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Controlled K-Theory

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Abstract

The controlled finiteness obstruction and torsion are defined using controlled algebra, giving a more algebraic proof of the topological invariance of torsion and the homotopy finiteness of compact *ANR*s.

Introduction

The Wall finiteness obstruction and Whitehead torsion are the traditional applications of algebraic K-theory to topology, relating geometric finiteness properties of spaces to the algebraic properties of modules over the fundamental group ring. The methods of controlled algebra developed by Connell and Hollingsworth [8], Chapman [6] and Quinn [15], [16] use a more refined version of algebraic K-theory in which the algebra is parametrized by a metric space. For any $\epsilon > 0$ there is a notion of ϵ-controlled K-theory, in which the size of any operation is restricted to be at most (some multiple of) ϵ in the metric space. In fact, only the controlled Whitehead group of automorphisms was defined directly, with the controlled reduced projective class group obtained from it by a version of the splitting theorem of Bass, Heller and Swan [2] which embeds the reduced projective class group $\widetilde{K}_0(\mathbb{Z}[\pi])$ of a group ring $\mathbb{Z}[\pi]$ as a direct summand in the Whitehead group $Wh(\pi \times \mathbb{Z})$.

In this paper we develop the controlled algebra of projections, define the $\widetilde{K}_0$-groups directly, and relate the controlled $\widetilde{K}_0$ and Wh-groups to each other by various exactness properties. The algebraic methods are used to give a self-contained treatment of the following results:

1. A homeomorphism of finite CW complexes is simple. This is the topological invariance of Whitehead torsion, originally proved by Chapman [3].
2. Every compact ANR has the homotopy type of a finite CW complex. This is the Borsuk conjecture, originally proved by West [24].

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3. The results of Ferry [10] and Chapman [4], [5] generalizing 1. and 2., by which an ϵ-domination (resp. ϵ-homotopy equivalence) for sufficiently small ϵ implies the vanishing of the ordinary Wall finiteness obstruction (resp. Whitehead torsion).

A PL homeomorphism is a simple homotopy equivalence; the combinatorial invariance of Whitehead torsion is proved by Milnor [15] using an induction argument, in which the key ingredient is the computation $Wh(\{1\}) = 0$ of Higman [13]. Similarly, the key ingredient in any proof of 1. is the computation $Wh(\mathbb{Z}^n) = 0$ of Bass, Heller and Swan [2]. The failure of the Hauptvermutung shows that a homeomorphism of finite CW complexes is not in general homotopic to a PL homeomorphism. However, a homeomorphism has zero controlled torsion and hence also zero Whitehead torsion. The proofs of 2. and 3. are closely related to the proof of 1: the connection between the three results has long been recognized.

In our proofs we have attempted to minimize the geometry and maximize the algebra. It should be noted that the "squeezing" arguments of Ferry and Pedersen provide an alternative algebraic method of proof of 1.,2.,3., using the bounded algebraic K-theory of Pedersen and Weibel. In this approach the controlled finiteness obstruction of a controlled finite domination over a finite-dimensional space X is identified with the bounded torsion of a bounded homotopy equivalence over the open cone $O(X)$, and a controlled torsion is identified with a bounded K_2-invariant. See Ferry, Hambleton and Pedersen [11,§7] for a brief account.

In subsequent work we shall extend the methods to controlled L-theory, giving a similarly direct proof of the topological invariance of the rational Pontrjagin classes, originally obtained by Novikov.

The plan of the paper is as follows. In §1 we first review the category of geometric modules and geometric morphisms due to Quinn, and then discuss "projective modules" and "projective module chain complexes" in this category. In §2 we introduce "control" into these. In §3 we define the controlled projective class groups $\widetilde{K}_0(X, p_X, n, \epsilon)$ using n-dimensional ϵ-controlled projective complexes on a control map $p_X : M \longrightarrow X$. In §4 we define similarly the controlled torsion groups $Wh(X, p_X, n, \epsilon)$ using n-dimensional ϵ-controlled contractible free complexes on a control map $p_X : M \longrightarrow X$. In §5 the $\widetilde{K}_0$ and Wh-groups are related by the "stably exact sequence" of a pair $(X, Y \subseteq X)$. The excision and Mayer-Vietoris properties of the controlled K-groups are developed more generally in §6. The controlled analogue of the decomposition

$$Wh(\pi \times \mathbb{Z}) \;=\; Wh(\pi) \oplus \widetilde{K}_0(\mathbb{Z}[\pi]) \oplus 2\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])$$

of Bass [1] is obtained in §7. This is used in §8 to establish a Vietoris-type property of controlled K-theory invariants: if sufficiently controlled they vanish in a less controlled K-group. Controlled finiteness and torsion invariants are defined in §9. The topological invariance and finiteness results 1., 2. and 3. are proved in §10. Finally, an appendix gives a brief account of controlled lower K-theory.

The referee asked if there is a categorical approach to our "stable isomorphisms" and "stably exact" sequences, and even went so far as to suggest an appropriate category. An object in this category should be a system of abelian groups $\{A_\epsilon | \epsilon \in (0,\infty)\}$, with A_ϵ mapping to A_δ if $\epsilon < \delta$, and a morphism $f : \{A_\epsilon\} \to \{B_\epsilon\}$ should be a collection of homomorphisms $f_\epsilon : A_\epsilon \to B_{k\epsilon}$ making some obvious diagrams commute. Here k is a constant independent of ϵ, but depending on f. A category for controlled algebra is certainly desirable. (For bounded algebra there are the categories of Pedersen and Weibel, as well as Anderson and Munkholm). Regrettably, we have not been able to provide such a categorical treatment in this paper.

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1. Geometric modules.

Basically our treatment of geometric modules and geometric morphisms follows the original work by Quinn [18]. Some notation and terminology is from the work of Connolly and Koźniewski [9,§4] and some is new.

Let M be a topological space, and let $|S|$ be a set together with a function

$$S : |S| \longrightarrow M.$$

In the following we identify a function with its graph; thus, S also represents a subset of $|S| \times M$. The first (resp. second) component of an element $s \in S \subset |S| \times M$ will be denoted $|s| \in |S|$ (resp. $[s] \in M$). The function S maps $|s|$ to $[s]$. The projection $|S| \times M \to M$ induces a bijection from the graph S to $|S|$.

Definition. The free $\mathbb{Z}$-module on the graph S is called the *geometric module on M generated by S*, and is denoted $\mathbb{Z}[S]$. A geometric module $\mathbb{Z}[S]$ is said to be *finitely generated* (*f.g.*) if $|S|$ (and hence S) is a finite set. The *direct sum* $\bigoplus_{\alpha \in A} \mathbb{Z}[S_\alpha]$ of a family $\{\mathbb{Z}[S_\alpha]\}_{\alpha \in A}$ of geometric modules on M indexed by a set A is defined as follows: First make a "copy" S'_α of each S_α by:

$$S'_\alpha : |S'_\alpha| = |S_\alpha| \times \{\alpha\} \approx |S_\alpha| \xrightarrow{S_\alpha} M.$$

The $|S'_\alpha|$'s are disjoint subsets of $\left(\bigcup_{\alpha \in A} |S_\alpha|\right) \times A$; the disjoint union of $|S'_\alpha|$'s will be denoted $\bigsqcup_{\alpha \in A} |S'_\alpha|$. The S'_α's define a unique function $\bigsqcup_{\alpha \in A} S'_\alpha : \bigsqcup_{\alpha \in A} |S'_\alpha| \to M$. Now $\bigoplus_{\alpha \in A} \mathbb{Z}[S_\alpha]$ is defined to be $\mathbb{Z}[\bigsqcup_{\alpha \in A} S'_\alpha]$. In this paper, we shall pretend that the $|S_\alpha|$'s are mutually disjoint, writing $\bigoplus_{\alpha \in A} \mathbb{Z}[S_\alpha] = \mathbb{Z}[\bigsqcup_{\alpha \in A} S_\alpha]$ without mentioning taking copies S'_α of S_α from now on.

Examples. (1) When $|S|$ is empty, $\mathbb{Z}[S]$ is denoted 0.
(2) Let M be a CW complex and fix an integer $n \geq 0$. Let $|S|$ be the set of the n-cells of M. For each n-cell $e \in |S|$, let $\varphi_e : D^n \to M$ be the characteristic map for e. The correspondence $S : |S| \to M; e \mapsto \varphi_e(O)$ defines a geometric module $\mathbb{Z}[S]$ on M, where O denotes the center of the n-disk D^n. As an abstract abelian group, it is the group of $\mathbb{Z}$-coefficient cellular n-chains of M.

Definitions. Let $\mathbb{Z}[S]$ and $\mathbb{Z}[T]$ be geometric modules on M. Consider triples (s, ρ, t) consisting of elements $s \in S$, $t \in T$ and a path $\rho : [0, \tau] \to M$ ($\tau \geq 0$) such that $\rho(0) = [s]$ and $\rho(\tau) = [t]$. Such a triple (s, ρ, t) will be called a *path from s to t*. A *geometric morphism* $f : \mathbb{Z}[S] \to \mathbb{Z}[T]$ is a formal linear combination

$$\sum_{\lambda \in \Lambda} m_\lambda (s_\lambda, \rho_\lambda : [0, \tau_\lambda] \to M, t_\lambda)$$

of paths from generators of $\mathbb{Z}[S]$ to generators of $\mathbb{Z}[T]$, with integer coefficients. Here Λ is some index set, and the number of paths starting from each generator is required to be finite. Two geometric morphisms $f = \sum_{\lambda \in \Lambda} m_\lambda (s_\lambda, \rho_\lambda, t_\lambda)$ and $f' = \sum_{\gamma \in \Gamma} m'_\gamma (s'_\gamma, \rho'_\gamma, t'_\gamma)$ from $\mathbb{Z}[S]$ to $\mathbb{Z}[T]$ are *equal* ($f = f'$) if there exists a bijection $\varphi : \Lambda \to \Gamma$ such that

$$m'_{\varphi(\lambda)} = m_\lambda \quad \text{and} \quad (s'_{\varphi(\lambda)}, \rho'_{\varphi(\lambda)}, t'_{\varphi(\lambda)}) = (s_\lambda, \rho_\lambda, t_\lambda) \quad (\text{for all } \lambda \in \Lambda),$$

after deleting terms with zero coefficients. The beginning and the end points of the paths in a geometric morphism $f = \sum_{\lambda \in \Lambda} m_\lambda (s_\lambda, \rho_\lambda, t_\lambda) : \mathbb{Z}[S] \longrightarrow \mathbb{Z}[T]$ determine a $\mathbb{Z}$-module homomorphism:

$$|f| : \mathbb{Z}[S] \longrightarrow \mathbb{Z}[T] \ ; \ s \longmapsto \sum_{s_\lambda = s} m_\lambda t_\lambda .$$

Examples. (1) A geometric morphism with no term is called the *zero geometric morphism*, and is denoted 0. $|0|$ is the zero homomorphism.
(2) Let $\mathbb{Z}[S]$ be a geometric module on M and define a "one-point" path $c_s : \{0\} \to M$ by $c_s(0) = [s]$, for $s \in S$. The geometric morphism

$$\sum_{s \in S} 1(s, c_s, s) : \mathbb{Z}[S] \longrightarrow \mathbb{Z}[S]$$

is called the *identity geometric morphism on* $\mathbb{Z}[S]$, and is denoted $1_{\mathbb{Z}[S]}$ or simply 1. For a geometric morphism $f : \mathbb{Z}[S] \longrightarrow \mathbb{Z}[T]$, the equalities $f 1_{\mathbb{Z}[S]} = f = 1_{\mathbb{Z}[T]} f$ hold. $|1_{\mathbb{Z}[S]}|$ is the ordinary identity homomorphism on $\mathbb{Z}[S]$.
(3) The geometric morphism $0(s, \rho, t)$ is equal to 0, for any path (s, ρ, t).
(4) The geometric morphisms $2(s, \rho, t) + 3(s, \rho, t)$ and $5(s, \rho, t)$ are not equal, because the numbers of terms with non-zero coefficients are different.

Definitions. The *sum* of two geometric morphisms is defined by formally combining the two linear combinations. The *integer multiplication* of a geometric morphism is defined by termwise integer multiplication. The *difference* $f-g$ of f and g is defined by $f+(-1)g$. The *composition* gf of two consecutive geometric morphisms

$$f=\sum_{\lambda\in\Lambda} m_\lambda(s_\lambda,\rho_\lambda,t_\lambda):\mathbb{Z}[S]\longrightarrow\mathbb{Z}[T],\quad g=\sum_{\gamma\in\Gamma} n_\gamma(t'_\gamma,\sigma_\gamma,u_\gamma):\mathbb{Z}[T]\longrightarrow\mathbb{Z}[U]$$

is defined to be

$$\sum_{\lambda\in\Lambda,\gamma\in\Gamma,t_\lambda=t'_\gamma} n_\gamma m_\lambda(s_\lambda,\sigma_\gamma\rho_\lambda,u_\gamma),$$

where $\sigma_\gamma\rho_\lambda:[0,\tau_\lambda+\tau'_\gamma]\to M$ is the composite path

$$\sigma_\gamma\rho_\lambda(x)=\begin{cases}\rho_\lambda(x) & \text{if } 0\le x\le\tau_\lambda,\\ \sigma_\gamma(x-\tau_\lambda) & \text{if } \tau_\lambda\le x\le\tau_\lambda+\tau'_\gamma,\end{cases}$$

of two paths $\rho_\lambda:[0,\tau_\lambda]\to M$, $\sigma_\gamma:[0,\tau'_\gamma]\to M$ with $\rho_\lambda(\tau_\lambda)=\sigma_\gamma(0)$. If $f_\alpha=\sum_{\lambda\in\Lambda_\alpha} m_{\alpha\lambda}(s_{\alpha\lambda},\rho_{\alpha\lambda},t_{\alpha\lambda})$ $(\alpha\in A)$, then

$$\bigoplus_{\alpha\in A} f_\alpha=\sum_{\alpha\in A,\ \lambda\in\Lambda_\alpha} m_{\alpha\lambda}(s_{\alpha\lambda},\rho_{\alpha\lambda},t_{\alpha\lambda})$$

is called the *direct sum* of the family $\{f_\alpha\}_{\alpha\in A}$ of geometric morphisms.

We shall often use matrices to present a geometric morphism between direct sums of geometric modules. Let

$$f=\sum_{\lambda\in\Lambda} m_\lambda(s_\lambda,\rho_\lambda,t_\lambda):\bigoplus_{j=1}^{n}\mathbb{Z}[S_j]\longrightarrow\bigoplus_{i=1}^{m}\mathbb{Z}[T_i]$$

be a geometric morphism. Define geometric morphisms $f_{ij}:\mathbb{Z}[S_j]\to\mathbb{Z}[T_i]$ $(1\le i\le m,\ 1\le j\le n)$ by:

$$f_{ij}=\sum_{\lambda\in\Lambda,s_\lambda\in S_j,t_\lambda\in T_i} m_\lambda(s_\lambda,\rho_\lambda,t_\lambda).$$

f is completely determined by f_{ij}'s; f is equal to the sum $\sum_{i,j} f_{ij}$ if we regard f_{ij}'s as geometric morphisms between $\mathbb{Z}[\bigsqcup S_j]$ and $\mathbb{Z}[\bigsqcup T_i]$ via the inclusions $S_j\subset\bigsqcup_{1\le j\le n} S_j$, $T_i\subset\bigsqcup_{1\le i\le m} T_i$. An $m\times n$ matrix $(f_{ij})_{1\le i\le m,1\le j\le n}$ will be used to express f using f_{ij}'s. For example, the direct sum $\bigoplus_{i=1}^n f_i:\bigoplus_{i=1}^n\mathbb{Z}[S_i]\to\bigoplus_{i=1}^n\mathbb{Z}[S_i]$ can be written as a diagonal matrix with entries $f_1,\ldots,f_n$.

Suppose $\pi : \widetilde{M} \to M$ is a covering map. Given a geometric module $\mathbb{Z}[S]$ on M, one can form the pullback geometric module $\mathbb{Z}[\widetilde{S}]$ on $\widetilde{M}$ by the standard pullback construction

$$\widetilde{S} : |\widetilde{S}| = \{(|s|, \tilde{m}) \in |S| \times \widetilde{M} \mid S(|s|) = \pi(\tilde{m})\} \longrightarrow \widetilde{M} \ ; \ (|s|, \tilde{m}) \ \mapsto \ \tilde{m} \ .$$

The induced covering

$$S^*\pi \ : \ |\widetilde{S}| \longrightarrow |S| \ ; \ (|s|, \tilde{m}) \ \mapsto \ |s|$$

determines a covering $\pi_S : \widetilde{S} \longrightarrow S$ of the graph S:

$$\pi_S \ : \ \widetilde{S} \approx |\widetilde{S}| \xrightarrow{S^*\pi} |S| \approx S.$$

Next we define the pullback of a geometric morphism with respect to π. Let $\mathbb{Z}[\widetilde{S}]$ and $\mathbb{Z}[\widetilde{T}]$ be the pullbacks of $\mathbb{Z}[S]$ and $\mathbb{Z}[T]$ with respect to $\pi : \widetilde{M} \to M$. Let (s, ρ, t) be a path in M from $s \in S$ to $t \in T$. If $\tilde{s} \in \widetilde{S}$ is mapped to s by $\pi_S : \widetilde{S} \to S$, there is a unique path $(\tilde{s}, \tilde{\rho}, \tilde{t})$ in $\widetilde{M}$ from $\tilde{s}$ to some element $\tilde{t} \in \widetilde{T}$ such that the composition $\pi\tilde{\rho}$ is equal to ρ. Such a path is called a *lift* of (s, ρ, t). Now, for a geometric morphism $f = \sum_\lambda m_\lambda(s_\lambda, \rho_\lambda, t_\lambda) : \mathbb{Z}[S] \to \mathbb{Z}[T]$, define its pullback geometric morphism $\tilde{f} : \mathbb{Z}[\widetilde{S}] \longrightarrow \mathbb{Z}[\widetilde{T}]$ by:

$$\tilde{f} = \sum_{\lambda \in \Lambda} \sum_{\pi_S(\tilde{s}_\lambda) = s_\lambda} m_\lambda(\tilde{s}_\lambda, \tilde{\rho}_\lambda, \tilde{t}_\lambda),$$

where $(\tilde{s}_\lambda, \tilde{\rho}_\lambda, \tilde{t}_\lambda)$ is the lift of $(s_\lambda, \rho_\lambda, t_\lambda)$ starting from $\tilde{s}_\lambda$. It is easily checked that $\widetilde{gf} = \tilde{g}\tilde{f}$.

Suppose π is a regular covering with the group of covering translations Π. Then $\pi_S : \widetilde{S} \longrightarrow S$ is also a regular Π-covering. Π acts freely on $\widetilde{S}$, and $\mathbb{Z}[\widetilde{S}]$ is freely generated as a $\mathbb{Z}[\Pi]$-module by any complete set of orbit representatives of $\widetilde{S}$. If $f : \mathbb{Z}[S] \to \mathbb{Z}[T]$ is a geometric morphism, the $\mathbb{Z}$-module homomorphism $|\tilde{f}|$ is actually a $\mathbb{Z}[\Pi]$-module homomorphism between the based free $\mathbb{Z}[\Pi]$-modules $\mathbb{Z}[\widetilde{S}]$ and $\mathbb{Z}[\widetilde{T}]$. For a fixed $\pi : \widetilde{M} \to M$ assembly is a functor

$$\begin{aligned} &\{\text{(f.g.) geometric modules on } M \text{ and geometric morphisms}\} \\ &\longrightarrow \{\text{(f.g.) free } \mathbb{Z}[\Pi]\text{-modules and homomorphisms}\} \ ; \ \mathbb{Z}[S] \ \to \ \mathbb{Z}[\widetilde{S}] \ . \end{aligned}$$

We shall be particularly concerned with the case when $\pi : \widetilde{M} \to M$ is the universal cover of M (assuming M is connected and locally 1-connected). A geometric module $\mathbb{Z}[S]$ determines a $\mathbb{Z}[\pi_1(M)]$-module $\mathbb{Z}[\widetilde{S}]$, which will be called the *assembly* of $\mathbb{Z}[S]$. Similarly, a geometric morphism $f : \mathbb{Z}[S] \to \mathbb{Z}[T]$ determines a $\mathbb{Z}[\pi_1(M)]$-module homomorphism $|\tilde{f}| : \mathbb{Z}[\widetilde{S}] \to \mathbb{Z}[\widetilde{T}]$, which will be called the *assembly* of f.

Definition. Two paths $(s, \rho : [0,\tau] \to M, t)$, $(s', \tau' : [0,\tau'] \to M, t')$ are *homotopic* if $s = s'$, $t = t'$, and there exist a non-negative continuous function $\tau(y)$ $(0 \le y \le 1)$ and a continuous map

$$h : \{(x,y) \in \mathbb{R}^2 | 0 \le x \le \tau(y),\ 0 \le y \le 1\} \longrightarrow M$$

such that $\tau(0) = \tau$, $\tau(1) = \tau'$, $h(x,0) = \rho(x)$ $(x \in [0,\tau])$, $h(x,1) = \rho'(x)$ $(x \in [0,\tau'])$, $h(0,y) = [s]$, $h(\tau(y), y) = [t] \in M$ $(y \in [0,1])$.

A *homotopy* $(\sim)$ of a geometric morphism is a finite sequence of the following operations:

1. homotopies of the paths,
2. combining two terms $m(s,\rho,t) + n(s,\rho,t)$ into $(m+n)(s,\rho,t)$, and its inverse.

For example, if a path (s,ρ,t) is homotopic to (s,ρ',t), then $(s,\rho,t) - (s,\rho',t)$ is homotopic to the zero geometric morphism:

$$(s,\rho,t) - (s,\rho',t) \overset{\text{operation 1}}{\sim} (s,\rho,t) - (s,\rho,t) \overset{\text{operation 2}}{\sim} 0(s,\rho,t) \ = \ 0.$$

The assemblies of homotopic geometric morphisms are the same homomorphisms. In fact, Quinn [18] has shown that, when M is connected and locally 1-connected, the assembly map $\mathbb{Z}[S] \mapsto \mathbb{Z}[\widetilde{S}]$ with respect to the universal cover $\widetilde{M}$ of M defines a natural equivalence between the category of geometric modules on M and homotopy classes of morphisms and the category of based free $\mathbb{Z}[\pi_1(M)]$-modules (with basis specified up to the action of $\pi_1(M)$).

Let $\varphi : M \to N$ be a continuous map. For a geometric module $A = \mathbb{Z}[S]$ on M, its *direct image* $\varphi_\sharp A$ is defined to be the geometric module $\mathbb{Z}[\varphi S : |S| \to M \to N]$ on N. Taking a direct image corresponds to changing the coefficient ring of the assembly from $\mathbb{Z}[\pi_1(M)]$ to $\mathbb{Z}[\pi_1(N)]$. For an element $s = (|s|, [s])$ of the graph S, φs will denote the element $(|s|, \varphi[s])$ of the graph of $\varphi S : |S| \to N$. If $f = \sum m_\lambda(s_\lambda, \rho_\lambda, t_\lambda) : A \to B$ is a geometric morphism between geometric modules A, B on M, then $\varphi_\sharp f : \varphi_\sharp A \to \varphi_\sharp B$ will denote the geometric morphism

$$\sum m_\lambda(\varphi s_\lambda, \varphi\rho_\lambda : [0,\tau_\lambda] \xrightarrow{\rho_\lambda} M \xrightarrow{\varphi} N, \varphi t_\lambda).$$

If $f \sim g$, then $\varphi_\sharp f \sim \varphi_\sharp g$.

Next we introduce "chain complexes" in the category of the geometric modules on M. Chain complexes will play a key role in this paper.

Definition. A *chain complex* on M is a sequence of morphisms of geometric modules on M

$$\{C, d\} : \ldots \to C_{r+1} \xrightarrow{d_{r+1}} C_r \xrightarrow{d_r} C_{r-1} \to \ldots$$

such that $d_r d_{r+1} \sim 0$.

Homotopies $d^2 \sim 0$ are used in the definition of chain complexes instead of the strict equalities, because boundary morphisms arising from CW complexes satisfy only $d^2 \sim 0$. See [17] for a detailed discussion. We shall actually need to work with chain complexes in the category of "projective modules".

Definitions. A geometric morphism $p : A \to A$ from a geometric module A on M to itself is a *projection* if $p^2 \sim p$. A *projective module* on M is a pair (A, p) consisting of a geometric module A on M and a projection $p : A \to A$. (A, p) is *finitely generated* if A is finitely generated. The *direct sum* $\bigoplus_i (A_i, p_i)$ of projective modules (A_i, p_i) is defined by $(\bigoplus_i A_i, \bigoplus_i p_i)$. A *morphism* $f : (A, p) \to (B, q)$ between two projective modules is a geometric morphism $f : A \to B$ satisfying $qf \sim f$ and $fp \sim f$. The morphism defined by the zero geometric morphism is called the *zero morphism* and is denoted 0.

For example, if (A, p) is a projective module on M, then the geometric morphism $p : A \to A$ defines a morphism from (A, p) to itself. This morphism p serves as the "identity" morphism up to homotopy. Thus projective modules on M and the homotopy classes of morphisms form a category. A morphism $f : (A, p) \to (B, q)$ is an *isomorphism* if there exists a morphism $g : (B, q) \to (A, p)$ such that $gf \sim p$ and $fg \sim q$; g is called the *inverse* of f.

