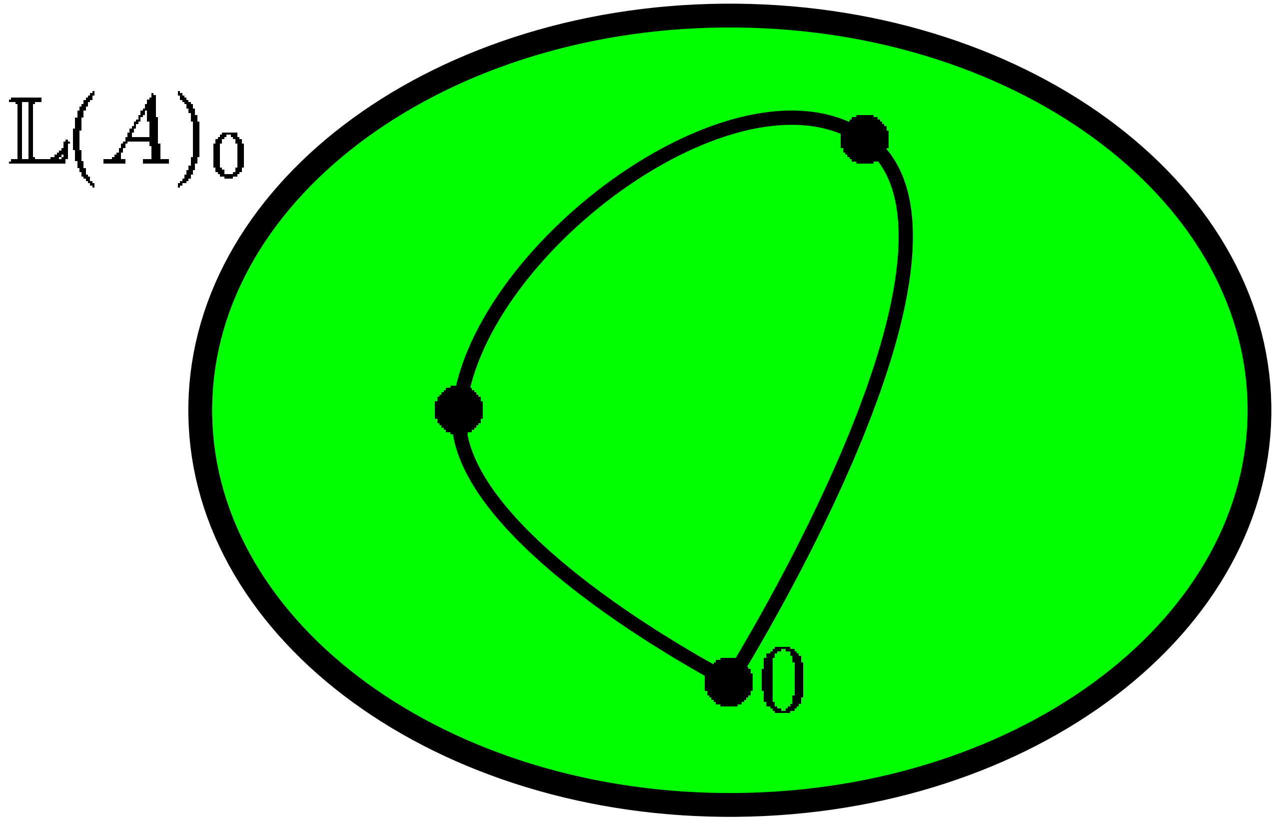
$= (\text{a } j\text{-dim cx } C, \{\psi_s : C^{j-s-*} \to C_*; s \geq 0\})$

s.t. $\psi_0 \pm \psi_0^* : C^{j-*} \to C_*$ a chain eq.

$\mathbb{L}(A)_k =$ the space of $(-k)$-dim QC's over A (!?)

$\Longrightarrow \Omega\mathbb{L}(A)_{k+1} \simeq \mathbb{L}(A)_k$

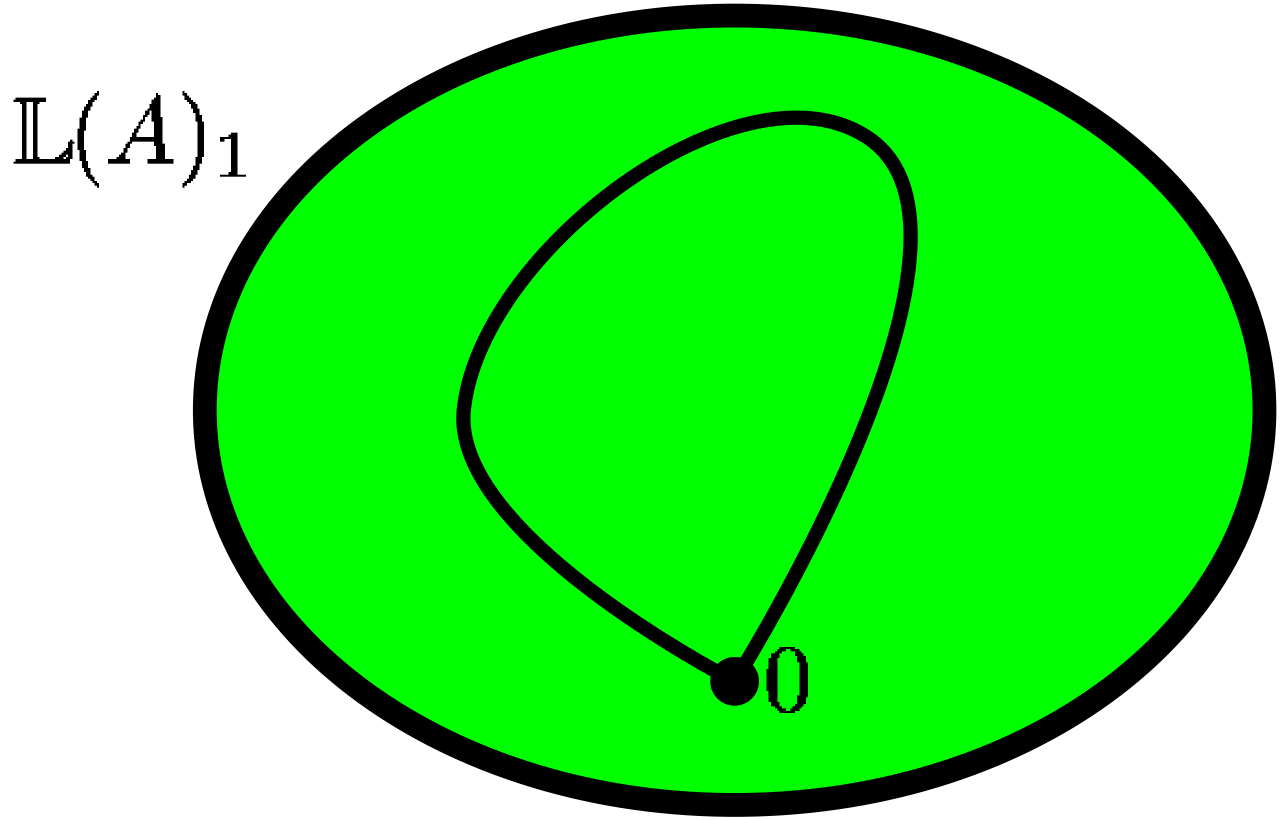
$\Longrightarrow \mathbb{L}(A) = \{\mathbb{L}(A)_k\}$ is an Ω-spectrum



0-simplex = 0-dim QC

1-simplex = 1-dim QC with boundary

...



the only 0-simplex $= 0$ (basepoint)

1-simplex $=$ 0-dim QC

2-simplex $=$ 1-dim QC with ∂, $\cdots$

Four-periodicity

- $L_i(\mathbb{Z}\pi) \cong L_{i+4}(\mathbb{Z}\pi) \quad (i \geqq 0)$

- $\mathcal{N}(X \times I^{i+4}, \partial) \cong \begin{cases} \mathcal{N}(X \times I^i, \partial) & (i \geqq 1) \\ \mathcal{N}(X) \oplus \mathbb{Z} & (i = 0) \end{cases}$

- $\mathcal{S}(X \times I^{i+4}, \partial) \cong \begin{cases} \mathcal{S}(X \times I^i, \partial) & (i \geqq 1) \\ \mathcal{S}(X) \text{ or } \mathcal{S}(X) \oplus \mathbb{Z} & (i = 0) \end{cases}$

Reason:

$$\begin{cases} \mathcal{N}(X \times I^i, \partial) \cong [X \times I^i, \partial\ ;\ G/\mathsf{TOP}, *] \\ \mathsf{GPC} \Longrightarrow \mathcal{S}(D^j, \partial) = 0 \quad (j \geqq 4) \end{cases}$$

$$\Longrightarrow \pi_{i+j}(G/\mathsf{TOP}) \cong L_{i+j}(\mathbb{Z})$$

In fact, the following is true:

$$\begin{cases} \mathbb{L} := \mathbb{L}(\mathbb{Z}) \simeq G/\mathsf{TOP} \times \mathbb{Z} \quad (L_0(\mathbb{Z}) = \mathbb{Z}) \\ G/\mathsf{TOP} \text{ is connected.} \end{cases}$$

For $i > 0$,

$$\begin{aligned} \mathcal{N}(X \times I^i, \partial) &= [X \times I^i, \partial\ ;\ \mathbb{L}, *] \\ &= H^0(X \times I^i\ ; \mathbb{L}) \\ &\cong H_{n+i}(X\ ;\ \mathbb{L}) \end{aligned}$$

4-Periodicity $\Longrightarrow$ $\boxed{\mathcal{S}(X \times I^4, \partial) = 0 \Longrightarrow \mathcal{S}(X) = 0}$

Study $\mathcal{N}(X \times I^i, \partial) \to L_{n+i}(\mathbb{Z}\pi)$, or

$\boxed{A : H_*(X; \mathbb{L}) \to L_*(\mathbb{Z}\pi)}$ (the assembly map).

will later interpret this map using <u>geometric control</u>

Novikov Conjecture

Γ : a discrete group

$B\Gamma$: the classifying space, *i.e.* $K(\Gamma, 1)$

N. C. for Γ

$\Longleftrightarrow$

$H_*(B\Gamma; \mathbb{L}) \xrightarrow{A} L_*(\mathbb{Z}\Gamma)$ is a rational split injection

Controlled L-Groups

A QC over $\mathbb{Z}\pi_1(X)$ is made up of points in X (basis elements of the modules) and paths connecting them (homomorphisms).

Can introduce 'fineness' using lengths of the paths.