A projective module of the form $(A, 1)$ can be identified with the geometric module A and is called a *free module*. The morphisms between free modules $(A, 1)$ and $(B, 1)$ are exactly the geometric morphisms between A and B.

Definition. A *projective chain complex on M* is a sequence of morphisms of projective modules on M

$$\{(C, p), d\} : \ldots \to (C_{r+1}, p_{r+1}) \xrightarrow{d_{r+1}} (C_r, p_r) \xrightarrow{d_r} (C_{r-1}, p_{r-1}) \to \ldots$$

such that $d_r d_{r+1} \sim 0$. When there is no ambiguity, we often omit the boundary morphisms d from notation. A projective chain complex (C, p) is *n-dimensional* if $C_r = 0$ for $r < 0$ and for $r > n$. If all p_r's are 1, then it is called a *free chain complex* and can be identified with the chain complex

$$C : \ldots \to C_{r+1} \to C_r \to C_{r-1} \to \ldots$$

of geometric modules. The *direct sum* of two projective chain complexes (C, p), (D, q) is defined by

$$(C, p) \oplus (D, q) : \cdots \to (C_r, p_r) \oplus (D_r, q_r) \xrightarrow{d_C \oplus d_D} (C_{r-1}, p_{r-1}) \oplus (D_{r-1}, q_{r-1}) \to \cdots.$$

A projective chain complex (C, p) is *finitely generated (f.g.)* if C_r is finitely generated for every r.

Definitions. (1) A *chain map* $f : (C, p) \to (D, q)$ between projective chain complexes is a collection $f = \{f_r\}$ of morphisms $f_r : (C_r, p_r) \to (D_r, q_r)$ such that $d_r f_r \sim f_{r-1} d_r$.
(2) A *chain homotopy* $h : f \simeq g$ between chain maps $f, g : (C, p) \to (D, q)$ is a collection $h = \{h_r\}$ of morphisms $h_r : (C_r, p_r) \to (D_{r+1}, q_{r+1})$ such that $d_{r+1} h_r + h_{r-1} d_r \sim g_r - f_r$.
(3) A chain map $f : (C, p) \to (D, q)$ is a *chain equivalence* if there exists a chain map $g : (D, q) \to (C, p)$, called a *chain homotopy inverse*, such that $gf \simeq p$ and $fg \simeq q$.
(4) Two projective chain complexes (C, p) and (D, q) are *chain equivalent*, $(C, p) \simeq (D, q)$, if there exists a chain equivalence between them.
(5) A projective chain complex (C, p) is *contractible* if it is chain equivalent to the zero chain complex. In this case, a chain homotopy $h : 0 \simeq p : (C, p) \to (C, p)$ is called a *chain contraction*.
(6) A chain map $f : (C, p) \to (D, q)$ is an *isomorphism*, $f : (C, p) \cong (D, q)$, if there exists a chain map $g : (D, q) \to (C, p)$, called an *inverse* of f, such that $gf \sim p$ and $fg \sim q$. Thus each f_r is an isomorphism of projective modules.
(7) The *algebraic mapping cone* $\mathcal{C}(f)$ of a chain map $f : (C, p) \to (D, q)$ is a projective chain complex defined by

$$\begin{aligned} d_{\mathcal{C}(f)} = \begin{pmatrix} d_D & (-)^{r-1} f \\ 0 & d_C \end{pmatrix} : \mathcal{C}(f)_r &= (D_r, q_r) \oplus (C_{r-1}, p_{r-1}) \\ \longrightarrow \mathcal{C}(f)_{r-1} &= (D_{r-1}, q_{r-1}) \oplus (C_{r-2}, p_{r-2}) \ . \end{aligned}$$

(A chain map f is a chain equivalence if and only if $\mathcal{C}(f)$ is contractible. See 2.4 below.)

Remark. Let $\{(C, p), d\}$ be a projective chain complex. The morphisms $d_r : (C_r, p_r) \to (C_{r-1}, p_{r-1})$ are, by definition, geometric morphisms $d_r : C_r \to C_{r-1}$. Thus we have a free chain complex

$$C = \{C_r, d_r\} : \ldots \longrightarrow C_{r+1} \xrightarrow{d_{r+1}} C_r \xrightarrow{d_r} C_{r-1} \longrightarrow \ldots .$$

The geometric morphisms $\{p_r\}$ form a chain map from C to itself.

2. Controlled algebra.

Now we introduce 'geometric control'. A continuous map $p_X : M \to X$ to a metric space is called a *control map*. If M is given a specific control map p_X, we say $\mathbb{Z}[S]$ is a geometric module *on* p_X. Suppose W is a subspace of X. The restriction of p_X to the subset $p_X^{-1}(W)$ of M will be denoted by $p_W : p_X^{-1}(W) \to W$. For $\epsilon \geq 0$, the closed ϵ neighborhood of W in X is denoted by W^ϵ. Obviously, $(W^\epsilon)^\delta \subset W^{\epsilon+\delta}$. For $\epsilon > 0$, $W^{-\epsilon}$ denotes the set $\{x \in W | d(x, X - W) \geq \epsilon\} \subset W$.

Given a control map p_X, we use the following convention for radii of geometric morphisms and homotopies. A geometric morphism f *has radius* ϵ if the image of the path $\rho : [0, \tau] \to M$ is contained in $p_X^{-1}(\{p_X\rho(0)\}^\epsilon \cap \{p_X\rho(\tau)\}^\epsilon)$ for each path (s, ρ, t) appearing with non-zero coefficient in f. A homotopy of geometric morphisms is an ϵ *homotopy* ($\sim_\epsilon$) if

1. in operation 1, each homotopy (even the constant one) of a path (s, ρ, t) has image in $p_X^{-1}(\{p_X\rho(0)\}^\epsilon \cap \{p_X\rho(\tau)\}^\epsilon)$, and
2. in operation 2, each path (s, ρ, t) in the combined terms (or split term) has image in $p_X^{-1}(\{p_X\rho(0)\}^\epsilon \cap \{p_X\rho(\tau)\}^\epsilon)$.

In other words, the morphism is required to have radius ϵ at every stage of the homotopy operations 1 and 2.

Proposition 2.1. *Assuming that the relevant operations on geometric morphisms are possible, the following hold true:*
(1) *If $f \sim_\epsilon f'$ and $f' \sim_\delta f''$, then $f \sim_{\max\{\epsilon,\delta\}} f''$.*
(2) *If $f \sim_\epsilon f'$ and $g \sim_\delta g'$, then $mf + ng \sim_{\max\{\epsilon,\delta\}} mf' + ng'$ for any $m, n \in \mathbb{Z}$.*
(3) *The composite gf of a geometric morphism f of radius δ and a geometric morphism g of radius ϵ has radius $\delta + \epsilon$.*
(4) *If $f \sim_\epsilon f'$ and $g \sim_\delta g'$, then $gf \sim_{\epsilon+\delta} g'f'$.*

Proof: Immediate from the definition. □

Let $p_X : M \to X$ be a control map for M. In the following definition, all geometric modules are on p_X.

Definitions. A projection $p : A \to A$ is an ϵ *projection* if $p^2 \sim_\epsilon p$. A projective module (A, p) is an ϵ *projective module* if p is an ϵ projection. A morphism $f : (A, p) \to (B, q)$ between projective modules is an ϵ *morphism* if f has radius ϵ and satisfies: $qf \sim_\epsilon f$, $fp \sim_\epsilon f$. An ϵ morphism $f : (A, p) \to (B, q)$ is an ϵ *isomorphism* if there exists an ϵ morphism $g : (B, q) \to (A, p)$ such that $gf \sim_{2\epsilon} p$ and $fg \sim_{2\epsilon} q$.

Remarks. (1) In the definition of ϵ morphisms and ϵ isomorphisms, we do not require (A, p) and (B, q) to be δ projective modules for any particular $\delta > 0$ so that the definition is as simple as possible. There is an extra advantage that the zero morphism is always a 0 morphism.
(2) The choice of coefficients of ϵ in the definition above looks rather arbitrary. Here is an explanation: First of all, we want an ϵ projection $p : A \to A$ to be an ϵ morphism between (A, p) and itself. Secondly, sizes should behave nicely with respect to composition. (See 2.2.) A different definition of ϵ projections will force a possibly different definition of ϵ morphisms and ϵ isomorphisms. One possibility is to use $p^2 \sim_{2\epsilon} p$, but this forces us to use $\sim_{3\epsilon}$ in the definition of ϵ isomorphisms, which is not so desirable. Anyway this is not a crucial problem.

Proposition 2.2. *The composition $gf : (A,p) \to (C,r)$ of a δ morphism (resp. isomorphism) $f : (A,p) \to (B,q)$ and an ϵ morphism (resp. isomorphism) $g : (B,q) \to (C,r)$ is a $\delta+\epsilon$ morphism (resp. isomorphism).*

Proof : Obviously gf has radius $\delta+\epsilon$, and

$$r(gf) = (rg)f \sim_{\delta+\epsilon} gf, \qquad (gf)p = g(fp) \sim_{\delta+\epsilon} gf.$$

So, gf is a $\delta+\epsilon$ morphism. If further f is a δ isomorphism with inverse f^{-1} and g is an ϵ isomorphism with inverse g^{-1}, then

$$(f^{-1}g^{-1})(gf) \sim_{2\delta+2\epsilon} f^{-1}qf \sim_{2\delta} f^{-1}f \sim_{2\delta} p \ ,$$

and similarly $(gf)(f^{-1}g^{-1}) \sim_{2\delta+2\epsilon} r$. □

Definition. A projective chain complex (C,p) on M is an ϵ *projective chain complex on* p_X if

1. each (C_r, p_r) is an ϵ projective module,
2. each $d_r : (C_r, p_r) \to (C_{r-1}, p_{r-1})$ is an ϵ morphism, and
3. $d_r d_{r+1} \sim_{2\epsilon} 0$ for each r.

A free ϵ projective chain complex will be called a *free ϵ chain complex*.

Definitions. (1) A chain map $f : (C,p) \to (D,q)$ is an ϵ chain map if $f_r : (C_r,p_r) \to (D_r,q_r)$ are ϵ morphisms such that $d_r f_r \sim_\epsilon f_{r-1}d_r$.
(2) A chain homotopy $h : f \simeq g$ between chain maps $f, g : (C,p) \to (D,q)$ is an ϵ *chain homotopy*, $h : f \simeq_\epsilon g$, if the h_r's are ϵ morphisms such that $d_{r+1}h_r + h_{r-1}d_r \sim_{2\epsilon} g_r - f_r$.
(3) An ϵ chain map $f : (C,p) \to (D,q)$ is an ϵ *chain equivalence* if there exists an ϵ chain map $g : (D,q) \to (C,p)$, called an ϵ *chain homotopy inverse*, such that $gf \simeq_\epsilon p$ and $fg \simeq_\epsilon q$.
(4) Two projective chain complexes (C,p) and (D,q) are ϵ *chain equivalent*, $(C,p) \simeq_\epsilon (D,q)$, if there exists an ϵ chain equivalence between them.
(5) A projective chain complex (C,p) is ϵ *contractible* if it is ϵ chain equivalent to the zero chain complex. In this case, an ϵ chain homotopy $h : 0 \simeq_\epsilon p : (C,p) \to (C,p)$ is called an ϵ *chain contraction*.
(6) An ϵ chain map $f : (C,p) \to (D,q)$ is an ϵ *isomorphism*, $(C,p) \cong_\epsilon (D,q)$ if there exists an ϵ chain map $g : (D,q) \to (C,p)$, called an ϵ *inverse*, such that $gf \sim_{2\epsilon} p$ and $fg \sim_{2\epsilon} q$. Thus each f_r is an ϵ isomorphism of projective modules.

An ϵ isomorphism of projective chain complexes is always an ϵ chain equivalence. For projective chain complexes of dimension 0, the converse is also true. The 'identity' chain map $p = \{p_r\}$ on an ϵ projective chain complex (C,p) is an ϵ isomorphism.

Proposition 2.3. (1) *The composition $f'f$ of an ϵ chain map $f : (C,p) \to (D,q)$ and an ϵ' chain map $f' : (D,q) \to (E,r)$ is an $\epsilon+\epsilon'$ chain map.*
(2) *The composition $f'f$ of an ϵ isomorphism $f : (C,p) \to (D,q)$ and an ϵ' isomorphism $f' : (D,q) \to (E,r)$ is an $\epsilon+\epsilon'$ isomorphism.*
(3) *The composition $f'f$ of an ϵ chain equivalence $f : (C,p) \to (D,q)$ and an ϵ' chain equivalence $f' : (D,q) \to (E,r)$ is an $\epsilon+\epsilon'$ chain equivalence.*

Proof : (1) and (2) are obvious. We prove (3): Let g and g' be chain homotopy inverses of f and f' with ϵ chain homotopies $h : gf \simeq p$, $k : fg \simeq q$ and ϵ' chain homotopies $h' : g'f' \simeq q$, $k' : f'g' \simeq r$. Then,

$$\begin{aligned} & d(f'kg'+k')+(f'kg'+k')d \sim_{\epsilon+2\epsilon'} f'(dk+kd)g'+(r-f'g') \\ & \sim_{2\epsilon+2\epsilon'} f'(q-fg)g'+(r-f'g') \sim_{2\epsilon+2\epsilon'} f'g'-f'fgg'+r-f'g' \\ & \sim_{2\epsilon+2\epsilon'} r-(f'f)(gg')\ , \end{aligned}$$

and similarly,

$$d(h+gh'f)+(h+gh'f)d \sim_{2\epsilon+2\epsilon'} p-(gg')(f'f).$$

□

Proposition 2.4. *Let $f : (C,p) \to (D,q)$ be an ϵ chain map. If the algebraic mapping cone $\mathcal{C}(f)$ is ϵ contractible, then f is a 2ϵ chain equivalence. If f is an ϵ chain equivalence, then $\mathcal{C}(f)$ is 3ϵ contractible.*

Proof : Given an ϵ chain contraction, $\Gamma : 0 \simeq_\epsilon q \oplus p : \mathcal{C}(f) \to \mathcal{C}(f)$, let g, h, k be the ϵ morphisms defined by

$$\begin{aligned} \Gamma \ = \ \begin{pmatrix} k & ? \\ (-)^r g & h \end{pmatrix} \ : \ \mathcal{C}(f)_r \ &= \ (D_r,q_r) \oplus (C_{r-1},p_{r-1}) \\ &\longrightarrow \mathcal{C}(f)_{r+1} \ = \ (D_{r+1},q_{r+1}) \oplus (C_r,p_r)\ . \end{aligned}$$

Then $g : (D,q) \to (C,p)$ is a chain homotopy inverse of f with ϵ chain homotopies $h : gf \simeq_\epsilon p : (C,p) \to (C,p)$, $k : fg \simeq_\epsilon q : (D,q) \to (D,q)$. Although the radius of g is ϵ, g is a 2ϵ chain map because we only have $dg \sim_{2\epsilon} gd$. Therefore f is a 2ϵ chain equivalence.

Conversely, suppose that f is an ϵ chain equivalence with ϵ chain homotopy inverse $g : (D,q) \to (C,p)$ and ϵ chain homotopies

$$\begin{aligned} h \ &: \ gf \ \simeq_\epsilon \ p \ : \ (C,p) \longrightarrow (C,p) \\ k \ &: \ fg \ \simeq_\epsilon \ q \ : \ (D,q) \longrightarrow (D,q)\ . \end{aligned}$$

A 3ϵ chain contraction of $\mathcal{C}(f)$ is given by:

$$\begin{aligned} &\Gamma \ = \ \begin{pmatrix} k+(fh-kf)g & (-)^r(fh-kf)h \\ (-)^r g & h \end{pmatrix} \ : \\ &\mathcal{C}(f)_r \ = \ (D_r,q_r) \oplus (C_{r-1},p_{r-1}) \longrightarrow \mathcal{C}(f)_{r+1} \ = \ (D_{r+1},q_{r+1}) \oplus (C_r,p_r)\ . \end{aligned}$$

□

3. Controlled finiteness obstruction.

We start with a brief review of the projective class group and finiteness obstruction in the uncontrolled case and then go on to deal with the controlled analogues.

Given a ring A and an integer $n \geq 0$, let $\widetilde{K}_0(A, n)$ be the quotient of the Grothendieck group of n-dimensional f.g. projective A-module chain complexes by the subgroup of f.g. free A-module chain complexes. For $n = 0$ this is the reduced projective class group of A

$$\widetilde{K}_0(A, 0) \ = \ \widetilde{K}_0(A) \ ,$$

the quotient of the Grothendieck group of f.g. projective A-modules by the subgroup of f.g. free A-modules. The reduced projective class of an n-dimensional f.g. projective A-module chain complex P

$$[P] \ = \ \sum_{r=0}^{n} (-)^r [P_r] \in \widetilde{K}_0(A)$$

is a chain homotopy invariant, such that $[P] = 0$ if and only if P is chain equivalent to a finite f.g. free A-module chain complex. The reduced projective class defines isomorphisms

$$\widetilde{K}_0(A, n) \longrightarrow \widetilde{K}_0(A) \ ; \ [P] \longrightarrow [P] \ .$$

The singular chain complex of the universal cover $\widetilde{M}$ of a finitely dominated space M is chain equivalent to a finite f.g. projective $\mathbb{Z}[\pi_1(M)]$-module chain complex $C(\widetilde{M})$. The finiteness obstruction of Wall [23] is the reduced projective class

$$[M] \ = \ [C(\widetilde{M})] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(M)]) \ ,$$

such that $[M] = 0$ if and only if M is homotopy equivalent to a finite CW complex.

We use ϵ projective chain complexes to define a controlled analogue analogue of the projective class groups. To obtain a correct analogue, we have to use chain complexes that are finitely generated.

Definition. Two finitely generated projective chain complexes (C, p) and (C', p') on p_X are *n-stable ϵ chain equivalent* if there exists an ϵ chain equivalence between $(C, p) \oplus (E, 1)$ and $(C', p') \oplus (E', 1)$ for some finitely generated n-dimensional free ϵ chain complexes $(E, 1)$, $(E', 1)$ on p_X.

For a fixed $\epsilon > 0$, n-stable ϵ chain equivalence is *not* an equivalence relation. If (C, p), (C', p') are n-stable ϵ chain equivalent and (C', p'), (C'', p'') are also n-stable ϵ chain equivalent, then (C, p) and (C'', p'') are only n-stable 2ϵ chain equivalent.

Definition. $\widetilde{K}_0(X, p_X, n, \epsilon)$ is the set of equivalence classes $[C, p]$ of finitely generated n-dimensional ϵ projective chain complexes on p_X. The equivalence relation is *generated* by n-stable ϵ chain equivalence. $\widetilde{K}_0(X, p_X, 0, \epsilon)$ will be denoted $\widetilde{K}_0(X, p_X, \epsilon)$.

We shall also need an analogue which uses projective chain complexes that are not necessarily finitely generated. Such an object arises naturally when we take a pullback of a finitely generated projective chain complex via an infinite-sheeted covering map.

Definition. A geometric module on a product space $M \times N$ is said to be *M-locally finite* if, for any $y \in N$, there is a neighbourhood U of y in N such that $M \times U$ contains only finitely many basis elements; a projective module (A, p) on $M \times N$ is said to be *M-locally finite* if A is M-locally finite; a projective chain complex (C, p) on $M \times N$ is *M-locally finite* if each (C_r, p_r) is M-locally finite.

For M-locally finite geometric modules, we only consider control maps of the form

$$q_X \;=\; p_X \times 1_N \;:\; M \times N \longrightarrow X \times N,$$

where $p_X : M \to X$ is a given control map for M, N is a metric space, and $X \times N$ is given a maximum product metric.

Definition. Two M-locally finite projective chain complexes (C, p) and (C', p') on q_X are *M-locally finitely n-stable ϵ chain equivalent* if there exists an ϵ chain equivalence between $(C, p) \oplus (E, 1)$ and $(C', p') \oplus (E', 1)$ for some M-locally finite n-dimensional free ϵ chain complexes $(E, 1)$, $(E', 1)$ on q_X.

Definition. $\widetilde{K}_0^M(X \times N, q_X, n, \epsilon)$ is the set of equivalence classes $[C, p]$ of M-locally finite n-dimensional ϵ projective chain complexes on q_X. The equivalence relation is *generated* by M-locally finitely n-stable ϵ chain equivalence. $\widetilde{K}_0^M(X, p_X, 0, \epsilon)$ will be denoted $\widetilde{K}_0^M(X, p_X, \epsilon)$.

Important Notice. In the rest of this section we mainly discuss $\widetilde{K}_0(X, p_X, n, \epsilon)$ and all the chain complexes are assumed to be finitely generated unless explicitly stated otherwise. But the argument carries over to the M-locally finite case without any modification.

Proposition 3.1. *Direct sum induces an abelian group structure on $\widetilde{K}_0(X, p_X, n, \epsilon)$. Further, if $[C, p] = [C', p'] \in \widetilde{K}_0(X, p_X, n, \epsilon)$, then there is a 3ϵ chain equivalence*

$$(C, p) \oplus (E, 1) \to (C', p') \oplus (F, 1)$$

for some n-dimensional free ϵ chain complexes $(E, 1)$, $(F, 1)$ on p_X. In particular, (C, p) and (C', p') are n-stable 3ϵ chain equivalent.

Proof : We shall show the existence of inverses. Note that if (A, p) is an ϵ projective module, then $(A, 1-p)$ is also an ϵ projective module and the direct sum $(A, p) \oplus (A, 1-p)$ is ϵ isomorphic to $(A, 1)$; the morphism $(p, 1-p) : (A, p) \oplus (A, 1-p) \to (A, 1)$ gives a desired ϵ isomorphism with an ϵ inverse ${}^t(p, 1-p) : (A, 1) \to (A, p) \oplus (A, 1-p)$.

Suppose $\{(C,p),d_C\}$ is an n-dimensional ϵ projective chain complex. The direct sum $\{(C,p),d_C\} \oplus \{(C,1-p),0\}$ with an n-dimensional ϵ projective chain complex:

$$\{(C,1-p),0\} : \ldots \to 0 \to (C_n, 1-p_n) \xrightarrow{0} \ldots \xrightarrow{0} (C_0, 1-p_0) \to 0$$

is ϵ isomorphic to the free ϵ chain complex:

$$\{(C,1),d_C\} : \ldots \to 0 \to (C_n, 1) \xrightarrow{d_C} \ldots \xrightarrow{d_C} (C_0, 1) \to 0.$$

Thus $[(C,1-p),0]$ is the inverse of $[(C,p),d_C]$.

Next suppose $[(C,p),d] = [(C',p'),d']$. We use the cancellation of inverses argument originally employed by Chapman to prove a similar result for controlled Whitehead groups [6, 3.5]. There is a sequence of n-dimensional ϵ projective chain complexes

$$\{(C,p),d\} = \{(C^{(1)},p^{(1)}),d\}, \{(C^{(2)},p^{(2)}),d\}, \ldots, \{(C^{(m)},p^{(m)}),d\} = \{(C',p'),d'\},$$

where

$$\{(C^{(k)},p^{(k)}),d\} \oplus \{(E^{(k)},1),d\} \simeq_\epsilon \{(C^{(k+1)},p^{(k+1)}),d\} \oplus \{(F^{(k)},1),d\}$$

for some n-dimensional free ϵ chain complexes $\{(E^{(k)},1),d\}$, $\{(F^{(k)},1),d\}$ on p_X. The following composition gives the desired 3ϵ chain equivalence:

$$\begin{aligned}
&\{(C,p),d\} \oplus \sum_{k=1}^{m-1}\{(E^{(k)},1),d\} \oplus \sum_{k=1}^{m}\{(C^{(k)},1),d\} \\
&\cong_\epsilon \{(C,p),d\} \oplus \sum_{k=1}^{m-1}\{(E^{(k)},1),d\} \oplus \sum_{k=1}^{m}\Big(\{(C^{(k)},1-p^{(k)}),0\} \oplus \{(C^{(k)},p^{(k)}),d\}\Big) \\
&= \{(C,p),d\} \oplus \{(C^{(1)},1-p^{(1)}),0\} \oplus \sum_{k=1}^{m-1}\Big(\{(C^{(k)},p^{(k)}),d\} \oplus \{(E^{(k)},1),d\}\Big) \\
&\qquad\qquad \oplus \sum_{k=2}^{m}\{(C^{(k)},1-p^{(k)}),0\} \oplus \{(C^{(m)},p^{(m)}),d\} \\
&\simeq_\epsilon \{(C,p),d\} \oplus \{(C^{(1)},1-p^{(1)}),0\} \oplus \sum_{k=1}^{m-1}\Big(\{(C^{(k+1)},p^{(k+1)}),d\} \oplus \{(F^{(k)},1),d\}\Big) \\
&\qquad\qquad \oplus \sum_{k=2}^{m}\{(C^{(k)},1-p^{(k)}),0\} \oplus \{(C',p'),d'\} \\
&= \sum_{k=1}^{m}\Big(\{(C^{(k)},p^{(k)}),d\} \oplus \{(C^{(k)},1-p^{(k)}),0\}\Big) \oplus \sum_{k=1}^{m-1}\{(F^{(k)},1),d\} \oplus \{(C',p'),d'\} \\
&\cong_\epsilon \{(C',p'),d'\} \oplus \sum_{k=1}^{m}\{(C^{(k)},1),d\} \oplus \sum_{k=1}^{m-1}\{(F^{(k)},1),d\}\ .
\end{aligned}$$

□

Remark. By the construction of the additive inverse, one can conclude that the class $[C,p] \in \widetilde{K}_0(X, p_X, n, \epsilon)$ depends only on the projective modules (C_i, p_i) and not on the boundary morphisms.

Next we discuss maps between control maps which induce homomorphisms of controlled $\widetilde{K}_0$-groups. Let $p_X : M \to X$ and $p_{X'} : M' \to X'$ be control maps. A *map* from p_X to $p_{X'}$ is a pair of continuous maps $\Phi = (\varphi : M \to M', \bar{\varphi} : X \to X')$ satisfying $p_{X'}\varphi = \bar{\varphi}p_X$. Let δ, ϵ be positive numbers and k be a positive integer. Consider the following condition on Φ:

$\mathbf{C}(\delta, \epsilon, k)$: if $x, y \in X$, and $d(x,y) \le k\delta$ then $d(\bar{\varphi}(x), \bar{\varphi}(y)) \le k\epsilon$.

Suppose that Φ satisfies the conditions $\mathbf{C}(\delta,\epsilon,1)$ and $\mathbf{C}(\delta,\epsilon,2)$ and apply $\varphi_\sharp$ to chain complexes: if (C,p) is δ projective chain complex on p_X, then $\varphi_\sharp(C,p) = (\varphi_\sharp C, \varphi_\sharp p)$ is an ϵ projective chain complex on $p_{X'}$, and if two δ projective chain complexes (C,p) and (C',p') on p_X are n-stable δ chain equivalent, then $\varphi_\sharp(C,p)$ and $\varphi_\sharp(C',p')$ are n-stable ϵ chain equivalent. Therefore Φ induces a homomorphism $\Phi_* : \widetilde{K}_0(X, p_X, n, \delta) \to \widetilde{K}_0(X', p_{X'}, n, \epsilon)$. The equality $(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*$ is easily verified. Note that the condition $\mathbf{C}(\delta,\epsilon,1)$ does not imply the other condition $\mathbf{C}(\delta,\epsilon,2)$. Also note that if X is compact, then for any $\epsilon > 0$, there exists a $\delta > 0$ satisfying these conditions.

Inclusion maps are typical examples of maps which satisfy $\mathbf{C}(\epsilon,\epsilon,k)$ for every positive number ϵ and every positive integer k. Let $i : Y \to X$ be an inclusion map and $\tilde{\imath} : p_X^{-1}(Y) \to M$ be the corresponding inclusion map. Then $(\tilde{\imath}, i)$ is a map from p_Y to p_X, where $p_Y : p_X^{-1}(Y) \to Y$ is the restriction of p_X, and it induces a homomorphism

$$(\tilde{\imath}, i)_* \ : \ \widetilde{K}_0(Y, p_Y, n, \epsilon) \longrightarrow \widetilde{K}_0(X, p_X, n, \epsilon)$$

for every $\epsilon > 0$ and every $n \ge 0$. This homomorphism will be called the *homomorphism induced by* i and will be denoted i_*.

More generally, if $\delta \le \epsilon$, then $(\tilde{\imath}, i)$ also satisfies $\mathbf{C}(\delta,\epsilon,k)$ for every $k \ge 1$ and induces a homomorphism

$$(\tilde{\imath}, i)_* \ : \ \widetilde{K}_0(Y, p_Y, n, \delta) \longrightarrow \widetilde{K}_0(X, p_X, n, \epsilon) \ .$$

This homomorphism will be called a *stabilization map*. It is also commonly called the *relaxation of control map*.

In the case of $\widetilde{K}_0^M$, we only consider maps between control maps of the form

$$\Phi \ = \ (\ 1_M \times \psi : M \times N \to M \times N', \quad 1_X \times \psi : X \times N \to X \times N' \ )$$

from $p_X \times 1_N$ to $p_X \times 1_{N'}$, where $\psi : N \to N'$ is a continuous map. Stabilizations are defined similarly.

Given an $\epsilon > 0$, we are interested not in the group $\widetilde{K}_0(X, p_X, n, \epsilon)$ itself but in the image of $\widetilde{K}_0(X, p_X, n, \delta)$ in it for sufficiently small $\delta > 0$. Such an image tends to get stable as δ gets smaller; e.g., see 8.2. Below we shall see that an analogue of the isomorphism $\widetilde{K}_0(A, n) \cong \widetilde{K}_0(A, 0)$ holds only stably in the controlled setting.

Let $n > 0$ be an integer. We shall study the relationship between $\widetilde{K}_0(X, p_X, n, \epsilon)$ and $\widetilde{K}_0(X, p_X, \epsilon)$. There is a homomorphism

$$\iota \ : \ \widetilde{K}_0(X, p_X, \epsilon) \longrightarrow \widetilde{K}_0(X, p_X, n, \epsilon)$$

obtained by viewing a 0-dimensional chain complex as an n-dimensional one. We shall see that this is onto.

Proposition 3.2. *An n-dimensional ϵ projective chain complex (C, p) is ϵ chain equivalent to an n-dimensional ϵ projective chain complex (D, q) such that $q_r = 1 : D_r \to D_r$ for $r > 0$ and $(D_0, q_0) = (\bigoplus_{r:\text{even}}(C_r, p_r)) \oplus (\bigoplus_{r:\text{odd}}(C_r, 1 - p_r))$.*

Proof : Let $(D_r, q_r) = \bigoplus_{i \geq r}(C_i, 1)$ for $r > 0$ and let (D_0, q_0) be as in the statement above. Define the boundary morphism d_D by:

$$(d_D)_r \ = \ \begin{pmatrix} d & 0 & 0 & 0 & \dots \\ 1 - p_r & 0 & 0 & 0 & \dots \\ 0 & p_{r+1} & 0 & 0 & \dots \\ 0 & 0 & 1 - p_{r+2} & 0 & \dots \\ 0 & 0 & 0 & p_{r+3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$D_r \ \longrightarrow \ \begin{cases} (C_{r-1}, 1) \oplus (C_r, 1) \oplus \cdots \oplus (C_n, 1), & \text{if } r > 1 \\ (C_0, p_0) \oplus (C_1, 1 - p_1) \oplus \cdots \oplus (C_n, p_n), & \text{if } r = 1 \text{ and } n \text{ is even,} \\ (C_0, p_0) \oplus (C_1, 1 - p_1) \oplus \cdots \oplus (C_n, 1 - p_n), & \text{if } r = 1 \text{ and } n \text{ is odd.} \end{cases}$$

Then $(D, q) = \{(D_r, q_r), d_D\}$ is an n-dimensional ϵ projective chain complex. The following ϵ chain maps give the desired ϵ chain equivalence and its ϵ chain homotopy inverse:

$$\tilde{p}_r = {}^t\,(\,p_r \quad 0 \quad \dots \quad 0\,) \ : \ (C_r, p_r) \longrightarrow (D_r, q_r)$$
$$\hat{p}_r = (\,p_r \quad 0 \quad \dots \quad 0\,) \ : \ (D_r, q_r) \longrightarrow (C_r, p_r).$$

□

Corollary 3.3. *The homomorphism $\iota : \widetilde{K}_0(X, p_X, \epsilon) \to \widetilde{K}_0(X, p_X, n, \epsilon)$ is onto.*

Proof : For an element $[C, p] \in \widetilde{K}_0(X, p_X, n, \epsilon)$, let (D, q) be as in 3.2. Then the sum of (C, p) and the 0-dimensional free chain complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow (D_0, 1) \longrightarrow 0 \longrightarrow \cdots$$

is ϵ chain equivalent to the sum of the 0-dimensional ϵ projective chain complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow (D_0, q_0) \longrightarrow 0 \longrightarrow \cdots$$

and the n-dimensional free ϵ chain complex

$$\cdots \longrightarrow (D_2, 1) \xrightarrow{d_D} (D_1, 1) \xrightarrow{d_D} (D_0, 1) \longrightarrow 0 \longrightarrow \cdots .$$

Here $d_D : (D_1, 1) \to (D_0, 1)$ denotes the morphism defined by the same geometric morphism which was used to define the morphism $d_D : (D_1, 1) \to (D_0, q_0)$. An ϵ chain equivalence is given by

$$\begin{aligned} \tilde{p}_r \ &: \ (C_r, p_r) \longrightarrow (D_r, 1) \qquad (r > 0), \\ \begin{pmatrix} 0 & q_0 \\ \tilde{p}_0 & 1 - q_0 \end{pmatrix} \ &: \ (C_0, p_0) \oplus (D_0, 1) \longrightarrow (D_0, q_0) \oplus (D_0, 1) \qquad (r = 0). \end{aligned}$$

□

Proposition 3.4. *This correspondence $(C, p) \mapsto (D_0, q_0)$ defines a well-defined homomorphism*

$$\sigma \ : \ \widetilde{K}_0(X, p_X, n, \epsilon) \longrightarrow \widetilde{K}_0(X, p_X, 9\epsilon) \ .$$

Proof : First suppose (C, p) is ϵ chain equivalent to 0. Let Γ be an ϵ chain contraction of (C, p). Define a 3ϵ chain contraction Γ' by: $\Gamma' = \Gamma d \Gamma$. This has a larger radius but the identity $(\Gamma')^2 \sim_{6\epsilon} 0$ holds. (Cf. Whitehead [25,(6.2)].) Using this one can show that

$$\begin{aligned} d + \Gamma' \ &: \ \bigoplus_{r:\text{even}} (C_r, p_r) \longrightarrow \bigoplus_{r:\text{odd}} (C_r, p_r) \\ d + \Gamma' \ &: \ \bigoplus_{r:\text{odd}} (C_r, p_r) \longrightarrow \bigoplus_{r:\text{even}} (C_r, p_r) \end{aligned}$$

are 3ϵ inverses of each other. Therefore $\bigoplus_{r:\text{odd}}(C_r, p_r)$ and $\bigoplus_{r:\text{even}}(C_r, p_r)$ represent the same element in $\widetilde{K}_0(X, p_X, 3\epsilon)$, and $[D_0, q_0] = 0 \in \widetilde{K}_0(X, p_X, 3\epsilon)$.

Next suppose $f : (C, p) \to (C', p')$ is an ϵ chain equivalence. By 2.4 its algebraic mapping cone $\mathcal{C}(f)$ is 3ϵ contractible. By the argument above, $\sum_r (-1)^r [\mathcal{C}(f)_r, p'_r \oplus p_{r-1}] = 0 \in \widetilde{K}_0(X, p_X, 9\epsilon)$. But this element is the same as $\sum_r (-1)^r [C'_r, p'_r] - \sum_r (-1)^r [C_r, p_r]$.

Since direct sum with free ϵ chain complexes corresponds to direct sum with free modules, this finishes the proof. □

The two homomorphisms ι and σ above are stable inverses: the following diagrams commute.

$$\begin{array}{ccc}
\widetilde{K}_0(X,p_X,\epsilon) & \xrightarrow{\iota} & \widetilde{K}_0(X,p_X,n,\epsilon) \\
\downarrow & & \| \\
\widetilde{K}_0(X,p_X,9\epsilon) & \xleftarrow[\sigma]{} & \widetilde{K}_0(X,p_X,n,\epsilon)
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{K}_0(X,p_X,9\epsilon) & \xleftarrow{\sigma} & \widetilde{K}_0(X,p_X,n,\epsilon) \\
\| & & \downarrow \\
\widetilde{K}_0(X,p_X,9\epsilon) & \xrightarrow[\iota]{} & \widetilde{K}_0(X,p_X,n,9\epsilon)
\end{array}$$

The following is a corollary to 3.2.

Corollary 3.5. *Let $n > 0$. If $[C,p] = 0 \in \widetilde{K}_0(X,p_X,n,\epsilon)$, then (C,p) is 60ϵ chain equivalent to an n-dimensional free 30ϵ chain complex on p_X.*

Proof : Let (D,q) be as in 3.2; (C,p) is ϵ chain equivalent to (D,q). By the previous proposition, $[D_0,q_0] = 0 \in \widetilde{K}_0(X,p_X,9\epsilon)$. By 3.1, there is a free module $(F,1)$ such that $(D_0,q_0)\oplus(F,1)$ is 27ϵ isomorphic to some free module $(G,1)$. The inclusion map of (D,q) into the sum (D',q') of (D,q) and the 1-dimensional free chain complex

$$\cdots \longrightarrow 0 \longrightarrow (F,1) \xrightarrow{1} (F,1) \longrightarrow 0 \longrightarrow \cdots$$

is an ϵ chain equivalence. Let $f : (D'_0,q'_0) = (D_0,q_0)\oplus(F,1) \to (G,1)$ be a 27ϵ isomorphism of projective modules. If we replace the boundary map $d_{D'} : (D'_1,1) \to (D'_0,q'_0)$ of (D',q') by $fd_{D'} : (D'_1,1) \to (G,1)$, then we get a free 28ϵ chain complex $(D'',1)$ with $D''_r = D'_r$ $(r > 0)$ and $D''_0 = G$. The isomorphisms

$$\begin{array}{ll}
1 : D'_r \longrightarrow D''_r & \quad (\ r > 0\ ) \\
f : D'_0 \longrightarrow D''_0 &
\end{array}$$

define a 28ϵ chain map from (D',q') to $(D'',1)$, and its inverse is a 55ϵ chain map $(f^{-1}(fd_{D'}) \sim_{(2\cdot 27+1)\epsilon} d_{D'}\ )$. Thus we get a 55ϵ isomorphism between (D',q') and $(D'',1)$. Composing these we get a 57ϵ chain equivalence from (C,p) to an n-dimensional free 28ϵ chain complex.

$\square$

4. Controlled Whitehead torsion.

We start with a brief review of Whitehead torsion in the uncontrolled case, and then go on to deal with the controlled analogues.

Given a group π and an integer $n \geq 1$ let $Wh(\pi,n)$ be the quotient of the Grothendieck group of n-dimensional contractible based f.g. free $\mathbb{Z}[\pi]$-module chain complexes by the subgroup of the elementary complexes. For $n = 1$ this is the Whitehead group of π

$$Wh(\pi,1)\ =\ Wh(\pi)\ ,$$

a quotient of the Grothendieck group of isomorphisms of based f.g. free $\mathbb{Z}[\pi]$-modules. The torsion of a contractible based f.g. free $\mathbb{Z}[\pi]$-module chain complex C is defined by

$$\tau(C) \;=\; \tau(d+\Gamma : C_{\mathrm{odd}} \longrightarrow C_{\mathrm{even}}) \in Wh(\pi)$$

using any chain contraction $\Gamma : 0 \simeq 1 : C \longrightarrow C$. (It is usually more convenient if we further require $\Gamma^2 = 0$. A corresponding requirement in the controlled case also helps size estimation. But this is not really necessary in the uncontrolled setting. See [7, p.52].) Torsion defines isomorphisms

$$Wh(\pi, n) \longrightarrow Wh(\pi) \; ; \; [C] \longrightarrow \tau(C) \; .$$

The Whitehead torsion of a homotopy equivalence $f : L \longrightarrow M$ of finite CW complexes

$$\tau(f) \;=\; \tau(\tilde{f} : C(\widetilde{L}) \longrightarrow C(\widetilde{M})) \in Wh(\pi_1(M))$$

is such that f is simple ($\tau(f) = 0$) if and only if f is homotopic to a deformation, that is a composite of elementary expansions and collapses - see Milnor [15] and Cohen [7] for detailed expositions.

We fix the control map $p_X : M \to X$ to a metric space X and an integer $n \geq 1$. Given a subspace $Y \subseteq X$ and $\epsilon > 0$, we define the relative controlled Whitehead group $Wh(X, Y, p_X, n, \epsilon)$, the controlled analogue of $Wh(\pi, \rho)$. In §5 this will be related to the controlled projective class groups $\widetilde{K}_0$ of §3 by a 'stably-exact sequence'. The controlled Whitehead groups $Wh(X, p_X, \epsilon) = Wh(X, \emptyset, p_X, 1, \epsilon)$ were previously defined by Quinn [18].

Important Notice. As in the previous section, all the modules and chain complexes will be assumed to be finitely generated. But the argument carries over to the M-locally finite case without any modification: one can define the M-locally finite controlled (relative) Whitehead groups $Wh^M(X \times N, Y \times N, p_X \times 1_N, n, \epsilon)$ using M-locally finite chain complexes and can prove analogous results.

Definition. A geometric morphism $f : \mathbb{Z}[S] \to \mathbb{Z}[S]$ is *elementary* if $\mathbb{Z}[S]$ is the direct sum of two geometric modules $\mathbb{Z}[S_1]$ and $\mathbb{Z}[S_2]$ and

$$f \;=\; \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} : \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2],$$

for some morphism $h : \mathbb{Z}[S_2] \to \mathbb{Z}[S_1]$. Such an f is an isomorphism and its inverse $f^{-1} = \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}$ is also elementary.

An isomorphism $f : \mathbb{Z}[S] \to \mathbb{Z}[S']$ between geometric modules of the same rank is *geometric* if there is a bijection $\varphi : S \to S'$ such that f has no paths from $s \in S$ to

$s' \in S'$ unless $s' = \varphi(s)$ and there is exactly one path from s to $\varphi(s)$ for each $s \in S$, whose coefficient is ± 1. Its inverse f^{-1} is obtained by reversing the orientation of the paths.

A *deformation* is a sequence

$$D \ : \ \mathbb{Z}[S_1] \xrightarrow{f_1} \mathbb{Z}[S_2] \xrightarrow{f_2} \ldots \xrightarrow{f_m} \mathbb{Z}[S_{m+1}]$$

of elementary automorphisms and geometric isomorphisms. D is an ϵ *deformation* if all composite geometric morphisms $f_j f_{j-1} \cdots f_i$, $f_i^{-1} f_{i+1}^{-1} \cdots f_j^{-1}$ have radius ϵ. (If $f_m f_{m-1} \cdots f_1$ has radius δ, then all the composites $f_j f_{j-1} \cdots f_i$ have radius 2δ, and similarly for the inverses.) When D is an ϵ deformation, the composite ϵ isomorphism $f = f_m f_{m-1} \cdots f_1$ is called an *ϵ-simple isomorphism*. The composite $f^{-1} = f_1^{-1} f_2^{-1} \cdots f_m^{-1}$ gives an ϵ inverse of f. The composite of an ϵ-simple isomorphism and a δ-simple isomorphism is an $(\epsilon + \delta)$-simple isomorphism.

Definition. A free chain complex of the form:

$$\ldots \to 0 \to 0 \to A \xrightarrow{1} A \to 0 \to 0 \to \ldots$$

is called an *elementary trivial chain complex*. A free chain complex T is *trivial* if it is the direct sum of elementary trivial chain complexes. A trivial chain complex is 0 contractible. An ϵ chain map $f = \{f_r\} : C \to D$ between free chain complexes on p_X is an *ϵ-simple isomorphism* if each f_r is an ϵ-simple isomorphism and $f^{-1} = \{f_r^{-1}\}$ is an ϵ chain map. We use the notation: $f : C \cong_{\epsilon,\Sigma} D$. The composite of an ϵ-simple isomorphism and a δ-simple isomorphism (of chain complexes) is an $(\epsilon + \delta)$-simple isomorphism. Let n be an integer. Two free chain complexes C and C' on p_X are *n-stable ϵ-simple equivalent* if there exists an ϵ-simple isomorphism between $C \oplus T$ and $C' \oplus T'$ for some n-dimensional trivial chain complexes T and T' on p_X.

Warning. Do not confuse "n-stable ϵ-simple equivalences" (of free chain complexes) with "n-stable ϵ chain equivalences" (of projective chain complexes) defined in the previous section. Any two free ϵ chain complexes of dimension $\leq n$ are always n-stable ϵ chain equivalent because they are free.

Let Y be a subspace of X. The *restriction to* Y of a geometric module $A = \mathbb{Z}[S]$ on p_X is the geometric module on p_Y generated by the elements $(|s|, [s])$ of S such that $[s] \in p_X^{-1}(Y)$ and is denoted $A(Y)$; i.e.,

$$A(Y) = \mathbb{Z}[S | S^{-1} p_X^{-1}(Y) : S^{-1} p_X^{-1}(Y) \longrightarrow p_X^{-1}(Y)].$$

(Recall that p_Y is a restriction of p_X.) The *restriction* to Y of a geometric morphism $f = \sum m_\lambda (s_\lambda, \rho_\lambda, t_\lambda) : A \to B$ is defined to be:

$$f|Y = \sum_{[s_\lambda] \in p_X^{-1}(Y)} m_\lambda (s_\lambda, \rho_\lambda, t_\lambda) : A \to B.$$

Note that f and $f|Y$ have the same domain. Of course one can also restrict the domain and the target: If f has radius ϵ, then $f|Y$ determines a geometric morphism from $A(Y)$ to $B(Y^\epsilon)$, which will be also denoted $f|Y$. Suppose f , $g : A \to B$ be two geometric morphisms. When $f|Y = g|Y$, we write $f = g$ *over* Y. When $f|Y \sim_\epsilon g|Y$, we write $f \sim_\epsilon g$ *over* Y.

Consider an elementary geometric morphism

$$f = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} : \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2].$$

If we replace h by the morphism $h|(X - Y)$, then we get another elementary geometric morphism $f_{[Y]}$ which coincides with f over $X - Y$ and is the identity over Y. $f_{[Y]}$ is called the *localization of* f *away from* Y. Note that $(f_{[Y]})^{-1} = (f^{-1})_{[Y]}$. For a deformation $D = (f_1, f_2, \dots, f_m)$, define the *localization of* D *away from* Y by $(f'_1, f'_2, \dots, f'_m)$, where $f'_j = (f_j)_{[Y]}$ if f_j is elementary and $f'_j = f_j$ if f_j is geometric. The composite $f'_m \cdots f'_1$ is called the *localization away from* Y of the simple isomorphism $f = f_m \cdots f_1$, and is denoted $f_{[Y]}$. If f is an ϵ-simple isomorphism, then $f_{[Y]}$ is also an ϵ-simple isomorphism, coincides with f over $X - Y^\epsilon$, and is geometric over $Y^{-\epsilon}$.

Definition. (1) Let $f,\ g : C \to D$ be chain maps between free chain complexes on p_X. A collection $h = \{h_r\}$ of ϵ morphisms is an ϵ *chain homotopy over* Y between f and g, $h : f \simeq_Y g$, if dh and hd both have radius 2ϵ and $dh + hd \sim_{2\epsilon} g - f$ over Y.
(2) An ϵ chain map $f : C \to D$ is an ϵ *chain equivalence over* Y if there exist an ϵ chain map $g : D \to C$ and ϵ chain homotopies over Y:$gf \simeq_Y p$ and $fg \simeq_Y q$.
(3) An ϵ chain homotopy $h : 0 \simeq_Y 1 : C \to C$ over Y is called an ϵ *chain contraction over* Y, and C is said to be ϵ contractible over Y.
(4) A *strong* ϵ *chain contraction* Γ *over* Y of C, will mean an ϵ chain contraction of C over Y which satisfies the additional condition $\Gamma_{r+1}\Gamma_r \sim_{2\epsilon} 0$ over Y. If such a Γ exists, we say C is *strongly* ϵ *contractible over* Y (or *strongly* ϵ *contractible* if $Y = X$). This extra condition can be achieved in the following way. Suppose Γ is an ϵ chain contraction of C over Y. Then $\Gamma' = \Gamma d \Gamma$ is a strong 3ϵ chain contraction of C over $Y^{-3\epsilon}$. (We used this construction in the proof of 3.4.)
(5) Two free chain complexes C and C' on p_X are said to be *n-stable ϵ-simple equivalent away from* Y if there exist n-dimensional free ϵ chain complexes D and D' on p_Y such that $C \oplus D$ and $C' \oplus D'$ are n-stable ϵ-simple equivalent. We use the notation: $C \underset{\epsilon}{\overset{n,Y}{\sim}} C'$. For example, an n-stable ϵ-simple equivalence away from the empty subset $\emptyset$ is the same as an n-stable ϵ-simple equivalence. For a fixed $\epsilon > 0$, n-stable ϵ-simple equivalence away from Y is *not* an equivalence relation; in general, $C \underset{\epsilon}{\overset{n,Y}{\sim}} C'$ and $C' \underset{\delta}{\overset{m,Z}{\sim}} C''$ imply $C \underset{\epsilon+\delta}{\overset{\max\{n,m\},Y\cup Z}{\sim}} C''$.

Definition. Let Y be a subspace of X. $Wh(X, Y, p_X, n, \epsilon)$ is defined to be the set of equivalence classes of n-dimensional free ϵ chain complexes on p_X which are strongly ϵ contractible over $X - Y$. The equivalence relation is generated by n-stable 40ϵ-simple equivalences away from $Y^{20\epsilon}$. If Y is the empty set, it will be omitted from the notation, and if $n = 1$, then n is omitted; e.g., $Wh(X, p_X, n, \epsilon) = Wh(X, \emptyset, p_X, n, \epsilon)$ and $Wh(X, Y, p_X, \epsilon) = Wh(X, Y, p_X, 1, \epsilon)$, etc.

Remark. "$Y^{20\epsilon}$" is used in this definition instead of "Y" so that any element of $Wh(X, Y, p_X, n, \epsilon)$ has an additive inverse. See 4.1 below. 4.1 also says that the equivalence relation generated by n-stable 40ϵ-simple equivalences away from $Y^{20\epsilon}$ implies n-stable 86ϵ-simple equivalence away from $Y^{20\epsilon}$. (Recall that n-stable 40ϵ-simple equivalences away from $Y^{20\epsilon}$ are not equivalence relations in general.) The proof of 4.1 is rather long and occupies the next several pages.