For $0 < \epsilon \leqq \delta$,

$$\boxed{L_j^{\epsilon,\delta}(X;\mathbb{Z}) := \{j\text{-dim } \epsilon \text{ QC on } X\}/\delta \text{ cobordism}}$$

Stability (Squeezing)

For $j \geq 0$ and a compact ANR X, there exist $\delta_0 > 0$ and $T \geq 1$ s.t. for all $\epsilon > 0$, $\delta > 0$ satisfying $T\epsilon \leq \delta \leq \delta_0$, the groups $L_j^{\epsilon,\delta}(X;\mathbb{Z})$ are all isomorphic.

Denote the common group by $L_j^c(X;\mathbb{Z})$, then

$$L_j^c(X;\mathbb{Z}) \cong H_j(X;\mathbb{L}) \ .$$

Generalization

$p : X \to B$: a 'control' map to a metric space B

A : a ring with involution

$0 < \epsilon \leqq \delta$

$$L_j^{\epsilon,\delta}(B; A, p) \ := \ \{\epsilon \text{ QC on } X\}/\delta \text{ cobordism}$$

'Fineness' is measured on B via p.

Assembly map via geometric control

$$L^c_*(X;\mathbb{Z}) = L^c_*(X;\mathbb{Z}, 1 : X \to X)$$
$$\longrightarrow L^c_*(\{*\};\mathbb{Z}, X \to \{*\}) = L_*(\mathbb{Z}\pi_1 X)$$

("Forget-Control Map")

Use **GEOMETRY** to understand this map!

Farrell and Jones' Works

Theorem (Farrell-Jones) Borel Conjecture is true for the following X^n $(n \neq 3, 4)$:

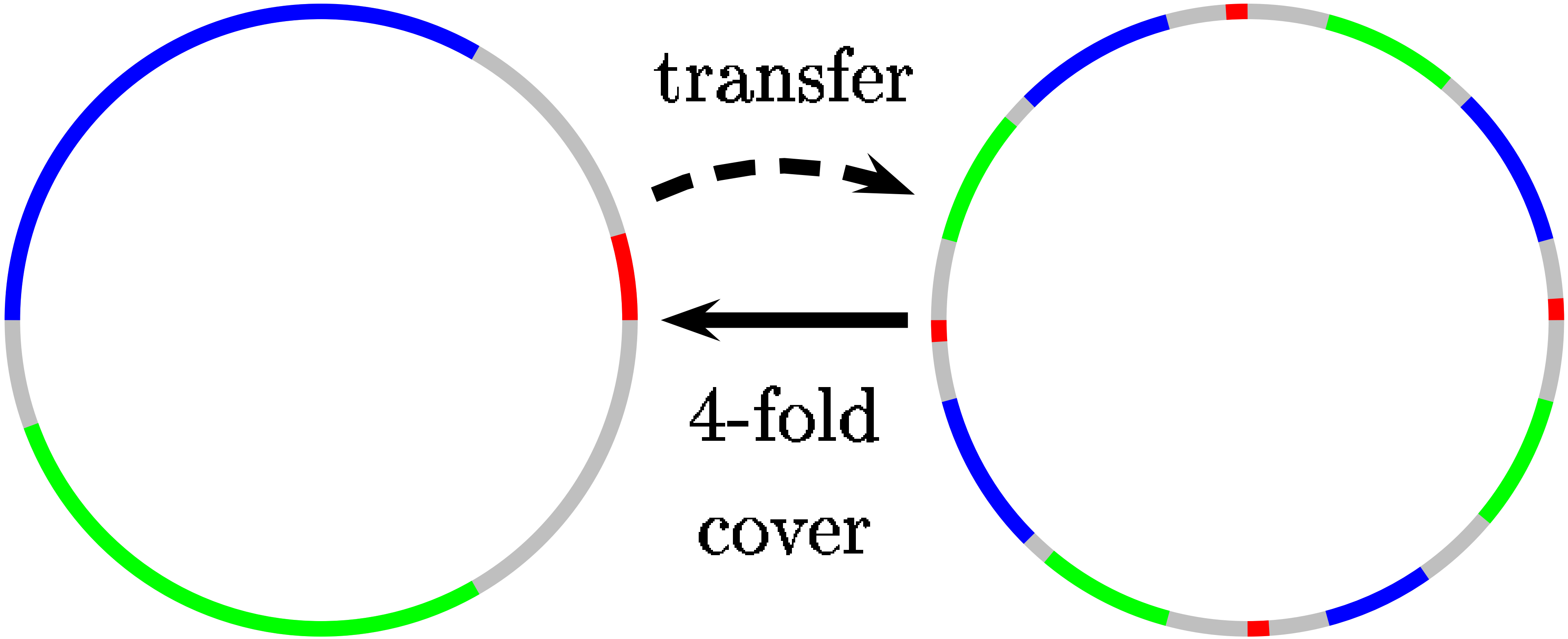
(1) X^n is a non-positively curved.

(2) $\pi_1(X^n) \subset GL_m(\mathbb{R})$ (a discrete subgp)

I.e., any htpy equiv. $M^n \to X$ is htpic to a homeo.

See Farrell's ICTP lecture.