Proposition 4.1. *Direct sum induces an abelian group structure on $Wh(X, Y, p_X, n, \epsilon)$. Further if $[C] = [C'] \in Wh(X, Y, p_X, n, \epsilon)$, then C and C' are n-stable 86ϵ-simple equivalent away from $Y^{20\epsilon}$.*

We need to show the existence of additive inverses. The next lemma shows that the suspension ΣC of C (or the suspension of anything which looks the same as C over $X - Y$) gives the additive inverse of $[C]$ at least when $\dim C < n$.

Lemma 4.2. *Suppose $C = \{C_r, d_r\}$ (resp. $C' = \{C'_r, d'_r\}$) is a free ϵ (resp. ϵ') chain complex of dimension m (resp. m') on p_X. Let Y be a subspace of X and assume that*

1. *C has an ϵ chain contraction Γ over $X - Y^\epsilon$,*
2. *$C_r(X - Y) = C'_r(X - Y)$ for all r, and*
3. *$d_r|X - Y^\epsilon = d'_r|X - Y^\epsilon : C_r(X - Y^\epsilon) \to C_{r-1}(X - Y)$ for all r.*

Let $\gamma = \max\{\epsilon, \epsilon'\}$, and $n = \max\{m + 1, m'\}$. Then there is a $(6\epsilon + \gamma)$-simple isomorphism from $C' \oplus \Sigma C$ to the direct sum of an n-dimensional free $4\epsilon + \gamma$ chain complex on $p_{Y^{11\epsilon+2\gamma}}$ and an n-dimensional trivial chain complex on p_X. In particular, $C' \oplus \Sigma C$ is n-stable $(6\epsilon + \gamma)$-simple equivalent to 0 away from $Y^{11\epsilon+2\gamma}$.

Proof : Define ϵ morphisms $\hat{d}_r : C'_r \to C_{r-1}$ and $\hat{\Gamma}_r : C_r \to C'_{r+1}$ by :

$$\hat{d}_r = \begin{cases} d_r & \text{over } X - Y^\epsilon \\ 0 & \text{over } Y^\epsilon \end{cases}$$

$$\widehat{\Gamma}_r = \begin{cases} \Gamma_r & \text{over } X - Y^\epsilon \\ 0 & \text{over } Y^\epsilon. \end{cases}$$

Consider the following 2ϵ-simple isomorphism and its inverse:

$$
\begin{aligned}
f_r \;&=\; \begin{pmatrix} (-)^r 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (-)^{r-1}\hat{d} & 1 \end{pmatrix} \begin{pmatrix} 1 & (-)^r \widehat{\Gamma} \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} (-)^r 1 & \widehat{\Gamma} \\ (-)^{r-1}\hat{d} & -\hat{d}\,\widehat{\Gamma}+1 \end{pmatrix} \\
&\quad : (C' \oplus \Sigma C)_r \;=\; C'_r \oplus C_{r-1} \longrightarrow C'_r \oplus C_{r-1} \ , \\
f_r^{-1} \;&=\; \begin{pmatrix} 1 & (-)^{r+1}\widehat{\Gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (-)^r \hat{d} & 1 \end{pmatrix} \begin{pmatrix} (-)^r 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (-)^r(1-\widehat{\Gamma}\hat{d}\,) & (-)^{r+1}\widehat{\Gamma} \\ \hat{d} & 1 \end{pmatrix} \\
&\quad : C'_r \oplus C_{r-1} \longrightarrow (C' \oplus \Sigma C)_r \;=\; C'_r \oplus C_{r-1} \ ,
\end{aligned}
$$

and define a new chain complex $\bar{C} = \{\bar{C}_r, \bar{d}_r\}$ by

$$
\begin{aligned}
\bar{C}_r \;&=\; C'_r \oplus C_{r-1} \ , \\
\bar{d}_r \;&=\; f_{r-1}(d'_r \oplus d_{r-1}) f_r^{-1} = \begin{pmatrix} -d' + (d'\widehat{\Gamma} + \widehat{\Gamma} d)\hat{d} & d'\widehat{\Gamma} + \widehat{\Gamma} d \\ \hat{d} d' + d\hat{d} - \hat{d}(d'\widehat{\Gamma} + \widehat{\Gamma} d)\hat{d} & d - \hat{d}(d'\widehat{\Gamma} + \widehat{\Gamma} d) \end{pmatrix} .
\end{aligned}
$$

Then $\bar{d}_r \bar{d}_{r+1} \sim_{8\epsilon+2\gamma} 0$, and $\bar{d}_r f_r \sim_{6\epsilon+\gamma} f_{r-1}(d'_r \oplus d_{r-1})$. Since $d'\widehat{\Gamma} + \widehat{\Gamma} d \sim_{2\epsilon} 1$ over $X - Y^{2\epsilon}$, we have

$$
\bar{d}_r \sim_{4\epsilon} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{over } \ X - Y^{3\epsilon} \ .
$$

Modify $\bar{d}_r$ over $X - Y^{3\epsilon}$ by a $3\epsilon + \gamma$ homotopy to get a $4\epsilon + \gamma$ chain complex $\widetilde{C} \;=\; \{\bar{C}_r, \tilde{d}_r\}$:

$$
\tilde{d}_r \;=\; \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{over } X - Y^{3\epsilon} \\ \bar{d}_r & \text{over } Y^{3\epsilon} \ . \end{cases}
$$

$f = \{f_r\}$ can be thought of as a $(6\epsilon + \gamma)$-simple isomorphism from $C' \oplus \Sigma C$ to $\widetilde{C}$. $\widetilde{C}$ is a direct sum of its restrictions

$$
\begin{aligned}
\widetilde{C}(Y^{11\epsilon+2\gamma}) &= \{C_r(Y^{11\epsilon+2\gamma}),\ d_r|Y^{11\epsilon+2\gamma}\ \}, \\
\widetilde{C}(X - Y^{11\epsilon+2\gamma}) &= \{C_r(X - Y^{11\epsilon+2\gamma}),\ d_r|X - Y^{11\epsilon+2\gamma}\ \}.
\end{aligned}
$$

$\widetilde{C}(X - Y^{11\epsilon+2\gamma})$ is a trivial complex, and $\widetilde{C}(Y^{11\epsilon+2\gamma})$ is a free $4\epsilon + \gamma$ chain complex on $p_{Y^{11\epsilon+2\gamma}}$. □

To get an n-dimensional inverse when $n > 1$, we use the following folding argument. A dual argument (folding up from the bottom dimension) was used in Yamasaki [26].

Lemma 4.3. *Let Y be a subspace of X and C be an n-dimensional free ϵ chain complex ($n > 1$) on p_X with a strong ϵ chain contraction Γ over $X - Y$. Then C is n-stable 16ϵ-simple equivalent away from $Y^{17\epsilon}$ to the $(n-1)$-dimensional free ϵ chain complex:*

$$\{\widehat{C}, \hat{d}\,\} \;:\; \ldots \to 0 \to C_{n-1} \xrightarrow{\binom{d}{\Gamma}} C_{n-2} \oplus C_n \xrightarrow{(d\;\;0)} C_{n-3} \xrightarrow{d} \ldots \xrightarrow{d} C_0 \to 0$$

which has a strong ϵ chain contraction over $X - Y$.

Proof : Let i, j (resp. r, q) be inclusion maps (resp. projections) of $C_n(Y)$, $C_n(X-Y)$ into C_n (resp. C_n to $C_n(Y)$, $C_n(X-Y)$). By assumption, we have a homotopy: $\Gamma dj \sim_{2\epsilon} j : C_n(X-Y) \to C_n$. Consider the following 3ϵ chain complex C':

$$\ldots \to 0 \to C_n(Y) \xrightarrow{\Delta} C_{n-1} \xrightarrow{\binom{d}{q\Gamma}} C_{n-2} \oplus C_n(X-Y) \xrightarrow{(d\;\;0)} C_{n-3} \xrightarrow{d} \ldots \xrightarrow{d} C_0 \to 0$$

where $\Delta = d_n i - d_n j q \Gamma d_n i$.

Then the following diagram gives an n-stable 4ϵ equivalence between C and C':

$$\begin{array}{ccccccc}
0 \longrightarrow C_n & \xrightarrow{\quad d \quad} & C_{n-1} & \xrightarrow{\quad d \quad} & C_{n-2} & \xrightarrow{\quad d \quad} & C_{n-3} \to \ldots \\
 & & \oplus & & \oplus & & \\
\Big\downarrow f_n & & C_n(X-Y) & \xrightarrow{\;1\;} & C_n(X-Y) & & \Big\| \\
 & & \Big\downarrow f_{n-1} & & \Big\downarrow f_{n-2} & & \\
0 \longrightarrow C_n(Y) & \longrightarrow & C_{n-1} & \longrightarrow & C_{n-2} \oplus C_n(X-Y) & \longrightarrow & C_{n-3} \to \ldots \\
\oplus & & \oplus & & & & \\
C_n(X-Y) & \xrightarrow[1]{} & C_n(X-Y) & & & &
\end{array}$$

where

$$\begin{aligned}
f_n &= \begin{pmatrix} 1 & 0 \\ q\Gamma di & 1 \end{pmatrix} \;:\; C_n \;=\; C_n(Y) \oplus C_n(X-Y) \longrightarrow C_n(Y) \oplus C_n(X-Y) \\
f_{n-1} &= \begin{pmatrix} 1 & -dj \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q\Gamma & 1 \end{pmatrix} \;:\; C_{n-1} \oplus C_n(X-Y) \longrightarrow C_{n-1} \oplus C_n(X-Y) \\
f_{n-2} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;:\; C_{n-2} \oplus C_n(X-Y) \longrightarrow C_{n-2} \oplus C_n(X-Y)\,.
\end{aligned}$$

Note the following:

1. $\widehat{C}$ has a strong ϵ chain contraction $\widehat{\Gamma}$ over $X - Y$ defined by:

$$\widehat{\Gamma}_{n-2} = (\,\Gamma_{n-2} \quad d\,) \quad , \quad \widehat{\Gamma}_{n-3} = \begin{pmatrix} \Gamma_{n-3} \\ 0 \end{pmatrix} \quad , \quad \widehat{\Gamma}_r = \Gamma_r \quad (r < n-3),$$

2. $\widehat{C}_r(X - Y) = C'_r(X - Y)$ for all r,
3. $\hat{d}_r|X - Y^\epsilon = d'_r|X - Y^\epsilon : C_r(X - Y^\epsilon) \to C_r(X - Y)$ for all r.

By 4.2, $C' \oplus \Sigma\widehat{C}$ is n-stable 9ϵ-simple equivalent to 0 away from $Y^{17\epsilon}$. Also note that $\widehat{C} \oplus \Sigma\widehat{C}$ is n-stable 7ϵ-simple equivalent to 0 away from $Y^{13\epsilon}$, again by 4.2. Therefore C and $\widehat{C}$ are n-stable 16ϵ-simple equivalent away from $Y^{17\epsilon}$:

$$C \underset{7\epsilon}{\overset{n,Y^{13\epsilon}}{\sim}} C' \oplus \Sigma\widehat{C} \oplus \widehat{C} \underset{9\epsilon}{\overset{n,Y^{17\epsilon}}{\sim}} \widehat{C}. \qquad \square$$

Corollary 4.4. *Let $n > 1$. Then $[\Sigma\widehat{C}]$ is the additive inverse of $[C]$ in $Wh(X, Y, p_X, n, \epsilon)$. In fact there is a n-stable 23ϵ-simple equivalence away from $Y^{17\epsilon}$ between $C \oplus \Sigma\widehat{C}$ and 0.*

Proof : $C \oplus \Sigma\widehat{C} \underset{16\epsilon}{\overset{n,Y^{17\epsilon}}{\sim}} \widehat{C} \oplus \Sigma\widehat{C} \underset{7\epsilon}{\overset{n,Y^{13\epsilon}}{\sim}} 0.$ $\square$

The existence of inverses when $n = 1$ is a special case ($C'_1 = C_0$, $C'_0 = C_1$, $d' = \Gamma$, $\epsilon = \epsilon' = \gamma$) of the next lemma.

Lemma 4.2$'$. *Let $C : 0 \to C_1 \xrightarrow{d} C_0 \to 0$ be a 1-dimensional free ϵ chain complex and $C' : 0 \to C'_1 \xrightarrow{d'} C'_0 \to 0$ be a 1-dimensional free ϵ' chain complex. Assume*
1. *C has an ϵ chain contraction Γ over $X - Y^\epsilon$.*
2. *$C'_1(X - Y) = C_0(X - Y), \quad C'_0(X - Y) = C_1(X - Y)$*
3. *$d'|X - Y^\epsilon \sim_\epsilon \Gamma|X - Y^\epsilon : C'_1(X - Y^\epsilon) \to C'_0(X - Y)$,*

and let $\gamma = \max\{\epsilon, \epsilon'\}$. Then the direct sum $C \oplus C'$ is $(5\epsilon+\gamma)$-simple isomorphic to the direct sum of a 1-dimensional free $3\epsilon+\gamma$ chain complex on $p_{Y^{5\epsilon+\gamma}}$ and a 1-dimensional trivial chain complex on p_X. In particular $C \oplus C'$ is 1-stable $(5\epsilon+\gamma)$-simple equivalent to 0 away from $Y^{5\epsilon+\gamma}$.

Proof : Define ϵ morphisms $\tilde{d} : C'_0 \to C_0$, $\widetilde{\Gamma} : C'_1 \to C_1$, $\widehat{\Gamma} : C_0 \to C'_0$ by d, Γ, Γ over $X - Y^\epsilon$ and by 0, 0, 0 over Y^ϵ. Define a 1-dimensional free $3\epsilon + \gamma$ chain complex E by:

$$E_1 = C'_1 \oplus C_1 \quad , \quad E_0 = C_0 \oplus C'_0$$

$$d_E = \begin{cases} \begin{pmatrix} d\widetilde{\Gamma} - \tilde{d}\,\widehat{\Gamma}d\widetilde{\Gamma} + \tilde{d}d' & d - \tilde{d}\,\widehat{\Gamma}d \\ \widehat{\Gamma}d\widetilde{\Gamma} - d' & \widehat{\Gamma}d \end{pmatrix} & \text{over } Y^{2\epsilon} \\ 1 & \text{over } X - Y^{2\epsilon}\ . \end{cases}$$

Define an ϵ-simple isomorphism $f_1 : C_1 \oplus C_1' \to E_1$ and a 2ϵ-simple isomorphism $f_0 : C_0 \oplus C_0' \to E_0$ by

$$f_1 = \begin{pmatrix} 0 & 1 \\ 1 & -\widetilde{\Gamma} \end{pmatrix}, \qquad f_0 = \begin{pmatrix} 1 & -\tilde{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \widehat{\Gamma} & -1 \end{pmatrix} .$$

A direct calculation shows that $d_E \sim_{3\epsilon+\gamma} f_0(d \oplus d')f_1^{-1}$, and one can check that $f = \{f_r\} : C \oplus C' \to E$ is a $(5\epsilon+\gamma)$-simple isomorphism. E is a direct sum of the free $3\epsilon+\gamma$ chain complex $E(Y^{5\epsilon+\gamma})$ on $p_{Y^{5\epsilon+\gamma}}$ and the trivial chain complex $E(X - Y^{5\epsilon+\gamma})$. □

This completes the proof of the existence of additive inverses. Next, suppose that $[C] = [C'] \in Wh(X, Y, p_X, n, \epsilon)$. One can argue as in the previous section using Chapman's trick to show that C and C' are n-stable 86ϵ-simple equivalent away from $Y^{20\epsilon}$: By definition there are elements $[C^{(i)}] \in Wh(X, Y, p_X, n, \epsilon)$ such that

$$C = C^{(0)} \underset{40\epsilon}{\overset{n,Y^{20\epsilon}}{\sim}} C^{(1)} \underset{40\epsilon}{\overset{n,Y^{20\epsilon}}{\sim}} \cdots \underset{40\epsilon}{\overset{n,Y^{20\epsilon}}{\sim}} C^{(m)} = C'.$$

By 4.4 and 4.2$'$, there are elements $[D^{(i)}] \in Wh(X, Y, p_X, n, \epsilon)$ such that

$$C^{(i)} \oplus D^{(i)} \underset{23\epsilon}{\overset{n,Y^{17\epsilon}}{\sim}} 0 \qquad \text{for } i = 0, 1, \ldots, m.$$

Then

$$\begin{aligned} C &\underset{23\epsilon}{\overset{n,Y^{17\epsilon}}{\sim}} C \oplus (C^{(0)} \oplus D^{(0)}) \oplus \ldots \oplus (C^{(m-1)} \oplus D^{(m-1)}) \oplus (C^{(m)} \oplus D^{(m)}) \\ &\underset{40\epsilon}{\overset{n,Y^{20\epsilon}}{\sim}} C \oplus (C^{(1)} \oplus D^{(0)}) \oplus \ldots \oplus (C^{(m)} \oplus D^{(m-1)}) \oplus (C^{(m)} \oplus D^{(m)}) \\ &= \quad (C \oplus D^{(0)}) \oplus (C^{(1)} \oplus D^{(1)}) \oplus \ldots \oplus (C^{(m)} \oplus D^{(m)}) \oplus C^{(m)} \\ &\underset{40\epsilon}{\overset{n,Y^{20\epsilon}}{\sim}} C'. \end{aligned}$$

Therefore $C \underset{86\epsilon}{\overset{n,Y^{20\epsilon}}{\sim}} C'$. This ends the proof of 4.1.

The next proposition gives a sufficient condition for two chain complexes to represent the same class in the relative controlled Whitehead group.

Proposition 4.5. *Suppose $[C, d]$ and $[C', d']$ are elements of $Wh(X, Y, p_X, n, \epsilon)$. If $C_r(X - Y) = C_r'(X - Y)$ and $d_r|X - Y^\epsilon = d_r'|X - Y^\epsilon$ for every r , then $[C] = [C']$.*

Proof : We first consider the case $n > 1$. Let Γ be a strong ϵ chain contraction over $X - Y$ of C. Define $\Gamma' : C_r' \to C_{r+1}'$ $(r \in \mathbb{Z})$ by

$$\Gamma' \; = \; \begin{cases} \Gamma & \text{over } X - Y^\epsilon \\ 0 & \text{over } Y^\epsilon \; . \end{cases}$$

Then Γ' is a strong ϵ chain contraction over $X - Y^{2\epsilon}$ of C'. Let $\widehat{C}$ and $\widehat{C}'$ be the $(n-1)$-dimensional chain complexes obtained by applying 4.3 to C, Γ and C', Γ' respectively. Then

$$C \underset{16\epsilon}{\overset{n,Y^{17\epsilon}}{\sim}} \widehat{C} \quad , \quad C' \underset{16\epsilon}{\overset{n,Y^{19\epsilon}}{\sim}} \widehat{C}' \ ,$$

and by 4.2 we have

$$\widehat{C} \oplus \Sigma\widehat{C} \underset{7\epsilon}{\overset{n,Y^{13\epsilon}}{\sim}} 0 \quad , \quad \widehat{C}' \oplus \Sigma\widehat{C} \underset{7\epsilon}{\overset{n,Y^{13\epsilon}}{\sim}} 0 \ .$$

Composing the equivalences:

$$C \underset{16\epsilon}{\overset{n,Y^{17\epsilon}}{\sim}} \widehat{C} \oplus \Sigma\widehat{C} \oplus \widehat{C}' \underset{16\epsilon}{\overset{n,Y^{19\epsilon}}{\sim}} C',$$

we get an n-stable 32ϵ-simple equivalence away from $Y^{19\epsilon}$ between C and C'.

The case $n = 1$ is similar; use the additive inverse of C (Lemma 4.2′) instead of $\Sigma\widehat{C}$ above. $\square$

Suppose $p_X : M \to X$ and $p_{X'} : M' \to X'$ are control maps and Y, Y' are subspaces of X and X'. If a map $\Phi = (\varphi, \bar{\varphi}) : p_X \to p_{X'}$ satisfies $\bar{\varphi}(Y) \subset Y'$ and the conditions $\mathbf{C}(\delta, \epsilon, 1)$, $\mathbf{C}(\delta, \epsilon, 2)$, $\mathbf{C}(\delta, \epsilon, 20)$, $\mathbf{C}(\delta, \epsilon, 40)$, then it induces a homomorphism $\Phi_* : Wh(X, Y, p_X, \delta) \to Wh(X', Y', p_{X'}, \epsilon)$. The equality $(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*$ holds.

As in the case of $\widetilde{K}_0$, an inclusion map $i : (A, B) \to (X, Y)$ induces a homomorphism

$$i_* \ : \ Wh(A, B, p_A, n, \epsilon) \longrightarrow Wh(X, Y, p_X, n, \epsilon) \ .$$

And more generally, if $\delta \le \epsilon$, there is a *stabilization map*

$$Wh(A, B, p_A, n, \delta) \longrightarrow Wh(X, Y, p_X, n, \epsilon) \ .$$

The groups $Wh(X, Y, p_X, n, \epsilon)$ and $Wh(X, Y, p_X, \epsilon)$ are only stably isomorphic. If $m < n$, there is a canonical homomorphism

$$\iota : Wh(X, Y, p_X, m, \epsilon) \to Wh(X, Y, p_X, n, \epsilon)$$

that sends $[C]$ to $[C]$. Fix an integer $n > 1$.

Proposition 4.6. *The map $\iota : Wh(X, Y, p_X, \epsilon) \to Wh(X, Y, p_X, n, \epsilon)$ is onto: if C is an n-dimensional free ϵ chain complex with a strong ϵ chain contraction Γ over $X - Y$, then the 1-dimensional free ϵ chain complex*

$$C_\Gamma \ : \ C_{\mathrm{odd}} \ = \ C_1 \oplus C_3 \oplus \dots \xrightarrow{d+\Gamma} C_{\mathrm{even}} \ = \ C_0 \oplus C_2 \oplus \dots$$

with the strong ϵ chain contraction $d + \Gamma : C_{\mathrm{even}} \to C_{\mathrm{odd}}$ over $X - Y$ represents the same element as C in $Wh(X, Y, p_X, n, \epsilon)$.

Proof : Lemma 4.3 says that any element $\tau(C) \in Wh(X, Y, p_X, n, \epsilon)$ comes from an element $\tau(\hat{C}) \in Wh(X, Y, p_X, n-1, \epsilon)$. So we can repeatedly use 4.3 to show the surjectivity.

$\square$

Proposition 4.7. *This correspondence $[C] \mapsto [C_\Gamma]$ defines a well-defined homomorphism*

$$\tau \ : \ Wh(X, Y, p_X, n, \epsilon) \longrightarrow Wh(X, Y^{(90n+250)\epsilon}, p_X, (90n+250)\epsilon) \ .$$

Proof : We first show that the class $[C_\Gamma]$ is independent of the choice of Γ. If Γ' is another strong ϵ chain contraction over $X - Y$ of C, there is a homotopy:

$$(1 + \Gamma'\Gamma)(d + \Gamma) \sim_{3\epsilon} (d + \Gamma')(1 + \Gamma'\Gamma) : C_{\text{odd}} \to C_{\text{even}} \qquad \text{over} \ \ X - Y^{\epsilon}.$$

Here the two morphisms $1 + \Gamma'\Gamma : C_{\text{odd}} \to C_{\text{odd}}$ and $1 + \Gamma'\Gamma : C_{\text{even}} \to C_{\text{even}}$ are $n\epsilon$-simple isomorphisms; in fact, they can be written as products:

$$\begin{array}{l} (1 + \Gamma'\Gamma|C_1)(1 + \Gamma'\Gamma|C_3) \cdots \\ (1 + \Gamma'\Gamma|C_0)(1 + \Gamma'\Gamma|C_2) \cdots \end{array}$$

respectively. Furthermore we have $n\epsilon$ homotopies

$$(1 + \Gamma'\Gamma)(1 + \Gamma'\Gamma)^{-1} \sim_{n\epsilon} 1, \qquad (1 + \Gamma'\Gamma)^{-1}(1 + \Gamma'\Gamma) \sim_{n\epsilon} 1.$$

(To see this, use the identity

$$(1 + \Gamma'\Gamma)^{-1} = 1 - \Gamma'\Gamma + (\Gamma'\Gamma)^2 - \cdots + (-1)^k(\Gamma'\Gamma)^k,$$

where $k = \left[\frac{n}{2}\right]$.) Define an $(n+3)\epsilon$ morphism $\Delta : C_{\text{odd}} \to C_{\text{even}}$ by:

$$\Delta \ = \ (1 + \Gamma'\Gamma)(d + \Gamma)(1 + \Gamma'\Gamma)^{-1}.$$

Then Δ, viewed as a 1-dimensional chain complex, represents an element of $Wh(X, Y^{n\epsilon}, p_X, (n+3)\epsilon)$; an $(n+3)\epsilon$ chain contraction over $X - Y^{n\epsilon}$ is given by

$$h \ = \ (1 + \Gamma'\Gamma)(d + \Gamma)(1 + \Gamma'\Gamma)^{-1}.$$

The identity above implies that C_Γ and Δ are $n\epsilon$-simple isomorphic; therefore, $[C_\Gamma] = [\Delta] \in Wh(X, Y^{(n+1)\epsilon}, p_X, (n+3)\epsilon)$. Next we compare Δ and $C_{\Gamma'}$. It turns out that

$$\Delta \sim_{(n+3)\epsilon} d + \Gamma' \qquad \text{over} \quad X - Y^{(n+1)\epsilon} \ .$$

Modify Δ by a homotopy over $X - Y^{(n+1)\epsilon}$ to get Δ' representing the same class as Δ and satisfying the strict identity:

$$\Delta' \ = \ d + \Gamma' \qquad \text{over} \quad X - Y^{(n+1)\epsilon} \ .$$

By 4.5, Δ' and $C_{\Gamma'}$ represent the same class in $Wh(X, Y^{n\epsilon}, p_X, (n+3)\epsilon)$. Therefore C_Γ and $C_{\Gamma'}$ represent the same class in $Wh(X, Y^{n\epsilon}, p_X, (n+3)\epsilon)$.

Next, suppose we are given two elements $[C]$, $[C'] \in Wh(X, Y, p_X, n, \epsilon)$ and assume that there is a 40ϵ-simple isomorphism $f : \bar{C} = C \oplus D \to \widehat{C} = C' \oplus D'$, where D and D' are n-dimensional free 40ϵ chain complexes on $Y^{20\epsilon}$. If Γ (resp. Γ') is a strong ϵ chain contraction of C (resp. C') over $X - Y$, then $\bar{\Gamma} = \Gamma \oplus 0$(resp. $\widehat{\Gamma} = \Gamma' \oplus 0$) is a strong ϵ chain contraction over $X - Y^{20\epsilon}$ of the free 40ϵ chain complex $\bar{C}$ (resp. $\widehat{C}$), and $\widehat{\Gamma}' = f\bar{\Gamma}f^{-1}$ is a strong 81ϵ chain contraction over $X - Y^{60\epsilon}$ of $\widehat{C}$. And f induces a 121ϵ-simple isomorphism between the 1-dimensional chain complexes $\widehat{C}_{\widehat{\Gamma}'}$ and $\bar{C}_{\bar{\Gamma}}$. C_Γ and $\bar{C}_{\bar{\Gamma}}$ (resp. $C'_{\Gamma'}$ and $\widehat{C}_{\widehat{\Gamma}}$) represent the same element in $Wh(X, Y^{20\epsilon}, p_X, 40\epsilon)$. Finally $\widehat{C}_{\widehat{\Gamma}}$ and $\widehat{C}_{\widehat{\Gamma}'}$ represent the same element in $Wh(X, (Y^{60\epsilon})^{81n\epsilon}, p_X, 81(n+3)\epsilon)$, by the argument in the preceding paragraph. Therefore $[C_\Gamma] = [C'_{\Gamma'}]$ in $Wh(X, Y^{(81n+60)\epsilon}, p_X, (81n+243)\epsilon)$.

Adding a trivial complex to C corresponds to adding a trivial complex to C_Γ, so it does not change the class of C_Γ. Thus τ is well-defined. It is obviously a homomorphism.

$\square$

The homomorphisms ι, τ are stable inverses as in the previous section, and we have similar commutative diagrams.

If $f : C \to D$ is an ϵ chain equivalence between n-dimensional free ϵ chain complexes on p_X, then $\mathcal{C}(f)$ is strongly 9ϵ contractible. We define the *torsion* $\tau(f)$ *of* f by

$$\tau(f) \;=\; [\mathcal{C}(f)] \in Wh(X, p_X, n+1, 9\epsilon)\ .$$

(When $n = 0$, i.e. f is an ϵ isomorphism, f is identified with $\mathcal{C}(f)$, and its torsion $\tau(f)$ is defined in $Wh(X, p_X, \epsilon)$.)

Proposition 4.8. *If $f : C \to D$ and $g : D \to E$ are ϵ chain equivalences between n-dimensional free ϵ chain complexes on p_X, then*

$$\tau(gf) \;=\; \tau(g) + \tau(f) \in Wh(X, p_X, n+1, 18\epsilon)\ .$$

When $n = 0$, the equality holds in $Wh(X, p_X, 2\epsilon)$.

Proof: Define a trivial chain complex $\{T, d_T\}$ by

$$(d_T)_r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : T_r = D_r \oplus D_{r-1} \longrightarrow D_{r-1} \oplus D_{r-2} = T_{r-1}.$$

Then a 5ϵ-simple isomorphism $\mathcal{C}(f) \oplus \mathcal{C}(g) \to \mathcal{C}(gf) \oplus T$ is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (-)^r 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (-)^r d & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ g & 0 & 1 & 0 \\ 0 & f & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & (-)^r k \\ 0 & 1 & 0 & f^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$: D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1} \longrightarrow D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1},$$

where f^{-1} is an ϵ chain homotopy inverse of f and k is an ϵ chain homotopy $ff^{-1} \simeq_\epsilon 1$. (Cf. Ranicki [20, 4.2 i)].) $\square$

5. Relative K-theory.

The homology groups $H_*(X,Y)$ of a pair of spaces $(X, Y \subseteq X)$ are such that there is defined an exact sequence

$$\ldots \longrightarrow H_n(Y) \longrightarrow H_n(X) \longrightarrow H_n(X,Y) \longrightarrow H_{n-1}(Y) \longrightarrow \ldots .$$

Analogously, for any group homomorphism $\rho \to \pi$, there is an exact sequence

$$Wh(\rho) \longrightarrow Wh(\pi) \longrightarrow Wh(\pi,\rho) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi]).$$

The relative group $Wh(\pi,\rho)$ is a quotient of the Grothendieck group of triples (C, D, f) with C a finite f.g. projective $\mathbb{Z}[\rho]$-module chain complex, D a finite based f.g. free $\mathbb{Z}[\pi]$-module chain complex, and $f : \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\rho]} C \simeq D$ a chain equivalence. We shall now use the algebraic theory of chain homotopy dominations [17] to obtain a 'stable-exact' sequence of the type

$$Wh(Y) \longrightarrow Wh(X) \longrightarrow Wh(X,Y) \longrightarrow \widetilde{K}_0(Y) \longrightarrow \widetilde{K}_0(X)$$

relating controlled finiteness obstruction and torsion groups, using the relative groups $Wh(X,Y,p_X,n,\epsilon)$ of §4.

Remark. If we replace $p_X : M \to X$, $Y \subset X$, and "finitely generated (f.g.)" below by $q_X = p_X \times 1_N : M \times N \to X \times N$, $X \times N' \subset X \times N$, and "$M$-locally finite", respectively, then we obtain an analogous result for Wh^M and $\widetilde{K}_0^M$.

Definition. An ϵ *domination* (D,f,g,h) of a free chain complex C on p_X is a free chain complex D on p_X together with ϵ chain maps $f : C \to D$, $g : D \to C$, and an ϵ chain homotopy $h : gf \simeq_\epsilon 1 : C \to C$. C is said to be *ϵ dominated by D.*

Proposition 5.1. *Let C be a free chain complex on p_X and let Y be a subspace of X. If $(C,1)$ is δ chain equivalent to a projective chain complex (D,r) on p_Y, then C is δ dominated by the free chain complex D obtained from (D,r) by forgetting the projection r. Conversely, if C is δ dominated by an n-dimensional (f.g.) free δ chain complex on p_Y, then $(C,1)$ is $(2n+5)\delta$ chain equivalent to an n-dimensional (f.g.) $(n+4)\delta$ projective chain complex on $p_{Y^{(n+4)\delta}}$.*

Proof : Let $f : (C,1) \to (D,r)$, $g : (D,r) \to (C,1)$ be inverse δ chain equivalences with δ chain homotopies

$$h \ : \ gf \simeq 1 \ : \ (C,1) \longrightarrow (C,1) \ \ , \ \ k \ : \ fg \simeq 1 : (D,r) \longrightarrow (D,r) \ .$$

Then (D, f, g, h) is the desired δ domination.

Next, suppose (D, f, g, h) is a δ domination of C, where D is an n-dimensional free δ chain complex on p_Y. Define an infinite $(n+2)\delta$ chain complex $\{C', d'\}$ on $p_{Y^{(n+4)\delta}}$ by

$$
\begin{aligned}
C'_i &= D_0 \oplus D_1 \oplus \ldots \oplus D_i \ , \\
d' &= \begin{pmatrix}
fg & -d & 0 & 0 & \ldots & 0 & 0 \\
-fhg & 1-fg & d & 0 & \ldots & 0 & 0 \\
fh^2g & fhg & fg & -d & \ldots & 0 & 0 \\
-fh^3g & -fh^2g & -fhg & 1-fg & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-fh^{2i-1}g & -fh^{2i-2}g & -fh^{2i-3}g & -fh^{2i-4}g & \ldots & 1-fg & d
\end{pmatrix} \\
&\quad : \ C'_{2i} = D_0 \oplus D_1 \oplus \ldots \oplus D_{2i} \longrightarrow C'_{2i-1} = D_0 \oplus D_1 \oplus \ldots \oplus D_{2i-1} \ , \\
d' &= \begin{pmatrix}
1-fg & d & 0 & 0 & \ldots & 0 & 0 \\
fhg & fg & -d & 0 & \ldots & 0 & 0 \\
-fh^2g & -fhg & 1-fg & d & \ldots & 0 & 0 \\
fh^3g & fh^2g & fhg & fg & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-fh^{2i}g & -fh^{2i-1}g & -fh^{2i-2}g & -fh^{2i-3}g & \ldots & 1-fg & d
\end{pmatrix} \\
&\quad : \ C'_{2i+1} = D_0 \oplus D_1 \oplus \ldots \oplus D_{2i+1} \longrightarrow C'_{2i} = D_0 \oplus D_1 \oplus \ldots \oplus D_{2i} \ .
\end{aligned}
$$

By a direct calculation, one can check that ${d'}^2 \sim_{(n+4)\delta} 0$. Here the following formulae may be useful:

$$
\begin{aligned}
&h^{2k}gf - h^{2k-1}gfh + h^{2k-2}gfh^2 - \ldots + gfh^{2k} \sim_{(2k+2)\delta} h^{2k} - dh^{2k+1} - h^{2k+1}d \ , \\
&h^{2k+1}gf - h^{2k}gfh + h^{2k-1}gfh^2 - \ldots - gfh^{2k+1} \sim_{(2k+3)\delta} dh^{2k+2} + h^{2k+2}d \ .
\end{aligned}
$$

These can be obtained by the substitution $gf \sim_{2\delta} 1 - dh - hd$. The δ chain map $f' : C \to C'$ and the $(n+1)\delta$ chain map $g' : C' \to C$ defined by

$$
\begin{aligned}
f' &= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix} : \ C_i \longrightarrow C'_i = D_0 \oplus \ldots \oplus D_i \qquad (d'f' \sim_{2\delta} f'd) \ , \\
g' &= \begin{pmatrix} h^ig & h^{i-1}g & \ldots & hg & g \end{pmatrix} : \ C'_i = D_0 \oplus \ldots \oplus D_i \longrightarrow C_i \ \ (dg' \sim_{(n+2)\delta} g'd')
\end{aligned}
$$

are inverse $(n+1)\delta$ chain equivalences, as there are defined chain homotopies

$$\begin{array}{l} h \ : \ g'f' \ = \ gf \simeq 1 \ : \ C \longrightarrow C \qquad (dh + hd \sim_{2\delta} 1 - g'f') \\ k' \ : \ f'g' \simeq 1 \ : \ C' \longrightarrow C' \qquad (d'k' + k'd' \ \sim_{(n+2)\delta} \ 1 - f'g') \end{array}$$

with

$$k' \ = \ \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

$$: \ C'_i = D_0 \oplus D_1 \oplus \ldots \oplus D_i \longrightarrow C'_{i+1} \ = \ D_0 \oplus D_1 \oplus \ldots \oplus D_{i+1} \ .$$

Let E be the n-skeleton of C':

$$E \ : \ \ldots \to 0 \to C'_n \xrightarrow{d'} C'_{n-1} \to \ldots \to C'_1 \xrightarrow{d'} C'_0 \ ,$$

and define $q = \{q_i : E_i \to E_i\}$ by

$$\begin{array}{l} q_i \ = \ 1 \ : \ E_i \ = \ C'_i \longrightarrow E_i \ = \ C'_i \qquad (0 \le i \le n-1) \\ q_n \ = \ \begin{cases} p \ : \ E_n \ = \ C'_n \longrightarrow E_n \ = \ C'_n & \text{if } n \text{ is even} \\ 1 - p \ : \ E_n \ = \ C'_n \longrightarrow E_n \ = \ C'_n & \text{if } n \text{ is odd ,} \end{cases} \end{array}$$

where

$$p \ = \ \begin{pmatrix} fg & -d & 0 & \ldots \\ -fhg & 1 - fg & d & \ldots \\ fh^2g & fhg & fg & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : \ C'_n \ = \ D_0 \oplus \ldots \oplus D_n \longrightarrow C'_n \ = \ D_0 \oplus \ldots \oplus D_n \ .$$

In other words, $q_n = 1 - d'_{n+1} : C'_{n+1} = C'_n \to C'_n$. Since $d'_i = p$ (*resp.* $d'_i = 1 - p$) if i is even (*resp.* odd) and $i > n$, the $(n+4)\delta$ homotopy ${d'}^2 \sim_{(n+4)\delta} 0$ implies the $(n+4)\delta$ homotopy $p^2 \sim_{(n+4)\delta} p$. Therefore q is an $(n+4)\delta$ projection. Furthermore $d'_n q_n \sim_{(n+4)\delta} d'_n$, because

$$0 \sim_{(n+4)\delta} d'_n d'_{n+1} \ = \ d'_n(1 - q_n) \ = \ d'_n - d'_n q_n.$$

Thus (E,q) is an n-dimensional $(n+4)\delta$ projective chain complex on $p_{Y^{(n+4)\delta}}$. The chain maps $I:(C',1)\to(E,q)$, $J:(E,q)\to(C',1)$ defined by

$$I \;=\; \begin{cases} 1 \;:\; C'_i \longrightarrow E_i \;=\; C'_i & \text{if } 0\le i\le n-1 \\ q_n \;:\; C'_i \longrightarrow E_i \;=\; C'_i & \text{if } i \;=\; n \\ 0 \;:\; C'_i \longrightarrow E_i \;=\; 0 & \text{if } i>n \end{cases}$$

$$J \;=\; \begin{cases} 1 \;:\; E_i \;=\; C'_i \longrightarrow C'_i & \text{if } 0\le i\le n-1 \\ q_n \;:\; E_i \;=\; C'_i \longrightarrow C'_i & \text{if } i \;=\; n \\ 0 \;:\; E_i \;=\; 0 \longrightarrow C'_i & \text{if } i>n \end{cases}$$

are inverse $(n+4)\delta$ chain equivalences. In fact

$$\begin{aligned} &IJ \sim_{(n+2)\delta} q \;:\; (E,q) \longrightarrow (E,q) \\ &K \;:\; JI\simeq 1 \;:\; (C',1)\longrightarrow (C',1) \qquad (d'K+Kd' \sim_{(n+2)\delta} 1-JI) \end{aligned}$$

where

$$K \;=\; \begin{cases} 0 \;:\; C'_i \longrightarrow C'_{i+1} & \text{if } 0\le i\le n-1 \\ 1 \;:\; C'_i \;=\; C'_n \longrightarrow C'_{i+1} \;=\; C'_n & \text{if } i\ge n. \end{cases}$$

Therefore $(C,1)$ and (E,q) are $(2n+5)\delta$ chain equivalent. □

Remark. This is a controlled version of Proposition 3.1 of Ranicki [19]. The first author would like to thank Erik Pedersen for correcting the error of sign in the formulae for d' and p in [19].

Proposition 5.2. *If a free chain complex $(C,1)$ on p_X is ϵ dominated by a free chain complex on p_Y for some $Y\subset X$, then C is ϵ contractible over $X-Y^\epsilon$. Conversely, if C is an n-dimensional (f.g.) free ϵ chain complex on p_X which is ϵ contractible over $X-Y$, then C is 3ϵ dominated by an n-dimensional (f.g.) free ϵ chain complex on $p_{Y^{(n+2)\epsilon}}$.*

Proof : Let (D,f,g,h) be an ϵ domination of C. Since the radius of f is ϵ, f restricts to 0 on $X-Y^\epsilon$. Therefore h is an ϵ chain contraction of C over $X-Y^\epsilon$.

Next let Γ be an ϵ chain contraction of C over $X-Y$. For each integer r, define a geometric module D_r by

$$D_r \;=\; C_r(Y^{(n-r+2)\epsilon}) \;.$$

The restriction of the boundary morphism $d_r: C_r\to C_{r-1}$ to $Y^{(n-r+2)\epsilon}$ can be viewed as a morphism $d_r: D_r\to D_{r-1}$, because d_r has radius ϵ. And obviously $d_r d_{r+1}\sim_{2\epsilon} 0: D_{r+1}\to D_{r-1}$. Therefore $\{D_r,d_r\}$ is a "subcomplex" of C; i.e., the inclusion map $i:D\to C$ is an ϵ chain map. By assumption $1-d\Gamma-\Gamma d: C_r\to C_r$

is 2ϵ homotopic to a morphism $F_r : C_r \to C_r$ which is 0 over $X - Y$. F_r has radius 2ϵ, so it defines a morphism $f_r : C_r \to D_r$ such that $F = if$. $f = \{f_r\}$ is a 3ϵ chain map, because

$$\begin{aligned} i(df - fd) &= dF - Fd \sim_{3\epsilon} d(1 - d\Gamma - \Gamma d) - (1 - d\Gamma - \Gamma d)d \\ &\sim_{3\epsilon} -dd\Gamma + \Gamma dd \sim_{3\epsilon} 0. \end{aligned}$$

Since we have a 2ϵ homotopy $d\Gamma + \Gamma d \sim_{2\epsilon} 1 - if$, (D, f, i, Γ) is a 3ϵ domination of C. □

Let $n > 0$. The inclusion maps $i : Y \to X$ and $j : (X, \emptyset) \to (X, Y)$ induce stabilization maps:

$$\begin{aligned} &Wh(Y, p_Y, n, \epsilon) \xrightarrow{i_*} Wh(X, p_X, n, \epsilon) \xrightarrow{j_*} Wh(X, Y, p_X, n, \epsilon), \\ &K_0(Y, p_Y, n, \epsilon) \xrightarrow{i_*} \widetilde{K}_0(X, p_X, n, \epsilon) \ . \end{aligned}$$

We construct a connecting homomorphism

$$\partial : Wh(X, Y, p_X, n, \epsilon) \longrightarrow \widetilde{K}_0(W, p_W, n, \epsilon')$$

for any subspace W of X containing $Y^{K_n \epsilon}$ and any number ϵ' greater than or equal to $K_n\epsilon$, where $K_n = 12n + 70$:if C is an n-dimensional f.g. free ϵ chain complex on p_X which is strongly ϵ contractible over $X - Y$, then

$$\partial([C]) \ = \ [E, q] \in \widetilde{K}_0(W, p_W, n, \epsilon') \ ,$$

where (E, q) is any n-dimensional $(3n + 12)\epsilon$ projective chain complex on $p_{Y^{(4n+14)\epsilon}}$ that is $(6n + 15)\epsilon$ chain equivalent to $(C, 1)$. Such a projective chain complex (E, q) exists, because C is 3ϵ dominated by an n-dimensional f.g. free ϵ chain complex D on $p_{Y^{(n+2)\epsilon}}$ by the proposition above, and then by 5.1 $(C, 1)$ is $(6n+15)\epsilon$ chain equivalent to an n-dimensional f.g. $(3n + 12)\epsilon$ projective chain complex (E, q) on $p_{Y^{(4n+14)\epsilon}}$.

We show that ∂ is well-defined. Suppose C' is another n-dimensional f.g. free ϵ chain complex on p_X representing the same element as C in $Wh(X, Y, p_X, n, \epsilon)$ and suppose $(C', 1)$ is $(6n + 15)\epsilon$ chain equivalent to an n-dimensional f.g. $(3n + 12)\epsilon$ projective chain complex (E', q') on $p_{Y^{(4n+14)\epsilon}}$. Without loss of generality, we may assume that there is a 40ϵ-simple isomorphism

$$C \oplus D \oplus T \cong_{40\epsilon, \Sigma} C' \oplus D' \oplus T'$$

where D, D' are n-dimensional f.g. free 40ϵ chain complexes on $p_{Y^{20\epsilon}}$ and T, T' are n-dimensional f.g. free trivial chain complexes on p_X. In particular, $(C, 1) \oplus (D, 1)$ and $(C', 1) \oplus (D', 1)$ are 40ϵ chain equivalent. Therefore $(E, q) \oplus (D, 1)$ and $(E', q') \oplus (D', 1)$ are $(12n + 70)\epsilon$ chain equivalent. Therefore (E, q) and (E', q') represent the same element in $\widetilde{K}_0(W, p_W, n, \epsilon')$.

Remark. $\partial[C]$ depends only on the behaviour of C near Y. More precisely, let (C, d), $(\bar{C}, \bar{d})$ be n-dimensional free ϵ chain complexes on p_X with strong ϵ chain contractions Γ, $\bar{\Gamma}$ over $X - Y$ respectively, and suppose

1. $C_r(Y^{(2n+5)\epsilon}) = \bar{C}_r(Y^{(2n+5)\epsilon})$
2. $d_r|Y^{(2n+4)\epsilon} = \bar{d}_r|Y^{(2n+4)\epsilon}$
3. $\Gamma_r|Y^{(2n+4)\epsilon} = \bar{\Gamma}_r|Y^{(2n+4)\epsilon}$

for all r. Then the construction above yields the same (E, q) for C and $\bar{C}$. Furthermore, note that $\bar{C}$ need not be ϵ contractible all over $X - Y$ for (E, q) to be defined. Thus, in order to compute $\partial([C])$, we may replace C by another n-dimensional free ϵ complex $\bar{C}$ which satisfies 1 and 2 above and use an ϵ chain contraction $\bar{\Gamma}$ over $Y^{(2n+4)\epsilon} - Y$ which satisfies 3 above.

Now, for $W \supset Y^{K_n\epsilon}$ and $\epsilon' \geq K_n\epsilon$, we have a sequence

$$Wh(Y, p_Y, n, \epsilon) \xrightarrow{i_*} Wh(X, p_X, n, \epsilon) \xrightarrow{j_*} Wh(X, Y, p_X, n, \epsilon)$$
$$\xrightarrow{\partial} \widetilde{K}_0(W, p_W, n, \epsilon') \xrightarrow{i_*} \widetilde{K}_0(X, p_X, n, \epsilon') ,$$

where $K_n = 12n + 70$. It is easy to verify that the compositions j_*i_*, ∂j_*, and $i_*\partial$ are 0.

Theorem 5.3. *Fix an integer $n \geq 1$. There exists a constant L_n (≥ 1) which depends only on n such that the followings hold:*
(1) Suppose $\bar{Y} \supset Y^{L_n\epsilon}$ and $\bar{\epsilon} \geq L_n\epsilon$. Then the stabilization map $Wh(X, p_X, n, \epsilon) \to Wh(X, p_X, n, \bar{\epsilon})$ maps the kernel of

$$j_* \ : \ Wh(X, p_X, n, \epsilon) \longrightarrow Wh(X, Y, p_X, n, \epsilon) ,$$

into the image of

$$i_* \ : \ Wh(\bar{Y}, p_{\bar{Y}}, n, \bar{\epsilon}) \longrightarrow Wh(X, p_X, n, \bar{\epsilon}) .$$

(2) Suppose $Z \supset W^{L_n\epsilon'}$ and $\delta \geq L_n\epsilon'$. Then the stabilization map $Wh(X, Y, p_X, n, \epsilon) \to Wh(X, Z, p_X, n, \delta)$ maps the kernel of

$$\partial \ : \ Wh(X, Y, p_X, n, \epsilon) \longrightarrow \widetilde{K}_0(W, p_W, n, \epsilon') ,$$

into the image of

$$j_* \ : \ Wh(X, p_X, n, \delta) \longrightarrow Wh(X, Z, p_X, n, \delta) .$$

(3) Suppose $Z \supset W^{L_n\epsilon'}$ and $\delta \geq L_n\epsilon'$, and also assume $V \supset Z^{K_n\delta}$, $\delta' \geq K_n\delta$ so that

$$\partial \ : \ Wh(X, Z, p_X, n, \delta) \longrightarrow \widetilde{K}_0(V, p_V, n, \delta')$$

is defined. Then the stabilization map $\widetilde{K}_0(W, p_W, n, \epsilon') \to \widetilde{K}_0(V, p_V, n, \delta')$ maps the kernel of

$$i_* \ : \ \widetilde{K}_0(W, p_W, n, \epsilon') \longrightarrow \widetilde{K}_0(X, p_X, n, \epsilon') ,$$

into the image of ∂.

We shall use the following lemma.

Lemma 5.4. *Let C and D be free δ chain complexes.*
(1) If two chain maps $f, f' : C \to D$ are δ chain homotopic, then there is a 2δ-simple isomorphism between $\mathcal{C}(f)$ and $\mathcal{C}(f')$
(2) There is a 2δ-simple isomorphism from $\mathcal{C}(1_C : C \to C)$ to a free trivial chain complex.