One Ingredient : Transfer



Squeezing in a General Setting

Theorem (Pedersen-Yamasaki)

For $j \geq 0$ and a finite polyhedron B, there exist $\delta_0 > 0$ and $T \geq 1$ s.t. if A is a ring with involution and $p : E \to B$ is a stratified system of fibrations, then for all $\epsilon > 0$, $\delta > 0$ satisfying $T\epsilon \leq \delta \leq \delta_0$, the groups $L_j^{\epsilon,\delta}(B; A, p)$ are all isomorphic.

Key Trick： **Eilenberg Swindle**

$1 = 0$!?

$$\begin{aligned} 1 &= 1 + ((-1) + 1) + ((-1) + 1) + \cdots \\ &= (1 + (-1)) + (1 + (-1)) + \cdots \\ &= 0 \end{aligned}$$

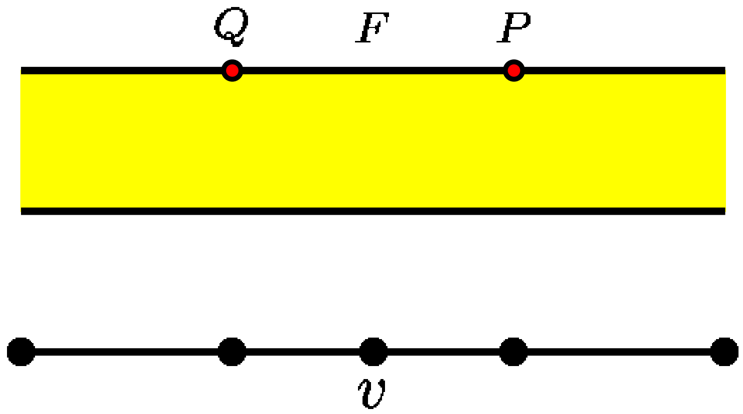
Split off a piece near v:

F : free, P, Q : projective

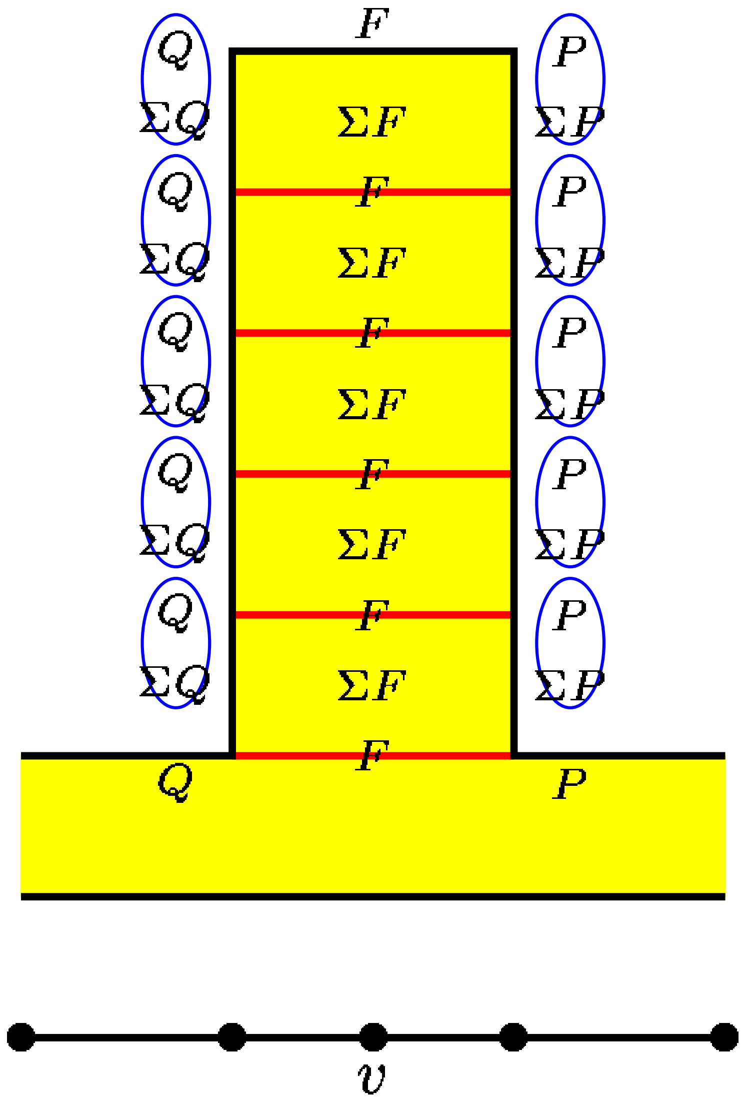
Poincaré Duality $\Longrightarrow$

$$[P] + [Q] = 0 \in \widetilde{K}_0 \text{ over } \mathrm{Star}(v)$$

$[P] = \sum(-1)^j [P_j], \quad [\Sigma P] = -[P]$



Make a Tower :