Proof : (1) Let $h : f \simeq f'$ be a δ chain homotopy. A desired 2δ-simple isomorphism is given by:

$$\begin{pmatrix} 1 & (-)^r h \\ 0 & 1 \end{pmatrix} \; : \; \mathcal{C}(f)_r \;=\; D_r \oplus C_{r-1} \longrightarrow \mathcal{C}(f')_r \;=\; D_r \oplus C_{r-1}.$$

(2) Define a free trivial chain complex $T = \{T_r, d_r\}$ by

$$\begin{aligned} T_r &= C_r \oplus C_{r-1} \;, \\ d_r &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; : \; C_r \oplus C_{r-1} \longrightarrow C_{r-1} \oplus C_{r-2} \;. \end{aligned}$$

The required 2δ-simple isomorphism from $\mathcal{C}(1_C)$ to T is given by:

$$\begin{pmatrix} (-)^r 1 & 0 \\ (-)^{r-1} d_C & 1 \end{pmatrix} \; : \; \mathcal{C}(1_C)_r \;=\; C_r \oplus C_{r-1} \longrightarrow T_r \;=\; C_r \oplus C_{r-1}. \qquad \square$$

Proof of 5.3: We show that $L_n = 27000(9n + 34)$ has the desired properties.
(1) Suppose $[C] \in Wh(X, p_X, n, \epsilon)$ is an element of the kernel of j_*. By 4.1 there exists an 86ϵ-simple isomorphism

$$f \; : \; C \oplus D \oplus T \longrightarrow D' \oplus T'$$

for some n-dimensional f.g. free 86ϵ chain complexes D, D' on $p_{Y^{20\epsilon}}$ and some n-dimensional f.g. free trivial complexes T, T' on p_X. Let $i : D \to C \oplus D \oplus T$ and $j : D' \to D' \oplus T'$ denote the inclusion maps, and $q : D' \oplus T' \to D'$ denote the projection map. The map i is the direct sum of the ϵ chain equivalence $0 : 0 \to C$, the 86ϵ chain equivalence $1 : D \to D$, and the 0 chain equivalence $0 : 0 \to T$, and hence is an 86ϵ chain equivalence. Similarly, q is an 86ϵ chain equivalence. Therefore the composite $g = qfi : D \to D'$ is a $3{\cdot}86\epsilon$ chain equivalence, and $\mathcal{C}(g)$ is 900ϵ contractible. $\mathcal{C}(g) \oplus T'$ is equal to $\mathcal{C}(jg : D \to D' \oplus T')$. Since T' is 0 contractible, there is a 0 chain homotopy $jq \simeq 1$, and it induces an 86ϵ chain homotopy $jg = (jq)fi \simeq fi$. By 5.4 $\mathcal{C}(jg)$ is $2 \cdot 86\epsilon$-simple isomorphic to $\mathcal{C}(fi)$. The 86ϵ chain map defined by

$$f \oplus 1 : \mathcal{C}(fi)_r = (C \oplus D \oplus T)_r \oplus D_{r-1} \longrightarrow (D' \oplus T')_r \oplus D_{r-1} = \mathcal{C}(i)_r$$

is a $2\cdot 86\epsilon$-simple isomorphism from $\mathcal{C}(fi)$ to $\mathcal{C}(i)$, because its inverse is a $2\cdot 86\epsilon$ chain map. And $\mathcal{C}(i)$ is equal to $C\oplus T\oplus \mathcal{C}(1_D : D\to D)$. Finally $\mathcal{C}(1_D)$ is $2\cdot 86\epsilon$-simple isomorphic to a trivial chain complex T''. By composing

$$C\oplus T\oplus T'' \cong_{192\epsilon,\Sigma} C\oplus T\oplus \mathcal{C}(1_D) \;=\; \mathcal{C}(i)\cong_{192\epsilon,\Sigma}\mathcal{C}(fi)\cong_{192\epsilon,\Sigma}\mathcal{C}(jg) \;=\; \mathcal{C}(g)\oplus T' ,$$

we get a 600ϵ-simple isomorphism between $C\oplus T\oplus T''$ and $\mathcal{C}(g)\oplus T'$. $\mathcal{C}(g)$ represents an element of $Wh(\bar{Y}, p_{\bar{Y}}, n+1, 2700\epsilon)$. ($\mathcal{C}(g)$ has a strong 2700ϵ chain contraction Γ, and the 5400ϵ homotopy $\Gamma^2\sim 0$ takes place over $\bar{Y}$.) By 4.3, this element comes from some element $[\bar{C}]\in Wh(\bar{Y}, p_{\bar{Y}}, n, 2700\epsilon)$. $\bar{C}$ and C may not represent the same element in $Wh(X, p_X, n, 2700\epsilon)$, but they do represent the same element in $Wh(X, p_X, n+1, 2700\epsilon)$:

$$\iota([\bar{C}]) = \iota([C]) \in Wh(X, p_X, n+1, 2700\epsilon),$$

and hence we have

$$\tau\iota([\bar{C}]) = \tau\iota([C]) \in Wh(X, p_X, (90(n+1)+250)\cdot 2700\epsilon).$$

Therefore

$$[\bar{C}] = \iota\tau\iota([\bar{C}]) = \iota\tau\iota([C]) = [C] \in Wh(X, p_X, n, 27000(9n+34)\epsilon).$$

(2) Suppose $[C]\in Wh(X, Y, p_X, n, \epsilon)$ is an element of the kernel of ∂. By definition, $\partial[C] = [E, q] \in \widetilde{K}_0(W, p_W, n, \epsilon')$ where (E, q) is an n-dimensional f.g. $(3n+12)\epsilon$ projective chain complex on $p_{Y^{(4n+14)\epsilon}}$ that is $(6n+15)\epsilon$ chain equivalent to $(C, 1)$. Since $[E, q] = 0$ in $\widetilde{K}_0(W, p_W, n, \epsilon')$, (E, q) is $60\epsilon'$ chain equivalent to an n-dimensional f.g. free $30\epsilon'$ chain complex $(D, 1)$ on p_W by 3.5. By composing these we obtain a $61\epsilon'$ chain equivalence $f : D\to C$. $\mathcal{C}(f)$ is an $(n+1)$-dimensional free $61\epsilon'$ chain complex on p_X and is $183\epsilon'$ contractible; hence it is strongly $549\epsilon'$ contractible and determines an element of $Wh(X, p_X, n+1, 549\epsilon')$. By 4.5, $\mathcal{C}(f)$ and C represents the same element in $Wh(X, W, p_X, n+1, 549\epsilon')$. By 4.3 there exists an element $[\bar{C}]\in Wh(X, p_X, n, 549\epsilon')$ which maps to $[\mathcal{C}(f)]$. One can use the homomorphisms ι and τ as in (1) to show that $\bar{C}$ and C represent the same element in $Wh(X, W^{549(90n+340)\epsilon'}, p_X, n, 549(90n+340)\epsilon')$ and hence in $Wh(X, Z, p_X, n, \delta)$.

(3) Suppose $[E, q]\in \widetilde{K}_0(W, p_W, n, \epsilon')$ is an element of the kernel of i_*. (E, q) is $60\epsilon'$ chain equivalent to an n-dimensional f.g. free $30\epsilon'$ chain complex $(C, 1)$ on p_X. C is $60\epsilon'$ contractible over $X - W^{60\epsilon'}$, and hence strongly $180\epsilon'$ contractible over $X - W^{240\epsilon'}$. Therefore C defines an element in $Wh(X, Z, p_X, n, \delta)$ and

$$\partial[C] \;=\; [E, q] \in \widetilde{K}_0(V, p_V, n, \delta'). \qquad \square$$

6. Excision and the Mayer-Vietoris sequence.

Throughout this section assume that $X = X_+ \cup X_-$ is the union of two closed subspaces X_+ and X_- with intersection $Y = X_+ \cap X_-$. The excision isomorphisms of ordinary homology

$$H_*(X_+, Y) \;\cong\; H_*(X, X_-)$$

and the Mayer-Vietoris exact sequence

$$\ldots \to H_n(Y) \to H_n(X_+) \oplus H_n(X_-) \to H_n(X) \to H_{n-1}(Y) \to \ldots$$

have various algebraic K-theory analogues. In this section we first discuss the excision maps of controlled Whitehead groups, and then use them to introduce a Mayer-Vietoris sequence in controlled K-theory of the type

$$Wh(Y) \to Wh(X_+) \oplus Wh(X_-) \to Wh(X) \to \widetilde{K}_0(Y) \to \widetilde{K}_0(X_+) \oplus \widetilde{K}_0(X_-) \ .$$

As in ordinary homology and the bounded K-theory exact sequences of Ranicki [22] the main ingredient is a chain level Mayer-Vietoris decomposition: a free ϵ chain complex C on X can be expressed as a sum $C = C_+ + C_-$ of complexes with $C_\pm$ defined on a neighbourhood of $X_\pm$ in X, and $C_+ \cap C_-$ defined on a neighbourhood of Y in X. If C is contractible then C_+, C_-, $C_+ \cap C_-$ are finitely dominated, but not in general contractible.

Remarks. (1) The assumption of X_+ and X_- being closed ensures that any path connecting a point in X_- and a point in X_+ passes through Y. More precisely, suppose $\gamma : [0, s] \to X$ is a path with $\gamma(0) \in X_-$ and $\gamma(s) \in X_+$, and suppose $\gamma([0, s]) \subset \{\gamma(0)\}^\delta$ for some δ. By the connectivity of the interval, there exists a $t \in [0, s]$ such that $\gamma(t) \in Y$. Since $\{\gamma(0)\}^\delta \subset \{\gamma(t)\}^{2\delta}$, $\gamma([0, s])$ is contained in $Y^{2\delta}$. This argument will be used in place of the relation $X_-^\delta = X_- \cup Y^\delta$, which is false in general. (This assumption is not essential. Without this, the argument in this section works if we replace sets of the form $X_\pm \cup V^\delta$ by $(X_\pm \cup V)^\delta$, etc.)
(2) If we replace $p_X : M \to X$, X_+, X_- by $p_X \times 1_N : M \times N \to X \times N$, $X \times N_+$, $X \times N_-$, respectively, and use M-locally finite chain complexes rather than finitely generated chain complexes, then we obtain an analogous result for Wh^M and $\widetilde{K}_0$, which will be used in the next section.

There is an inclusion-induced homomorphism

$$i_* \ : \ Wh(X_+, Y, p_{X_+}, n, \epsilon) \longrightarrow Wh(X, X_-, p_X, n, \epsilon) \ .$$

We construct its stable inverse

$$\mathrm{exc} \ : \ Wh(X, X_-, p_X, n, \epsilon) \longrightarrow Wh(X_+, X_+ \cap Y^{(n+300)\epsilon}, p_{X_+}, n, 90\epsilon) \ .$$

For a chain complex $\{C, d\}$ representing an element of $Wh(X, X_-, p_X, n, \epsilon)$, let $\{C_+, d_+\}$ be any n-dimensional f.g. free 90ϵ chain complex on p_{X_+} such that

1. $(C_+)_r(X_+ - Y^{(n+180)\epsilon}) = C_r(X_+ - Y^{(n+180)\epsilon})$, and
2. $d_+ = d_C$ over $X_+ - Y^{(n+270)\epsilon}$.

(Such a C_+ can be constructed by letting $(C_+)_r = C_r(X_+ - Y^{r\epsilon})$, for example.) If Γ is a strong ϵ chain contraction of C over $X - X_-$, then

$$\Gamma_+ \;=\; \begin{cases} \Gamma & \text{over } X_+ - Y^{(n+270)\epsilon} \\ 0 & \text{over } X_+ \cap Y^{(n+270)\epsilon} \end{cases}$$

is a strong 90ϵ chain contraction of C_+ over $X_+ - Y^{(n+300)\epsilon}$. Thus C_+ represents an element of $Wh(X_+, X_+ \cap Y^{(n+300)\epsilon}, p_{X_+}, n, 90\epsilon)$. This element is independent of the choice of C_+ by 4.5. We claim that this correspondence $[C] \mapsto [C_+]$ defines the desired well-defined homomorphism exc. It suffices to show that, given 40ϵ-simple isomorphic complexes C and C', one can choose C_+ and C'_+ which are 3600ϵ-simple isomorphic to each other. Let $f : C \to C'$ be the 40ϵ-simple isomorphism and define C_+ by $(C_+)_r = C_r(X_+ - Y^{(r+40)\epsilon})$. Consider the localization $g = f_{[X_- \cup Y^{(n+160)\epsilon}]}$. As the subspaces $X_\pm$ are closed in X, g is geometric over $X_- \cup Y^{(n+80)\epsilon}$ and $g = f$ over $X_+ - Y^{(n+200)\epsilon}$. Let C'' be the chain complex obtained by replacing the boundary map of C' with gdg^{-1}, where d denotes the boundary map of C. C'' and C' are the same (up to 81ϵ homotopy of boundary maps) over $X_+ - Y^{(n+250)\epsilon}$. The 120ϵ-simple isomorphism $g : C \to C''$ (of radius 40ϵ) restricts to a 40ϵ-simple isomorphism from C_+ onto a geometric module subcomplex C'_+ of C''. Then $C'_r(X_+ - Y^{(n+80)\epsilon}) = (C'_+)_r(X_+ - Y^{(n+80)\epsilon})$, and $d_{C'_+} = d_{C''} \sim_{81\epsilon} d_{C'}$ over $X_+ - Y^{(n+250)\epsilon}$. Therefore C_+ and C'_+ have the desired properties (up to homotopy), and exc is well-defined.

The homomorphisms i_* and exc are stable inverses; i.e., we have commutative diagrams:

$$\begin{tikzcd}
Wh(X_+, Y, p_{X_+}, n, \epsilon) \arrow[r, "i_*"] \arrow[d] & Wh(X, X_-, p_X, n, \epsilon) \arrow[d, equal] \\
Wh(X_+, X_+ \cap Y^{(n+300)\epsilon}, p_{X_+}, n, 90\epsilon) & Wh(X, X_-, p_X, n, \epsilon) \arrow[l, "\mathrm{exc}"']
\end{tikzcd}$$

$$\begin{array}{ccc}
Wh(X_+, X_+ \cap Y^{(n+300)\epsilon}, p_{X_+}, n, 90\epsilon) & \xleftarrow{\text{exc}} & Wh(X, X_-, p_X, n, \epsilon) \\
\| & & \downarrow \\
Wh(X_+, X_+ \cap Y^{(n+300)\epsilon}, p_{X_+}, n, 90\epsilon) & \xrightarrow[i_*]{} & Wh(X, X_-^{(n+300)\epsilon}, p_X, n, 90\epsilon)
\end{array}$$

where the vertical arrows are the stabilization maps.

Now let $\kappa_n = 90K_n + n + 300 = 1081n + 6600$, and let W be any closed subset of X containing $Y^{\kappa_n \epsilon}$ and ϵ' be any number greater than or equal to $\kappa_n \epsilon$. We define a homomorphism $\partial_+ : Wh(X, p_X, n, \epsilon) \to \widetilde{K}_0(W, p_W, n, \epsilon')$ by the composition:

$$Wh(X, p_X, n, \epsilon) \to Wh(X, X_-, p_X, n, \epsilon) \xrightarrow{\text{exc}} Wh(X_+, X_+ \cap Y^{(n+300)\epsilon}, p_{X_+}, n, 90\epsilon)$$
$$\xrightarrow{\partial} \widetilde{K}_0(X_+ \cap W, p_{X_+ \cap W}, n, \epsilon') \to \widetilde{K}_0(W, p_W, n, \epsilon') \ .$$

(Note that $W \supset (Y^{(n+300)\epsilon})^{K_n \cdot 90\epsilon}$ and $\epsilon' \geq K_n \cdot 90\epsilon$.) Similarly, define a homomorphism $\partial_- : Wh(X, p_X, n, \epsilon) \to \widetilde{K}_0(W, p_W, n, \epsilon')$ by exchanging the roles of X_+ and X_-.

Proposition 6.1. $\partial_+ + \partial_- \ = \ 0$.

Proof : $\partial_+ + \partial_-$ factors as follows:

$$Wh(X, p_X, n, \epsilon) \to Wh(X, p_X, n, 90\epsilon) \to Wh(X, Y^{(n+300)\epsilon}, p_X, n, 90\epsilon)$$
$$\xrightarrow{\partial} \widetilde{K}_0(W, p_W, n, \epsilon')$$

and the composition of the last two maps is 0. □

Let us summarize the situation: $X = X_- \cup X_+$, $Y = X_- \cap X_+$, $W \supset Y^{\kappa_n \epsilon}$, $\epsilon' \geq \kappa_n \epsilon$. Given these data, we have the Mayer-Vietoris sequence:

$$Wh(Y, p_Y, n, \epsilon) \xrightarrow{\begin{pmatrix} -i_- \\ i_+ \end{pmatrix}} Wh(X_-, p_{X_-}, n, \epsilon) \oplus Wh(X_+, p_{X_+}, n, \epsilon)$$
$$\xrightarrow{(j_- \ \ j_+)} Wh(X, p_X, n, \epsilon) \xrightarrow{\partial_+ = -\partial_-} \widetilde{K}_0(W, p_W, n, \epsilon')$$
$$\xrightarrow{\begin{pmatrix} -i_- \\ i_+ \end{pmatrix}} \widetilde{K}_0(X_- \cup W, p_{X_- \cup W}, n, \epsilon') \oplus \widetilde{K}_0(X_+ \cup W, p_{X_+ \cup W}, n, \epsilon')$$

where $i_\pm$'s and $j_\pm$'s are the stabilization maps induced by the inclusion maps. The compositions $(j_- \ \ j_+) \begin{pmatrix} -i_- \\ i_+ \end{pmatrix}$, $\partial_+(j_- \ \ j_+)$ are obviously 0, and $\begin{pmatrix} -i_- \\ i_+ \end{pmatrix} \partial_+$ is also 0 by 6.1. This sequence is stably exact in the following sense. (The p's will denote appropriate restrictions of p_X.)

Theorem 6.2. *Fix an integer $n \geq 1$. There exists a constant M_n (≥ 1) which depends only on n such that the followings hold:*
(1) Suppose $\bar{Y} \supset Y^{M_n\epsilon}$ and $\bar{\epsilon} \geq M_n\epsilon$. Then the stabilization map $Wh(X_-, p, n, \epsilon) \oplus Wh(X_+, p, n, \epsilon) \to Wh(X_- \cup \bar{Y}, p, n, \bar{\epsilon}) \oplus Wh(X_- \cup \bar{Y}, p, n, \bar{\epsilon})$ maps the kernel of

$$(j_- \ \ j_+) \ : \ Wh(X_-, p, n, \epsilon) \oplus Wh(X_+, p, n, \epsilon) \longrightarrow Wh(X, p_X, n, \epsilon)$$

into the image of

$$\begin{pmatrix} -i_- \\ i_+ \end{pmatrix} \ : \ Wh(\bar{Y}, p, n, \bar{\epsilon}) \longrightarrow Wh(X_- \cup \bar{Y}, p, n, \bar{\epsilon}) \oplus Wh(X_+ \cup \bar{Y}, p, n, \bar{\epsilon}).$$

(2) Suppose $Z \supset W^{M_n\epsilon'}$ and $\delta \geq M_n\epsilon'$. Then the stabilization map $Wh(X, p_X, n, \epsilon) \to Wh(X, p_X, n, \delta)$ maps the kernel of

$$\partial_+ \ : \ Wh(X, p_X, n, \epsilon) \longrightarrow \widetilde{K}_0(W, p_W, n, \epsilon')$$

into the image of

$$(j_- \ \ j_+) \ : \ Wh(X_- \cup Z, p, n, \delta) \oplus Wh(X_+ \cup Z, p, n, \delta) \longrightarrow Wh(X, p_X, n, \delta).$$

(3) Suppose $\delta \geq M_n\epsilon'$, and also assume that $V \supset W^{\kappa_n\delta}$, $\delta' \geq \kappa_n\delta$ so that the map

$$\partial_+ : Wh(X, p_X, n, \delta) \longrightarrow \widetilde{K}_0(V, p_V, n, \delta')$$

associated with the two subsets $X_\pm \cup W$ is defined. Then the stabilization map $\widetilde{K}_0(W, p_W, n, \epsilon') \to \widetilde{K}_0(V, p_V, n, \delta')$ maps the kernel of

$$\begin{pmatrix} -i_- \\ i_+ \end{pmatrix} \ : \ \widetilde{K}_0(W, p_W, n, \epsilon') \longrightarrow \widetilde{K}_0(X_- \cup W, p, n, \epsilon') \oplus \widetilde{K}_0(X_+ \cup W, p, n, \epsilon')$$

into the image of ∂_+.

Proof : We show that $M_n = \max\{1200, L_n + L_n^2, 46(L_n + K_nL_n)\}$ has the desired properties, where K_n and L_n are as in the previous section.

(1) Let $([C_-], [C_+]) \in Wh(X_-, p, n, \epsilon) \oplus Wh(X_+, p, n, \epsilon)$ be an element of the kernel of $(j_- \ \ j_+)$. By 4.1, there is an 86ϵ-simple isomorphism $f : C_- \oplus C_+ \oplus T' \to T$ for some trivial complexes T, T'. Replacing C_- and C_+ by $C_- \oplus T'(X_-)$ and $C_+ \oplus T'(X - X_-)$ respectively, we may assume that f is an 86ϵ-simple isomorphism between $C_- \oplus C_+$ and T. Let f' denote the localization of f away from $X_- \cup Y^{2\cdot 86\epsilon}$, then f' is geometric over $X_- \cup Y^{86\epsilon}$ and $f' = f$ over $X_+ - Y^{3\cdot 86\epsilon}$. Replace the boundary map of T by $f'(d_{C_-} \oplus d_{C_+})(f')^{-1}$. This produces a 200ϵ chain complex E whose boundary

map is 172ϵ homotopic to d_T over $X_+ - Y^{400\epsilon}$. Therefore f' defines a 259ϵ-simple isomorphism from $C_- \oplus C_+$ to the direct sum of a 200ϵ chain complex on $X_- \cup Y^{800\epsilon}$ and a trivial complex. Since f' is geometric over $X_- \cup Y^{86\epsilon}$ and has radius 86ϵ, we can discard the paths in f' starting from the basis elements of C_- to obtain a 259ϵ-simple isomorphism $g : C_+ \to D_+ \oplus T''$, where D_+ is a 200ϵ chain complex on $p_{Y^{800\epsilon}}$ and T'' is the trivial complex $T(X_+ - Y^{600\epsilon})$. D_+ is strongly 1200ϵ contractible and defines the same element as C_+ in $Wh(X_+ \cup Y^{800\epsilon}, p, n, 1200\epsilon)$.

The 345ϵ-simple isomorphism $f(1_{C_-} \oplus g^{-1}) : C_- \oplus D_+ \oplus T'' \to C_- \oplus C_+ \to T$ is homotopic to the identity on T'', so we can discard this portion to obtain a 345ϵ-simple isomorphism form $C_- \oplus D_+$ to the trivial complex $T(X_- \cup Y^{600\epsilon})$. Therefore $-[D_+] = [C_-]$ in $Wh(X_- \cup Y^{800\epsilon}, p, n, 1200\epsilon)$. Thus $([C_-], [C_+]) \in Wh(X_- \cup \bar{Y}, p, n, \bar{\epsilon}) \oplus Wh(X_+ \cup \bar{Y}, p, n, \bar{\epsilon})$ is the image of $[D_+] \in Wh(\bar{Y}, p, n, \bar{\epsilon})$ by $\begin{pmatrix} -i_- \\ i_+ \end{pmatrix}$.

(2) Suppose $[C] \in Wh(X, p_X, n, \epsilon)$ is an element of the kernel of ∂_+. Let C_+ be as in the definition of the excision map. Then $[C_+]$ in the second row of the following diagram is in the kernel of ∂:

$$\begin{array}{ccccc}
[C_+] \in Wh(X_+, X_+ \cap Y^{(n+300)\epsilon}, p, n, 90\epsilon) & \xrightarrow{\partial} & \widetilde{K}_0(X_+ \cap W, p, n, \epsilon') & \ni & \partial[C_+] \\
\downarrow & & \downarrow & & \\
[C_+] \in Wh(X_+ \cup W, Y^{(n+300)\epsilon}, p, n, 90\epsilon) & \xrightarrow{\partial} & \widetilde{K}_0(W, p, n, \epsilon') & \ni & \partial_+[C] = 0
\end{array}$$

where the vertical maps are induced by inclusion maps. By 5.3(2), there exists an element $[\hat{C}_+] \in Wh(X_+ \cup W, p, n, \gamma)$ such that $[\hat{C}_+] = [C_+]$ in $Wh(X_+ \cup W, (X_+ \cup W) \cap W^\gamma, p, n, \gamma)$ and hence also in $Wh(X, W^\gamma, p, n, \gamma)$, where $\gamma = L_n\epsilon'$. As $\partial_- = -\partial_+$, there exists an element $[\hat{C}_-] \in Wh(X_- \cup W, p, n, \gamma)$ such that $[\hat{C}_-] = [C_-]$ in $Wh(X_- \cup W, (X_- \cup W) \cap W^\gamma, p, n, \gamma)$, where $\{C_-, d_-\}$ is an n-dimensional free 90ϵ chain complex on p_{X_-} such that $(C_-)_r(X_- - Y^{(n+180)\epsilon}) = C_r(X_- - Y^{(n+180)\epsilon})$, and $d_- = d_C$ over $X_- - Y^{(n+270)\epsilon}$. By 4.5, $[C] = [C_-] + [C_+]$ in $Wh(X, W^\gamma, p, n, \gamma)$. Therefore $[C] = [\hat{C}_-] + [\hat{C}_+]$ in $Wh(X, W^\gamma, p, n, \gamma)$. Apply 5.3(1) to $[C] - [\hat{C}_-] - [\hat{C}_+] \in Wh(X, p, n, \gamma)$ to obtain an element $[D] \in Wh(W^{\gamma + L_n\gamma}, p, n, L_n\gamma)$ which maps via i_* to $[C] - [\hat{C}_-] - [\hat{C}_+] \in Wh(X, p, n, L_n\gamma)$. As $Z \supset W^{\gamma + L_n\gamma}$ and $\delta \geq L_n\gamma$, $([\hat{C}_-] + [D], [\hat{C}_+])$ defines an element of $Wh(X_- \cup Z, p, n, \delta) \oplus Wh(X_+ \cup Z, p, n, \delta)$, and $(j_-\ \ j_+)$ maps this element to $[C] \in Wh(X, p, n, \delta)$.

(3) Suppose $[E, q] \in \widetilde{K}_0(W, p_W, n, \epsilon')$ is an element of the kernel of $\begin{pmatrix} -i_- \\ i_+ \end{pmatrix}$. By

5.3(3), there exist elements

$$[C_+] \in Wh(X_+ \cup W, (X_+ \cup W) \cap W^{L_n\epsilon'}, p, n, L_n\epsilon')$$
$$[C_-] \in Wh(X_- \cup W, (X_- \cup W) \cap W^{L_n\epsilon'}, p, n, L_n\epsilon')$$

such that

$$\begin{aligned}\partial[C_+] &= [E,q] \in \widetilde{K}_0((X_+ \cup W) \cap W', p, n, \gamma)\\ \partial[C_-] &= [E,q] \in \widetilde{K}_0((X_- \cup W) \cap W', p, n, \gamma)\end{aligned}$$

where $W' = (W^{L_n\epsilon'})^{K_nL_n\epsilon'}$ and $\gamma = (1+K_n)L_n\epsilon'$. By definition, $\partial[C_+]$ (resp. $\partial[C_-]$) is represented by a γ projective chain complex (E_+, q_+) (resp. (E_-, q_-)) on $p_{W'}$ which is γ chain equivalent to $(C_+, 1)$ (resp. $(C_-, 1)$). By applying 3.1 to $[E_+, q_+] = [E_-, q_-] \in \widetilde{K}_0(W', p, n, \gamma)$, we obtain free γ chain complexes F, G on $p_{W'}$ such that

$$(E_+, q_+) \oplus (F, 1) \simeq_{3\gamma} (E_-, q_-) \oplus (G, 1)\ .$$

Therefore, there is a 5γ chain equivalence $f : C_- \oplus G \to C_+ \oplus F$. $\mathcal{C}(f)$ is an $(n+1)$-dimensional strongly 45γ contractible chain complex. Apply 4.3 to obtain an n-dimensional strongly 45γ contractible chain complex $\{\widehat{\mathcal{C}(f)}, \hat{d}\}$. By construction $\widehat{\mathcal{C}(f)}_r(X_+ - W') = (C_+)_r(X_+ - W')$ and $\hat{d}_r = d_{C_+}$ over $X_+ - (W')^{45\gamma}$. As $W^{(n+180)\delta} \supset W'$ and $W^{(n+270)\delta} \supset (W')^{45\gamma}$, the excision map

$$\text{exc} \ :\ Wh(X, X_- \cup W, p, n, \delta) \longrightarrow Wh(X_+ \cup W, (X_+ \cup W) \cap W^{(n+300)\delta}, p, n, 90\delta)$$

used to define $\partial_+ : Wh(X, p_X, n, \delta) \to \widetilde{K}_0(V, p, n, \delta')$ maps $[\widehat{\mathcal{C}(f)}]$ to $[C_+]$. Therefore ∂_+ maps $[\widehat{\mathcal{C}(f)}] \in Wh(X, p_X, n, \delta)$ to $[E, q]$. $\square$

7. Controlled Whitehead group of $M \times S^1$.

In this section we establish a controlled analogue of the split exact sequence of Bass [1, XII] for the Whitehead group of $\pi \times \mathbb{Z}$

$$\begin{aligned}0 \longrightarrow Wh(\pi) \xrightarrow{\ i_!\ } Wh(\pi \times \mathbb{Z}) &\xrightarrow{B \oplus N_+ \oplus N_-} \\ &\widetilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi]) \longrightarrow 0\end{aligned}$$

with $i_!$ induced by the inclusion $i : \pi \longrightarrow \pi \times \mathbb{Z}$. In the controlled analogue there are no $\widetilde{\mathrm{Nil}}$-terms, and the sequence is only stably exact. Geometrically, B sends the torsion $\tau(f) \in Wh(\pi \times \mathbb{Z})$ of a homotopy equivalence $f : M \longrightarrow X \times S^1$ between a compact manifold M and the product of a finite Poincaré complex X and S^1 to the

Siebenmann end obstruction of one of the two ends $\bar{M}^{\pm} = \bar{f}^{-1}(\mathbb{R}^{\pm})$ of the infinite cyclic cover $\bar{M} = f^*(X \times \mathbb{R})$ of M

$$B\tau(f) \;=\; [\bar{M}^+] \;=\; -[\bar{M}^-] \in \widetilde{K}_0(\mathbb{Z}[\pi]) \; .$$

B is split by the injection

$$\bar{B} \;:\; \widetilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi \times \mathbb{Z}) \; ; \; [P] \longrightarrow \tau(z : P[z, z^{-1}] \longrightarrow P[z, z^{-1}]) \; .$$

In the terminology of Ranicki [21] this is the 'algebraically significant injection' of $\widetilde{K}_0(\mathbb{Z}[\pi])$ in $Wh(\pi \times \mathbb{Z})$, to be distinguished from the 'geometrically significant injection'

$$\bar{B}' \;:\; \widetilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi \times \mathbb{Z}) \; ; \; [P] \longrightarrow \tau(-z : P[z, z^{-1}] \to P[z, z^{-1}])$$

with image the subgroup of transfer invariant elements of $Wh(\pi \times \mathbb{Z})$.

Theorem 7.1. *Let $p_X : M \to X$ be a control map. For any $n > 0$, $\delta > 0$ and $\epsilon \geq 18\delta$, there is a commutative diagram*

$$\begin{array}{ccc}
Wh(X, p'_X, n, 18\delta) & \xleftarrow{\;\;\bar{B}\;\;} & \widetilde{K}_0(X, p_X, n, \delta) \\
\big\downarrow & & \big\downarrow \\
Wh(X, p'_X, n, \epsilon) & \xrightarrow[\;\;B\;\;]{} & \widetilde{K}_0(X, p_X, n, \kappa_n \epsilon)
\end{array}$$

where p'_X denotes the following control map:

$$p'_X \;:\; M \times S^1 \xrightarrow{\text{projection}} M \xrightarrow{p_X} X \; ,$$

the vertical maps are stabilization maps, and $\kappa_n = 1081n + 6600$.

κ_n above is the constant which was used when we defined the connecting homomorphism ∂_+ for the Mayer-Vietoris sequence in the previous section.

To prove 7.1, it will be useful to consider a control map of the following form:

$$p_X \times 1_\Delta \;:\; M \times \Delta \longrightarrow X \times \Delta \; ,$$

where $\Delta \subset \mathbb{R}$ or S^1. We shall consider S^1 as the quotient $\mathbb{R}/\mathbb{Z}$ and use the metric induced from that of $\mathbb{R}$. The projection map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ will be denoted by π. We shall always use the maximum metric for a product of metric spaces.

The hypothesis of the following lemma is satisfied by any simplex Δ in euclidean space $\mathbb{R}^k$, or a Hilbert cube I^∞. In the application in this section, Δ will be an interval $[-s, s] \subset \mathbb{R}$.

Lemma 7.2. *Let Δ be a compact metric space and assume that there is a strong deformation retraction $\{r_t\}_{0\le t\le 1}$ of Δ to a point $v \in \Delta$ such that $d(r_t(x), r_t(y)) \le d(x,y)$ for all x, $y \in \Delta$ and $t \in [0,1]$. Suppose that $p_{X\times\Delta} : N \to X \times \Delta$ is a control map such that there is a strong deformation retraction $\{\widetilde{R}_t\}$ of N to $p^{-1}_{X\times\Delta}(X \times \{v\})$ which covers the strong deformation retraction $\{R_t = 1_X \times r_t\}$ of $X \times \Delta$ to $X \times \{v\}$, and let*

$$p_X \ = \ p_{X\times\Delta}| \ : \ (p_{X\times\Delta})^{-1}(X \times \{v\}) \to X \times \{v\} \ = \ X \ .$$

Then there are isomorphisms:

$$\begin{aligned}\widetilde{K}_0(X \times \Delta, p_{X\times\Delta}, n, \epsilon) \ &\cong \ \widetilde{K}_0(X, p_X, n, \epsilon) \\ Wh(X \times \Delta, p_{X\times\Delta}, n+1, \epsilon) \ &\cong \ Wh(X, p_X, n+1, \epsilon)\end{aligned}$$

for every $n \ge 0$ and $\epsilon > 0$.

Proof : We consider the $\widetilde{K}_0$ case. The isomorphism is given by:

$$(\widetilde{R}_1, R_1)_* \ : \ \widetilde{K}_0(X \times \Delta, p_{X\times\Delta}, n, \epsilon) \longrightarrow \widetilde{K}_0(X, p_X, n, \epsilon)$$

with the inverse i_* induced by the inclusion $i : X \to X \times \Delta$. The composition $(\widetilde{R}_1, R_1)_* i_*$ is obviously the identity map. To prove that $i_*(\widetilde{R}_1, R_1)_*$ is the identity, we need to show the equivalence of (E, q) and $(\widetilde{R}_1)_\sharp(E, q)$ for every element $[E, q] \in \widetilde{K}_0(X \times \Delta, p_{X\times\Delta}, n, \epsilon)$, but this is obvious because there exists a sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $(\widetilde{R}_{t_i})_\sharp(E, q)$ and $(\widetilde{R}_{t_{i+1}})_\sharp(E, q)$ are ϵ isomorphic for each $i = 0, \cdots, m-1$. The isomorphisms are given by tracks of $\{\widetilde{R}_t\}_{t_i \le t \le t_{i+1}}$. The proof for Wh is similar and is omitted. □

Proof of 7.1 : We define the homomorphism

$$B \ : \ Wh(X, p'_X, n, \epsilon) \longrightarrow \widetilde{K}_0(X, p_X, n, \kappa_n\epsilon)$$

for every $n > 0$ and $\epsilon > 0$. Let C be an n-dimensional strongly ϵ contractible f.g. free ϵ chain complex on $p'_X : M \times S^1 \to X$. Let $\tilde{C}$ denote the pullback of C via the map $1_M \times \pi : M \times \mathbb{R} \to M \times S^1$. $\tilde{C}$ is not finitely generated, but is M-locally finite in the sense of §3. $\tilde{C}$ is strongly ϵ contractible measured in X, but not necessarily so when measured via $p_X \times 1_{\mathbb{R}} : M \times \mathbb{R} \to X \times \mathbb{R}$. Let K be a positive number and consider the linear map $\varphi^K : \mathbb{R} \to \mathbb{R}$ defined by $\varphi^K(x) = x/K$. If K is sufficiently large, then $\varphi^K_\sharp(\tilde{C})$ is an M-locally finite n-dimensional strongly ϵ contractible free ϵ chain complex on $p_X \times 1_{\mathbb{R}}$, thus it represents an element in $Wh^M(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, n, \epsilon)$. We define $B([C])$ to be the image of this element by the composition:

$$\begin{aligned}Wh^M(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, n, \epsilon) &\xrightarrow{\partial_+} \widetilde{K}_0^M(X \times J, p_X \times 1_J, n, \kappa_n\epsilon) \\ &= \ \widetilde{K}_0(X \times J, p_X \times 1_J, n, \kappa_n\epsilon) \xrightarrow{\cong} \widetilde{K}_0(X, p_X, n, \kappa_n\epsilon) \ ,\end{aligned}$$

where ∂_+ is the connecting homomorphism in the Mayer-Vietoris sequence for the triad $X \times (\mathbb{R}; (-\infty, 0], [0, \infty))$ and J is some interval $[-s, s]$, and the last map is the isomorphism of 7.2 induced by the retraction $M \times J \to M$. Because of this retraction at the end, the image of $[\varphi^K_\sharp(\tilde{C})]$ in $\widetilde{K}_0(X, p_X, n, \kappa_n \epsilon)$ is independent of the choice of K used for shrinking for a given C. Suppose $[C] = [C']$ in $Wh(X, p'_X, n, \epsilon)$. If we use a sufficiently large K, then $[\varphi^K_\sharp(\tilde{C})] = [\varphi^K_\sharp(\tilde{C}')]$ in $Wh^M(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, n, \epsilon)$. Therefore B is well-defined. It is obviously a homomorphism. We shall give an alternative description of B later.

Next we define

$$\bar{B} \ : \ \widetilde{K}_0(X, p_X, n, \delta) \longrightarrow Wh(X, p'_X, n, 18\delta)$$

for every $n > 0$ and $\delta > 0$. Let (A, p) be a δ projective module on p_X, and consider a geometric module $D = \mathbb{Z}[\{P\}]$ on S^1 generated by $P = \pi(0) \in S^1$. Define a path $t : [0, 1] \to \mathbb{R}$ by $t(\theta) = \theta$ $(0 \le \theta \le 1)$, and let z denote the path $(P, \pi \circ t : [0, 1] \to S^1, P)$ from P to P. Define a homomorphism

$$\begin{aligned} \bar{B}_0 \ : \ \widetilde{K}_0(X, p_X, \delta) &\longrightarrow Wh(X, p'_X, 2\delta) \ ; \\ [A, p] &\longrightarrow [f_p = (1 - p) \otimes 1 + p \otimes z \ : \ A \otimes D \to A \otimes D] \ . \end{aligned}$$

Tensor products of geometric modules and tensor products of geometric morphisms are defined in Yamasaki [26]. For the convenience of the reader, we give a brief review. Let $\mathbb{Z}[R]$ and $\mathbb{Z}[S]$ be geometric modules on M and N respectively. Their *tensor product* $\mathbb{Z}[R] \otimes \mathbb{Z}[S]$ is defined to be $\mathbb{Z}[R \times S : |R| \times |S| \to M \times N]$. For $r = (|r|, [r]) \in R$ and $s = (|s|, [s]) \in S$, $r \otimes s$ will denote the element $((|r|, |s|), ([r], [s]))$ of $R \times S$. If $(r, \rho : [0, \tau] \to M, r')$ is a path from $r \in R$ to $r' \in R'$ and $(s, \sigma : [0, \tau'] \to N, s')$ is a path from $s \in S$ to $s' \in S'$, then their *tensor product* $(r, \rho, r') \otimes (s, \sigma, s')$ is the path $(r \otimes s, \rho \otimes \sigma, r' \otimes s')$, where $\rho \otimes \sigma : [0, \tau + \tau'] \to M \times N$ is the following composite path:

$$\rho \otimes \sigma(x) = \begin{cases} (\rho(x), \sigma(0)) & \text{if } 0 \le x \le \tau, \\ (\rho(\tau), \sigma(x - \tau)) & \text{if } \tau \le x \le \tau + \tau'. \end{cases}$$

Tensor products of geometric morphisms are defined by bilinearly extending this. In general we have a homotopy $(f' \otimes g')(f \otimes g) \sim f'f \otimes g'g$ instead of a strict equality.

Let us go back to the definition of $\bar{B}_0$. It is easy to check that f_p is a δ isomorphism measured in X; its inverse is given by $(1 - p) \otimes 1 + p \otimes z^{-1}$. If we add a free module $(E, 1_E)$ to (A, p), then $f_{p \oplus 1_E} = f_p \oplus (1_E \otimes z)$ represent the same class as f_p. Next suppose that $g : (A, p) \to (A', p')$ is a δ isomorphism of δ projective modules, with inverse g^{-1}. Define a δ isomorphism $F : (A \otimes D) \oplus (A' \otimes D) \to (A \otimes D) \oplus (A' \otimes D)$ by

$$F \ = \ \begin{pmatrix} (1 - p) \otimes 1 & g^{-1} \otimes 1 \\ g \otimes 1 & (1 - p') \otimes 1 \end{pmatrix} \qquad (F^2 \sim_{2\delta} 1) \ ,$$

then

$$(1 \oplus f_{p'})F \sim_{5\delta} F(f_p \oplus 1) .$$

Now by 4.8, $[f'_p] = [f_p]$ in $Wh(X, p'_X, 2\delta)$ and hence $\bar{B}_0$ is well-defined. The desired $\bar{B}$ is defined by the composition:

$$\widetilde{K}_0(X, p_X, n, \delta) \xrightarrow{\sigma} \widetilde{K}_0(X, p_X, 9\delta) \xrightarrow{\bar{B}_0} Wh(X, p'_X, 18\delta) \xrightarrow{\iota} Wh(X, p'_X, n, 18\delta) .$$

The commutativity of the diagram of 7.1 is easily verified. $\square$

We rewrite 6.2(2) using 7.1. Let p_X, X_+, X_-, Y be as in §6. For a given $\epsilon > 0$, let W be a closed subspace of X containing $Y^{\kappa_n \epsilon}$ and γ be any number $\geq 18\kappa_n\epsilon$. Let p'_W be the composition:

$$p_X^{-1}(W) \times S^1 \xrightarrow{\text{projection}} p_X^{-1}(W) \xrightarrow{p_X} W .$$

Define $\bar{\partial}_+ : Wh(X, p_X, n, \epsilon) \to Wh(W, p'_W, n, \gamma)$ by the following composition:

$$Wh(X, p_X, n, \epsilon) \xrightarrow{\partial_+} \widetilde{K}_0(W, p_W, n, \gamma/18) \xrightarrow{\bar{B}} Wh(W, p'_W, n, \gamma) .$$

The following composition is 0:

$$\begin{aligned} Wh(X_-, p_{X_-}, n, \epsilon) \oplus Wh(X_+, p_{X_+}, n, \epsilon) &\xrightarrow{(j_-\ j_+)} Wh(X, p_X, n, \epsilon) \\ &\xrightarrow{\bar{\partial}_+} Wh(W, p'_W, n, \gamma) . \end{aligned}$$

Furthermore we have:

Corollary 7.3. *Fix an integer $n \geq 1$. There exists a constant M_n (≥ 1) which depends only on n such that, if $Z \supset W^{M_n \kappa_n \gamma}$ and $\delta \geq M_n \kappa_n \gamma$, the stabilization map $Wh(X, p_X, n, \epsilon) \to Wh(X, p_X, n, \delta)$ maps the kernel of*

$$\bar{\partial}_+ \ : \ Wh(X, p_X, n, \epsilon) \longrightarrow Wh(W, p'_W, n, \gamma)$$

into the image of

$$(j_-\ j_+) \ : \ Wh(X_- \cup Z, p, n, \delta) \oplus Wh(X_+ \cup Z, p, n, \delta) \longrightarrow Wh(X, p_X, n, \delta) .$$

Proof : Immediate from 6.2 and 7.1. The same constant M_n as in 6.2 can be used. $\square$

This will be used in the next section for a stable vanishing result for controlled Whitehead torsion.

Our next aim in this section is to study $Wh(X \times S^1, p_X \times 1_{S^1}, n, \epsilon)$. Define subspaces S^1_+, S^1_- $(\subset S^1)$ by $S^1_+ = \pi([0,1/2])$, $S^1_- = \pi([-1/2,0])$ and let $P = \pi(0)$ (as before), $Q = \pi(1/2)$, $N = \pi(1/4)$, $S = \pi(-1/4)$. When ϵ is sufficiently small ($\kappa_n \epsilon < 1/8$), one can use 7.2 to rewrite the Mayer-Vietoris sequence for the triad $X \times (S^1; S^1_-, S^1_+)$ as follows:

$$
\begin{aligned}
&Wh(X, p_X, n, \epsilon) \oplus Wh(X, p_X, n, \epsilon) \xrightarrow{J} Wh(X, p_X, n, \epsilon) \oplus Wh(X, p_X, n, \epsilon) \\
&\xrightarrow{(i^-_* \ i^+_*)} Wh(X \times S^1, p_X \times 1, n, \epsilon) \xrightarrow{\partial'_+} \widetilde{K}_0(X \times \{P\}, p_X, n, \kappa_n\epsilon) \oplus \widetilde{K}_0(X \times \{Q\}, p_X, n, \kappa_n\epsilon) \\
&\xrightarrow{J'} \widetilde{K}_0(X = X \times \{S\}, p_X, n, \kappa_n\epsilon) \oplus \widetilde{K}_0(X = X \times \{N\}, p_X, n, \kappa_n\epsilon) \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
i^- \ &: \ X \ = \ X \times \{S\} \longrightarrow X \times S^1 \\
i^+ \ &: \ X \ = \ X \times \{N\} \longrightarrow X \times S^1
\end{aligned}
$$

are inclusion maps, and

$$
J \ = \ \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \ , \ J' \ = \ \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \ .
$$

If we further assume that $M_n \kappa_n \epsilon < 1/8$, then this sequence is stably exact.

Define $B' : Wh(X \times S^1, p_X \times 1, n, \epsilon) \to \widetilde{K}_0(X, p_X, n, \kappa_n \epsilon)$ by composing ∂'_+ with the projection onto the first direct summand, and consider

$$
0 \to Wh(X, p_X, n, \epsilon) \xrightarrow{i^+_*} Wh(X \times S^1, p_X \times 1, n, \epsilon) \xrightarrow{B'} \widetilde{K}_0(X, p_X, n, \kappa_n\epsilon) \to 0.
$$

The composition $B' i^+_*$ is zero. The map i^+_* is injective: the projection $p_M : M \times S^1 \to M$ induces the left inverse of i^+_*. Let $\delta = M_n \kappa_n \epsilon$. If $\kappa_n \delta < 1/8$, then from the stable exactness of the Mayer-Vietoris sequence above one can deduce that this sequence is also stably exact:

(1) the stabilization image of $\ker B'$ in $Wh(X \times S^1, p_X \times 1, n, \delta)$ is contained in the image of

$$
i^+_* \ : \ Wh(X, p_X, n, \delta) \longrightarrow Wh(X \times S^1, p_X \times 1, n, \delta) \ ,
$$

(2) the stabilization image of $\widetilde{K}_0(X, p_X, n, \kappa_n\epsilon)$ in $\widetilde{K}_0(X, p_X, n, \kappa_n\delta)$ is contained in the image of

$$
B' \ : \ Wh(X \times S^1, p_X \times 1, n, \delta) \longrightarrow \widetilde{K}_0(X, p_X, n, \kappa_n\delta) \ .
$$

By the remark preceding 5.3, the following diagram commutes.

$$\begin{array}{ccc} Wh(X\times S^1, p_X\times 1, n,\epsilon) & \xrightarrow{B'} & \widetilde{K}_0(X,p_X,n,\kappa_n\epsilon) \\ {\scriptstyle F}\big\downarrow & & \big\| \\ Wh(X,p'_X,n,\epsilon) & \xrightarrow[B]{} & \widetilde{K}_0(X,p_X,n,\kappa_n\epsilon), \end{array}$$

where F denotes the 'forget-control-in-S^1' map induced by $\Phi = (1 : M\times S^1 \to M\times S^1, \mathrm{projection} : X\times S^1 \to X)$.

Let $[C]\in Wh(X,p'_X,n,\epsilon)$. For a positive integer k, let $\widetilde{C}^k$ denote the pullback of C via the k-fold covering

$$M\times S^1 \longrightarrow M\times S^1 \ ;\ (m,\pi(\theta)) \longrightarrow (m,\pi(k\theta))\ .$$

If k is sufficiently large, then $\widetilde{C}^k$ represents an element of $Wh(X\times S^1, p_X\times 1, n,\epsilon)$. Again by the remark preceding 5.3, we have the equality : $B([C]) = B'([\widetilde{C}^k])$. This is the alternative description of B mentioned before.

Furthermore, we can use pullback to construct a stable right inverse of $\bar{B}'$ of B'. For an integer $k\geq 1/\gamma$, define:

$$\bar{B}'_{0,k}\ :\ \widetilde{K}_0(X,p_X,\gamma) \longrightarrow Wh(X\times S^1, p_X\times 1, 8\gamma)\ ;\ [A,p] \longrightarrow [\widetilde{(f_p)}^k]\ .$$

Here ($f_p = (1-p)\otimes 1 + p\otimes z$) is regarded as a 1-dimensional chain complex. If we define $\bar{B}'_k$ by:

$$\bar{B}'_k\ =\ \iota\bar{B}'_{0,k}\sigma\ :\ \widetilde{K}_0(X,p_X,n,\delta) \longrightarrow Wh(X\times S^1, p_X\times 1, n, 72\delta)$$

then $B'\bar{B}'_k$ is equal to the stabilization map. Therefore $Wh(X\times S^1, p_X\times 1, n,\epsilon)$ is stably a direct sum of $Wh(X,p_X,n,\epsilon)$ and $\widetilde{K}_0(X,p_X,n,\epsilon)$.

This stable splitting does depend on the integer k. But stably it depends only on k mod 2. Suppose $l > k \geq 1/\gamma$. Stretch a portion of $\widetilde{(f_p)}^l$ along an arc $\Delta\subset S^1$ to match with $\widetilde{(f_p)}^k$ over $X\times\Delta'$ for some subarc $\Delta'\subset\Delta$ and then use 5.3(1) to conclude that $\widetilde{(f_p)}^l - \widetilde{(f_p)}^k$ lies in the image of $Wh(X,p_X,n,72L_n\delta)$, where L_n is the constant given in 5.3. But this element must be zero, because

$$(p_M)_*[\widetilde{(f_p)}^l] = \tau\begin{pmatrix} 1-p & 0 & 0 & \dots & p \\ p & 1-p & 0 & \dots & 0 \\ 0 & p & 1-p & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1-p \end{pmatrix}$$

is equal to $(p_M)_*[\widetilde{(f_p)}^k]$ if $l\equiv k \pmod 2$. Therefore $\bar{B}'_k : \widetilde{K}_0(X,p_X,n,\delta)\to Wh(X\times S^1, p_X\times 1, n, 72L_n\delta)$ depends only on k mod 2. (If we use the geometrically significant $g_p = (1-p)\otimes 1 - p\otimes z$ of Ranicki [21] instead of f_p, then $\bar{B}'_k$ is independent of $k (\gg 0)$.)