F
Q
P
ΣQ
ΣF
ΣP
Q
F
P
ΣQ
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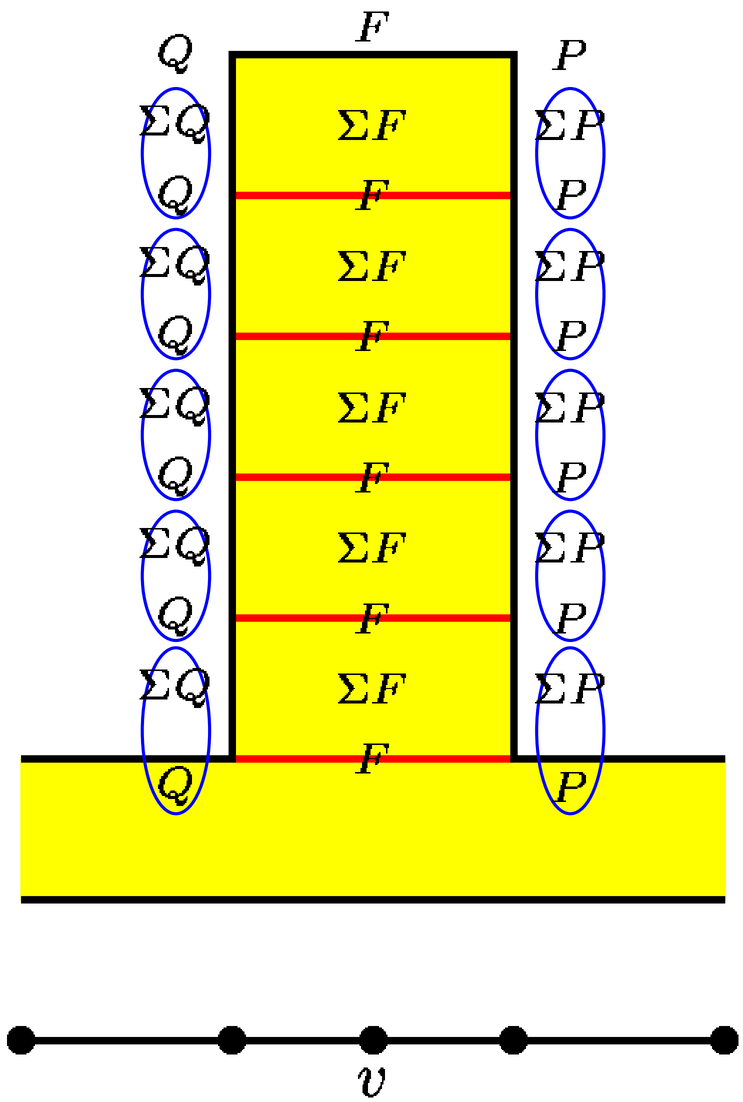
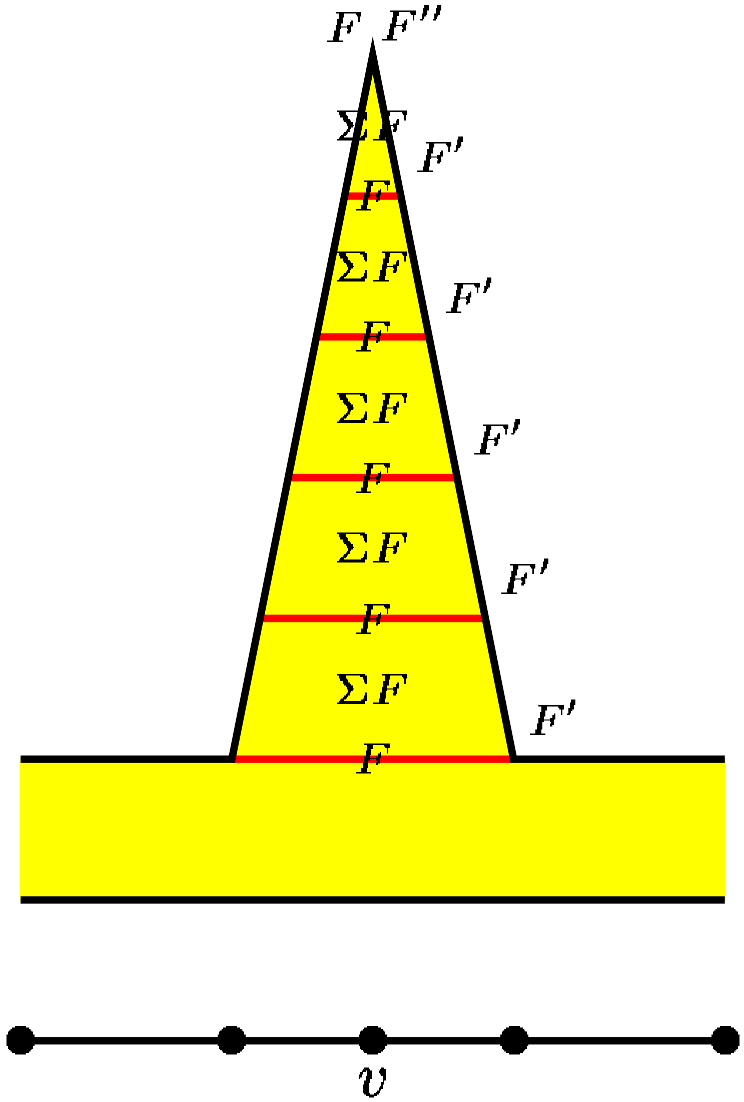
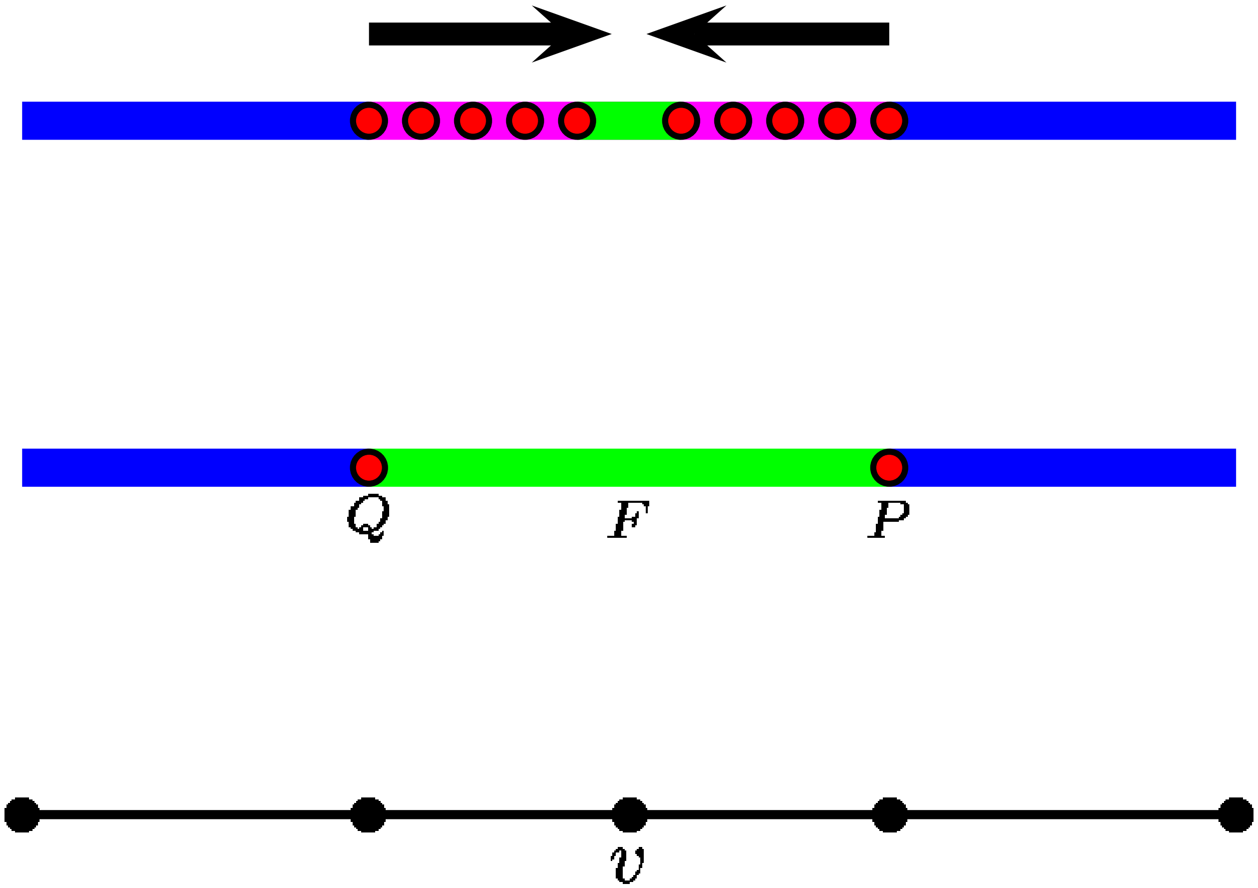
Finite Stage Eilenberg Swindle:



Squeeze toward v:



Squeeze toward v:



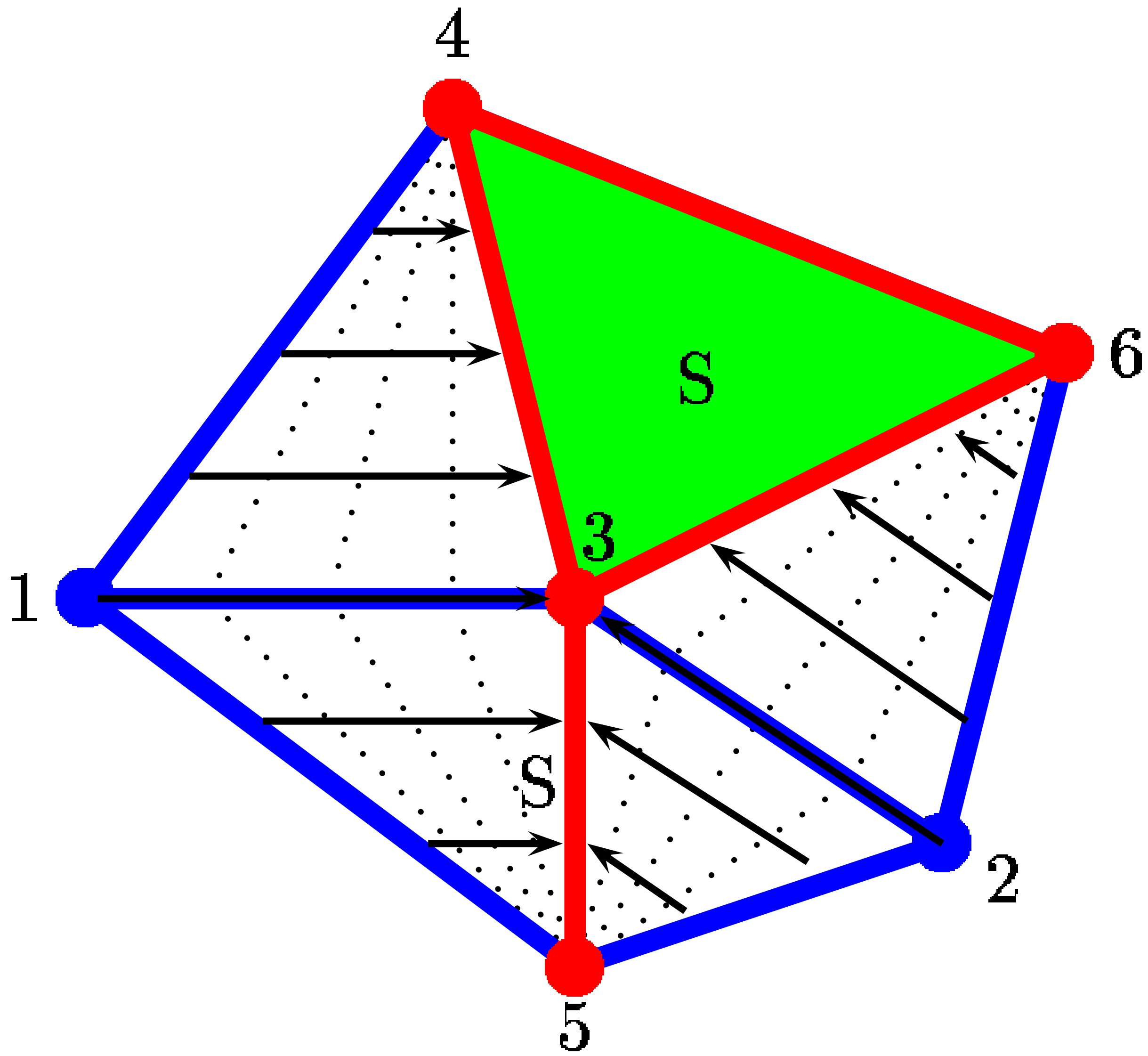
General Case:

Choose an order of the vertices of B, so that larger vertices are in the lower strata.

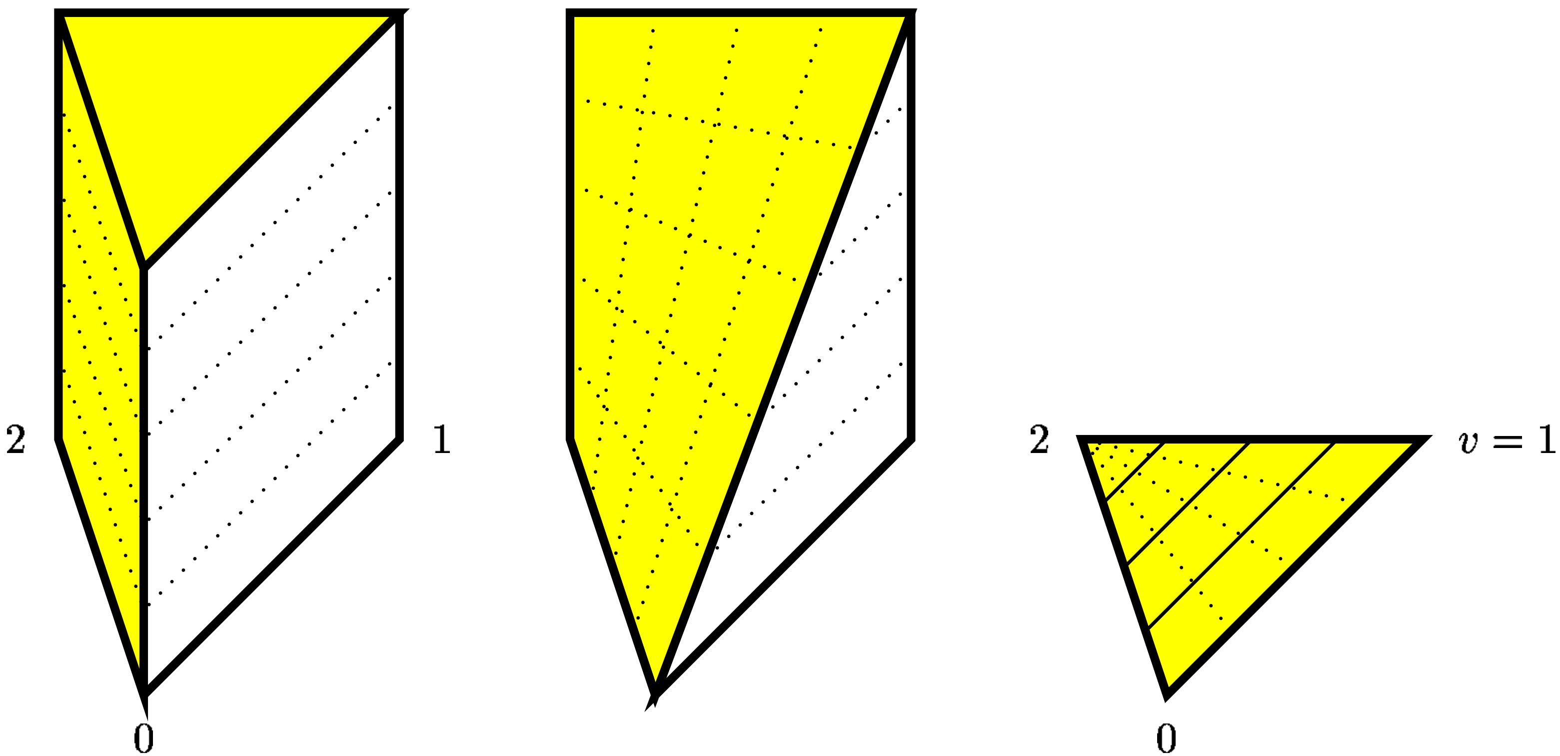
Starting from the smallest vertex, we ‘squeeze’ toward the vertices:

Take a vertex v, and consider the ‘stable set’ S at v which is spanned by the vertices $\geqq v$.

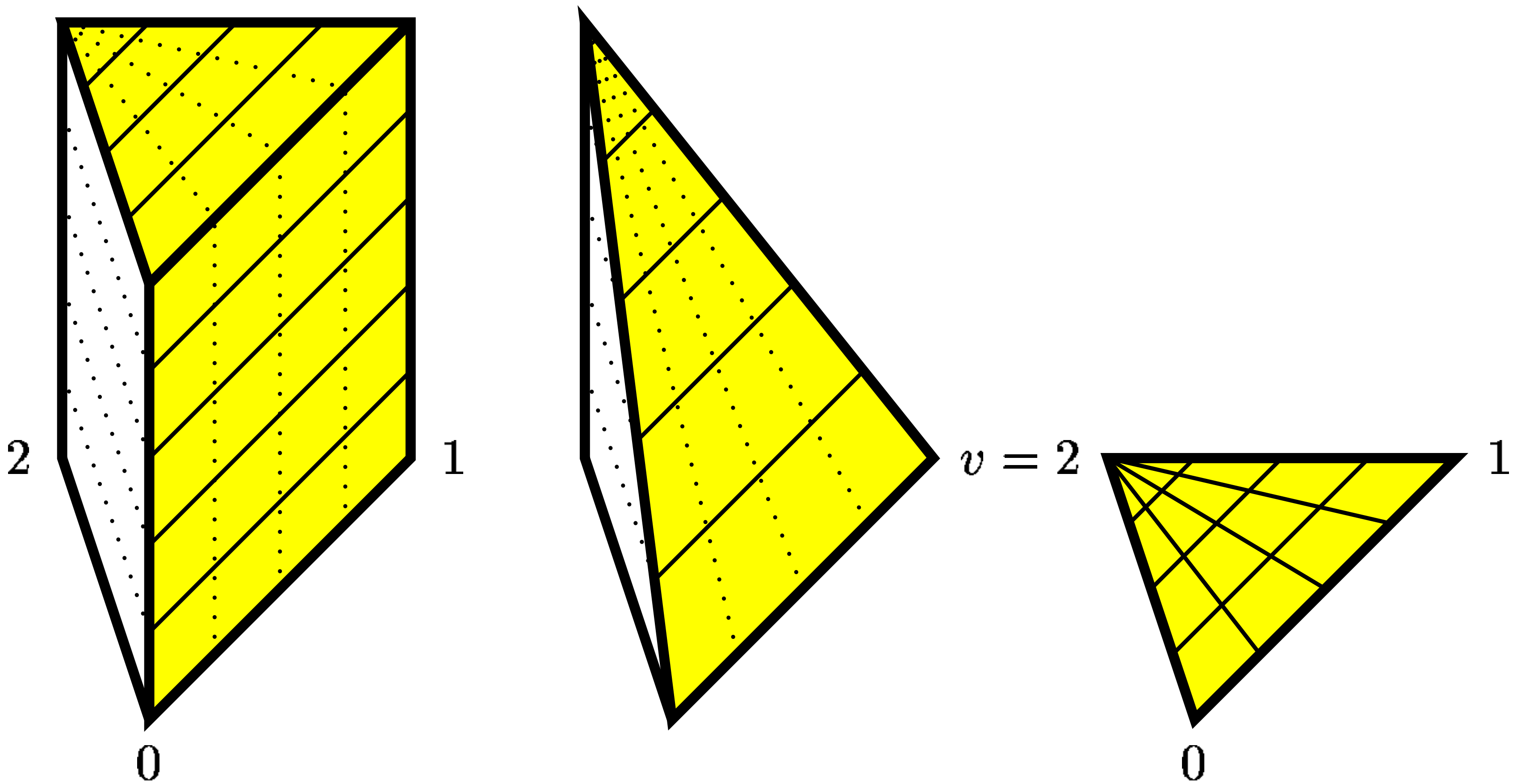
The star nbhd of v deformation retracts into S:



Construct a tower over the star nbhd, and apply the squeezing corresponding to the defirmation retraction to S.



$v = 2$



Rem. In general $L^c(B; A, p)$ is not homology. For this we encounter controlled K-theoretic obstruction.

Topics Related to Squeezing

1. Controlled Surgery Exact Sequence

$p : X^n \to B : UV^1$, $\epsilon < \delta$: sufficiently small

$$H_{n+1}(B;\mathbb{L}) \to \mathcal{S}^{\epsilon,\delta}(X;p) \to \mathcal{N}(X) \to H_n(B;\mathbb{L})$$

2. Homology Manifold Surgery Exact Sequence

$$L_{n+1}(\mathbb{Z}\pi) \to \mathcal{S}^H(X) \to H_n(X;\mathbb{L}) \to L_n(\mathbb{Z}\pi)$$

4-periodic : $\mathcal{S}^H(X) \cong \mathcal{S}^H(X \times I^n, \partial)$