8. The eventual Vietoris theorem.

A version of the Vietoris theorem appropriate to a non-connective generalized homology theory h_* states that if a map $p : M \longrightarrow K$ of reasonable spaces (such as polyhedra) has h_*-acyclic point inverses in dimensions ≤ 1

$$h_{-k}(p^{-1}(v) \longrightarrow \{v\}) \;=\; 0 \;\; (v \in K \;,\; k \geq -1)$$

then p is an h_*-isomorphism in dimensions ≤ 1

$$h_{-k}(p) \;=\; 0 \;\; (k \geq -1) \;.$$

There is an eventual Vietoris theorem for controlled torsion: if a reasonable control map $p : M \longrightarrow K$ is such that

$$Wh_{-k}(\pi_1(p^{-1}(v))) \;=\; 0 \;\; (v \in K \;,\; k \geq -1)$$

then for every $\epsilon > 0$, $n > 0$ there exists a $\delta > 0$ such that the stabilization maps

$$Wh_{-k}(K, p, n, \delta) \longrightarrow Wh_{-k}(K, p, n, \epsilon) \;\; (k \geq -1)$$

are zero. See the appendix. In fact, we shall avoid the overt use of the condition $Wh_{-k}(\pi_1(p^{-1}(v))) = 0$ involving the lower Wh-groups Wh_{-k} $(k \geq 0)$ by using the controlled version in §7 of the Bass-Heller-Swan splitting, crossing with the k-tori $T^k = (S^1)^k$ and using the stronger hypothesis $Wh(\pi_1(p^{-1}(v)) \times \mathbb{Z}^k) = 0$ for all $v \in K$, $k \geq 0$.

As an application we study the 'forget-control' assembly maps. For any control map $p_X : M \to X$ and any $\delta > 0$, there is a 'forget-control' map:

$$Wh(X, p_X, n, \delta) \longrightarrow Wh(X, p_X, n, +\infty) \;,$$

as an extreme of stabilization maps. If M is connected and locally 1-connected, then the assembly map gives an isomorphism

$$Wh(X, p_X, n, +\infty) \xrightarrow{\cong} Wh(\pi_1(X)).$$

The composite of these is the 'forget-control' assembly map. 'Forget-control' assembly maps for $\widetilde{K}_0$ are also defined similarly.

The Vietoris theorem for controlled torsion implies that, if we further assume that X is a connected compact metric ANR and $n \geq 0$, then there exists a $\delta > 0$ such that the image of the 'forget-control' assembly map

$$Wh(X, p_X, n, \delta) \longrightarrow Wh(\pi_1(M))$$

is contained in the kernel of

$$(p_X)_* \ : \ Wh(\pi_1(M)) \longrightarrow Wh(\pi_1(X)) \ .$$

Similarly, for $n \geq 0$, there exists a $\delta > 0$ such that the image of the 'forget-control' assembly map

$$\widetilde{K}_0(X, p_X, n, \delta) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi_1(M)])$$

is contained in the kernel of

$$(p_X)_* \ : \ \widetilde{K}_0(\mathbb{Z}[\pi_1(M)]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \ .$$

These results were originally obtained by Chapman and Ferry, using more geometric methods.

Let K be a finite polyhedron, and suppose that the control map $p_K : M \to K$ has an iterated mapping cylinder structure (Hatcher [12]), and that

$$Wh(\pi_1(p_K^{-1}(v)) \times \mathbb{Z}^k) \ = \ 0 \ \ (k \geq 0)$$

for every vertex $v \in K$. For each $k \geq 0$, let $p_K^{(k)}$ denote the composition:

$$p_K^{(k)} \ : \ M \times T^k \xrightarrow{\text{projection}} M \xrightarrow{p_K} K \ .$$

Then $p_K^{(k)}$ also has an iterated mapping cylinder structure induced from that of p_K and satisfies the same Whitehead group condition.

Theorem 8.1. *Let p_K be as above. For any $n > 0$ and $\epsilon > 0$, there exists a $\delta > 0$ such that the stabilization map*

$$Wh(K, p_K^{(k)}, n, \delta) \longrightarrow Wh(K, p_K^{(k)}, n, \epsilon)$$

is the zero map for every $k \geq 0$.

Proof : In the following proof, we do not distinguish a simplicial complex from its underlying polyhedron. For a simplicial complex L, $\sharp(L)$ will denote the number of simplices in L. Fix p_K and $n > 0$. We inductively show that there exists a sequence

$$(\epsilon \geq)\, \delta_1(\epsilon) \geq \delta_2(\epsilon) \geq \delta_3(\epsilon) \geq \cdots (> 0)$$

of positive functions such that if L is a subcomplex of K with $\sharp(L) \leq l$, then the stabilization maps

$$Wh(L, p_L^{(k)}, n, \delta_l(\epsilon)) \longrightarrow Wh(L, p_L^{(k)}, n, \epsilon)$$

are 0 for all $k \geq 0$ and all $\epsilon > 0$. Here $p_L^{(k)}$ is the restriction of $p_K^{(k)}$ to L. The theorem is a special case of this.

When $l = 1$, i.e., L is a point $\{v\}$, $\delta_1(\epsilon) = \epsilon$ works, because

$$Wh(\{v\}, p_{\{v\}}^{(k)}, n, \gamma) \;=\; Wh(\pi_1(p_K^{-1}(v) \times T^k)) \;=\; 0$$

for every $\gamma > 0$. Assume we have constructed $\delta_1, \ldots, \delta_{l-1}$. Let L be a subcomplex of K with $\sharp(L) \leq l$. Let Δ be a simplex of L that is not a face of any other simplex of L. Then L is the union of $L_+ = \Delta$ and $L_- = L - \text{interior}(\Delta)$ with intersection $L_0 = \partial\Delta$. Since $\sharp(L_0) < l$ and $\sharp(L_-) < l$, the stabilization maps

$$Wh(L_0, p_{L_0}^{(k)}, n, \delta_{l-1}(\epsilon)) \longrightarrow Wh(L_0, p_{L_0}^{(k)}, n, \epsilon)$$
$$Wh(L_-, p_{L_-}^{(k)}, n, \delta_{l-1}(\epsilon)) \longrightarrow Wh(L_-, p_{L_-}^{(k)}, n, \epsilon)$$

are 0 for all $\epsilon > 0$ and $k \geq 0$ by induction hypothesis. Note that this is also true for L_+, because

$$Wh(L_+, p_{L_+}^{(k)}, n, \gamma) \;\cong\; Wh(\{v\}, p_{\{v\}}^{(k)}, n, \gamma) \;=\; 0$$

for all $\gamma \geq 0$ and $k \geq 0$, by 7.2.

Now fix $\epsilon > 0$. Let $\hat{N}$ denote a regular neighbourhood of L_0 in L. Here and in the rest of the proof, a 'regular neighbourhood' of a subcomplex means a star neighbourhood of some iterated barycentric subdivision of the original simplicial structure. This is to ensure that there exists a strong deformation retraction of the regular neighbourhood of the subcomplex which can be covered by a strong deformation retraction of the preimage by p_K. Thus one can choose a strong deformation retraction $\{r_t\}_{0 \leq t \leq 1}$ of $L_- \cup \hat{N}$ to L_- so that it is covered by a strong deformation retraction $\{\tilde{r}_t\}$ of $p_K^{-1}(L_- \cup \hat{N})$ to $p_K^{-1}(L_-)$. This induces a strong deformation retraction $\{\tilde{r}_t^{(k)} = \tilde{r}_t \times 1_{T^k}\}$ of $(p_K^{(k)})^{-1}(L_- \cup \hat{N})$ to $(p_K^{(k)})^{-1}(L_-)$. Unlike 7.2, r_t may increase the distance. But, by the compactness of $L_+ \cap \hat{N}$, there exists a positive number $\delta^-(\epsilon)$ $(\leq \delta_{l-1}(\epsilon))$ which makes the following diagram commute for all $k \geq 0$

$$\begin{tikzcd}
Wh(L_- \cup \hat{N}, p^{(k)}, n, \delta^-(\epsilon)) \arrow[r] \arrow[d, "(\tilde{r}_1^{(k)}, r_1^{(k)})_*"'] & Wh(L_- \cup \hat{N}, p^{(k)}, n, \epsilon) \\
Wh(L_-, p^{(k)}, n, \delta_{l-1}(\epsilon)) \arrow[r] & Wh(L_-, p^{(k)}, n, \epsilon) \arrow[u, "i_*"']
\end{tikzcd}$$

Since the second row is the zero map, the top row is also the zero map for all $k \geq 0$. Similarly, there exists a positive number $\delta^+(\epsilon)$ such that

$$Wh(L_+ \cup \hat{N}, p^{(k)}, n, \delta^+(\epsilon)) \longrightarrow Wh(L_+ \cup \hat{N}, p^{(k)}, n, \epsilon)$$

is the zero map for all $k \geq 0$. Let $\hat{\delta}(\epsilon) = \min\{\delta^+(\epsilon), \delta^-(\epsilon)\}$, and choose a positive number γ sufficiently small so that

1. $M_n \kappa_n \gamma \leq \hat{\delta}(\epsilon)$, and
2. there exists a smaller regular neighbourhood N of L_0 in L such that $N^{M_n \kappa_n \gamma} \subset \hat{N}$, where M_n is the constant given in 7.3 and κ_n is the constant defined in §6.

As in the case of $L_- \cup \hat{N}$, there exists a positive function $\delta^0(\alpha)$ such that

$$Wh(N, p^{(k)}, n, \delta^0(\alpha)) \longrightarrow Wh(N, p^{(k)}, n, \alpha)$$

is the zero map for every $\alpha > 0$ and $k \geq 0$. Now choose $\delta^L(\epsilon) > 0$ sufficiently small so that

1. $\delta^L(\epsilon) < \delta^0(\gamma)/18\kappa_n$, and
2. $N \supset L_0^{\kappa_n \delta^L(\epsilon)}$,

where γ is as above.

Consider the following commutative diagram.

$$\begin{tikzcd}
 & Wh(L, p_L^{(k)}, n, \delta^L(\epsilon)) \arrow[r, "\bar{\partial}_+"] \arrow[d, equal] & Wh(N, p_N^{(k+1)}, n, \delta^0(\gamma)) \arrow[d, "0"] \\
 & Wh(L, p_L^{(k)}, n, \delta^L(\epsilon)) \arrow[r, "\bar{\partial}_+"] \arrow[dr] & Wh(N, p_N^{(k+1)}, n, \gamma) \\
\multicolumn{2}{c}{Wh(L_- \cup \hat{N}, p^{(k)}, n, \hat{\delta}(\epsilon)) \oplus Wh(L_+ \cup \hat{N}, p^{(k)}, n, \hat{\delta}(\epsilon))} \arrow[r] \arrow[d, "0"'] & Wh(L, p_L^{(k)}, n, \hat{\delta}(\epsilon)) \arrow[d] \\
Wh(L_- \cup \hat{N}, p^{(k)}, n, \epsilon) \oplus Wh(L_+ \cup \hat{N}, p^{(k)}, n, \epsilon) \arrow[r] & & Wh(L, p_L^{(k)}, n, \epsilon)
\end{tikzcd}$$

A simple diagram chase shows that the stabilization map

$$Wh(L, p_L^{(k)}, n, \delta^L(\epsilon)) \longrightarrow Wh(L, p_L^{(k)}, n, \epsilon)$$

is the zero map for all $k \geq 0$. Since there are only finitely many subcomplexes L with $\sharp(L) \leq l$, we can define $\delta_l(\epsilon)$ to be $\min\{\delta^L(\epsilon) | \sharp(L) \leq l\}$. This completes the inductive step and the theorem is proved. $\square$

Corollary 8.2. *Let p_K be as above. For any $n \geq 0$ and $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\widetilde{K}_0(K, p_K^{(k)}, n, \delta) \longrightarrow \widetilde{K}_0(K, p_K^{(k)}, n, \epsilon)$$

is the zero map for every $k \geq 0$.

Proof : When $n > 0$, this follows immediately from 8.1 and 7.1. The $n = 0$ case follows from the $n = 1$ case. $\square$

The following is an algebraic version of Ferry [10, Cor.3.2]:

Corollary 8.3. *Let X be a connected compact metric ANR embedded in the Hilbert cube I^∞. For any $n \geq 0$ and $\epsilon > 0$, there exists a $\delta > 0$ such that the stabilization maps*

$$\widetilde{K}_0(X, 1_X, n, \delta) \to \widetilde{K}_0(X, 1_X, n, \epsilon) \quad , \quad Wh(X, 1_X, n+1, \delta) \to Wh(X, 1_X, n+1, \epsilon)$$

are both zero. Consequently, there exists a $\delta_{X,n} > 0$ such that the 'forget-control' assembly maps

$$\widetilde{K}_0(X, 1_X, n, \delta_{X,n}) \to \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \quad , \quad Wh(X, 1_X, n+1, \delta_{X,n}) \to Wh(\pi_1(X))$$

are both zero.

Proof : X has a neighbourhood U with a retraction $r : U \to X$. We may assume that U is of the form $K \times I^{\infty - N}$, where K is a codimension 0 PL submanifold of I^N. Let $m = n + 1$. By the compactness of U there is a $\gamma > 0$ such that $(r, r) : 1_U \to 1_X$ induces a homomorphism

$$(r, r)_* \ : \ Wh(U, 1_U, m, \gamma) \longrightarrow Wh(X, 1_X, m, \epsilon) \ .$$

Since $Wh(\mathbb{Z}^k) = 0$ (Bass-Heller-Swan [2]), we can apply 8.1 to $1_K : K \to K$; there exists a $\delta > 0$ such that the homomorphism $Wh(K, 1_K^{(k)}, m, \delta) \to Wh(K, 1_K^{(k)}, m, \gamma)$ is the zero map for every $k \geq 0$. Let $r' : U = K \times I^{\infty - N} \to K$ denote the projection and $i' : K = K \times (0, 0, \ldots) \to U$ denote the inclusion map. These induce isomorphisms in Wh which are inverses of each other by 7.2. The following diagram commutes:

$$\begin{array}{ccccc}
Wh(X, 1_X^{(k)}, m, \delta) & \xrightarrow{\ i_*\ } & Wh(U, 1_U^{(k)}, m, \delta) & \xrightarrow[\cong]{(r', r')_*} & Wh(K, 1_K^{(k)}, m, \delta) \\
\big\downarrow & & \big\downarrow & & \big\downarrow 0 \\
Wh(X, 1_X^{(k)}, m, \epsilon) & \xleftarrow[(r, r)_*]{} & Wh(U, 1_U^{(k)}, m, \gamma) & \xleftarrow[i'_*]{\cong} & Wh(K, 1_K^{(k)}, m, \gamma)
\end{array}$$

where the vertical maps are stabilization maps. Therefore, all the vertical maps are zero maps.

Let $\epsilon = 1$, $k = 0$ and let $\delta_{X,n}$ be the corresponding δ. Since the forget-control map $Wh(X, 1_X, n+1, \delta_X) \to Wh(\pi_1(X))$ factors through $Wh(X, 1_X, n+1, 1)$, it is the zero map.

The claim for $\widetilde{K}_0$ (with a smaller $\delta_{X,n}$) follows from the $k = 1$ case and 7.1. $\square$

The following is an algebraic version of Chapman [5, Theorem 1′]:

Corollary 8.4. *Suppose $p_X : M \to X$ is a control map of a connected locally 1-connected space M to a connected compact metric ANR X embedded in I^∞. For any $n \geq 0$, there exists a $\delta > 0$ such that the images of the 'forget-control' assembly maps*

$$Wh(X, p_X, n+1, \delta) \longrightarrow Wh(\pi_1(M))$$

$$\widetilde{K}_0(X, p_X, n, \delta) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi_1(M)])$$

are contained in the kernels of

$$(p_X)_* \; : \; Wh(\pi_1(M)) \longrightarrow Wh(\pi_1(X))$$

$$(p_X)_* \; : \; \widetilde{K}_0(\mathbb{Z}[\pi_1(M)]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

respectively.

Proof : Let $\delta_{X,n}$ be as in 8.3. The claim for Whitehead groups is immediate from the following commutative diagram.

$$\begin{tikzcd}
Wh(X, p_X, n+1, \delta_{X,n}) \arrow[r, "(p_X, 1_X)_*"] \arrow[d] & Wh(X, 1_X, n+1, \delta_{X,n}) \arrow[d, "0"] \\
Wh(\pi_1(M)) \arrow[r, "(p_X)_*"'] & Wh(\pi_1(X))
\end{tikzcd}$$

The $\widetilde{K}_0$ case is similar. □

9. Controlled finiteness obstruction and torsion.

We shall now use the theory of transverse CW complexes to define controlled finiteness obstruction and torsion using the algebraically defined value groups of §3 and §4. Previously, Chapman [6, §§5,7] had defined controlled finiteness obstruction and torsion using geometrically defined value groups. The geometric invariants determine the algebraic invariants - we shall not need this, and for our purposes it suffices to consider only the algebraic ones, since these assemble to the uncontrolled finiteness obstruction and torsion respectively.

Let K be a CW complex. $K^{(k)}$ will denote its k-skeleton. A map $f : (M^k, \partial M) \to (K^{(k)}, K^{(k-1)})$ from a smooth k-dimensional manifold (possibly with boundary) is said to be *transverse to the k-cells* if for each open k-cell e^k of K, $f^{-1}(e^k)$ is a disjoint union of the interiors of finitely many closed k-balls B_i^k in M such that there exists a homeomorphism $\psi_i : B_i^k \to D^k$ to the k-ball D^k with $\theta_{e^k} \circ \psi_i = f|B_i^k$ for each i. Here $\theta_{e^k}; D^k \to K$ denotes the characteristic map for the closed k-cell $\bar{e}^k$. Any continuous map $f : (M^k, \partial M) \to (K^{(k)}, K^{(k-1)})$ is homotopic rel ∂ to one that is transverse to the k-cells.

A CW complex K is *transverse* if the attaching maps $\varphi : S^k \to K^{(k)}$ of the $(k+1)$-cells are all transverse to the k-cells for every k. Any finite CW complex is simple homotopy equivalent to a transverse CW complex. A subdivision (Milnor [15]) of a transverse CW complex is also transverse.

A map $f : K \to L$ between transverse CW complexes is *t-cellular* if it is cellular and for each cell e^k of K, the composition

$$(D^k, S^{k-1}) \xrightarrow{\theta} (K^{(k)}, K^{(k-1)}) \xrightarrow{f} (L^{(k)}, L^{(k-1)})$$

is transverse to the k-cells, where θ is the characteristic map for e^k. Any map can be homotoped to a t-cellular map.

A t-cellular map $f : K \to L$ induces a chain map $f_\% : f_\sharp C(K) \to C(L)$. Here $C(-)$ denotes the geometric cellular chain complex defined by Quinn [18] and $f_\sharp(-)$ denotes the geometric module chain complex obtained by applying $f_\sharp$ to the modules and morphisms. $f_\sharp C_k(K)$ is generated by the images of the centers of k-cells of K in L. For each k-cell of K, consider the characteristic map $\theta : D^k \to K$ and take the radial paths in D^k starting at the center of D^k and ending at the preimages by $f\theta$ of the centers of the k-cells of L. A path is assigned a $+$ sign (resp. a $-$ sign) if $f\theta$ is orientation-preserving (resp. orientation-reversing) about its endpoint. The chain map $f_\%$ is defined by the sum of the images of these paths in L with assigned sign. To see that this actually defines a chain map, let $\bigsqcup U_i$ be the preimage of the open k-cells of L via $f\theta : (D^k, S^{k-1}) \to (L^{(k)}, L^{(k-1)})$. Make $f\theta : D^K - \bigsqcup U_i \to L^{(k-1)}$ transverse to the centers $\{v_\alpha\}$ of the $(k-1)$-cells of L keeping the boundary fixed. Then the preimage of $\{v_\alpha\}$ in $D^k - \bigsqcup U_i$ is the disjoint union of circles and arcs. Using these arcs, we can make the paths in $f_\% d - d f_\%$ into pairs of opposite sign so that the paired paths are homotopic in L rel ∂.

If two t-cellular maps f, g are homotopic, then there is a diagram

$$\begin{tikzcd}
f_\sharp C(K) \arrow[dd, "\cong"'] \arrow[dr, "f_\%"] & \\
\arrow[r, phantom, "\simeq"] & C(L) \\
g_\sharp C(K) \arrow[ur, "g_\%"'] &
\end{tikzcd}$$

that commutes up to chain homotopy. Here the vertical map is the geometric isomorphism given by the homotopy.

If $f : K \to L$ and $g : L \to M$ are both t-cellular, then so is the composition $gf : K \to M$, and $(gf)_\% \sim g_\% g_\sharp(f_\%)$.

We shall now define the controlled torsion of a controlled homotopy equivalence. See Chapman [6, p.2] for the terminology.

Let K and L be n-dimensional transverse finite CW complexes, such that L is equipped with a control map $p_X : L \to X$ to a metric space X. The torsion of a $p_X^{-1}(\epsilon)$-equivalence $f : K \to L$ will be defined to be an element $\tau(f) \in Wh(X, p_X, n+1, 360\epsilon)$. Let $g : L \to K$ be a $p_X^{-1}(\epsilon)$-homotopy inverse of f. Subdivide K and L if necessary, and assume that the diameter of the image in X of each cell of K and L via $p_X f$ and p_X is smaller than $\epsilon/10$. Transverse CW complexes are "saturated" in the sense of Quinn [18]. Therefore f is $p_X^{-1}(\epsilon/10)$-homotopic to a t-cellular map $f' : K \to L$, and g is $(p_X f)^{-1}(\epsilon/10)$-homotopic to a t-cellular map $g' : L \to K$. Then $f'g'$ is t-cellularly $p_X^{-1}(2\epsilon)$-homotopic to 1_L, and $g'f'$ is t-cellularly $(p_X f')^{-1}(4\epsilon)$-homotopic to 1_K. $f'_\sharp C(K)$ and $C(L)$ are both free ϵ chain complexes. Consider the induced ϵ chain map

$$F = f'_\% : f'_\sharp C(K) \longrightarrow C(L)$$

and the composite 7ϵ chain map

$$G : C(L) \xrightarrow{\cong} f'_\sharp g'_\sharp C(L) \xrightarrow{f'_\sharp(g'_\%)} f'_\sharp C(K) ,$$

where the first map is the geometric 2ϵ isomorphism induced by the homotopy $1 \simeq f'g'$. Then FG is 9ϵ chain homotopic to $1 : C(L) \to C(L)$. This 9ϵ comes from the size estimate of the trace of each cell of L by the 2ϵ-homotopy $p_X \simeq p_X f'g'$. Similarly the $(p_X f')^{-1}(4\epsilon)$-homotopy gives a 17ϵ chain homotopy $GF\lambda \simeq 1$, where $\lambda : f'_\sharp C(K) \to f'_\sharp C(K)$ is the geometric 6ϵ isomorphism induced by the $p_X^{-1}(6\epsilon)$-homotopy

$$f' = f'1_{L'} \simeq f'(g'f') = (f'g')f' \simeq 1_{L'}f' = f' .$$

This homotopy induces a 13ϵ chain homotopy $f'_\%\lambda \simeq f'_\%$. Thus $GF\lambda \simeq_{20\epsilon} GF$, and $GF \simeq_{37\epsilon} 1$. F is a 40ϵ chain equivalence, and its torsion is defined in $Wh(X, p_X, n+1, 360\epsilon)$. This class is independent of the choice of f'. The forget-control assembly image of this class in $Wh(\pi_1(X))$ is the ordinary Whitehead torsion $\tau(f)$.

Next, we define the controlled finiteness obstruction of a controlled dominated space.

Let K and M be n-dimensional transverse CW complexes, and let $K \underset{u}{\overset{d}{\rightleftarrows}} M$ be a $p_X^{-1}(\epsilon)$-domination of M with respect to a control map $p_X : M \to X$. Assume that K is finite, then we may assume that the image of each cell of K by the map $p_X d$ has diameter $\le \epsilon$, by subdividing K if necessary. Also assume that the CW decomposition of M is sufficiently fine so that the image of each cell of M by p_X has diameter $\le \epsilon$.

Then $C(M)$ is δ dominated by $d_\sharp C(K)$ for some $\delta > 0$, and hence by 5.1 $(C(M), 1)$ is $(2n+5)\delta$ chain equivalent to an n-dimensional $(n+4)\delta$ projective chain complex. (Alternatively, apply the instant finiteness obstruction formula of Lück and Ranicki [14] to the controlled chain homotopy idempotent induced by $ud \simeq (ud)^2 : K \to K$. If we take this approach, then the assumption above on the CW structure on M is unnecessary.) The reduced projective class of this complex is the controlled finiteness obstruction $[M] \in \widetilde{K}_0(X, p_X, n, (4n+10)\delta)$. The forget-control assembly image in $\widetilde{K}_0(\mathbb{Z}[\pi_1(M)])$ is the ordinary Wall finiteness obstruction $[M]$.

10. The topological invariance and finiteness theorems.

We shall now use the Vietoris-type theorem of §8 and the controlled torsion and finiteness obstruction of §9 to prove that the torsion of a homeomorphism is zero, and that the finiteness obstruction of a compact ANR is zero.

Theorem 10.1. (Topological Invariance of Torsion) *A homeomorphism between finite CW complexes is simple.*

Theorem 10.2. (Borsuk Conjecture) *A compact metric ANR is homotopy equivalent to a finite polyhedron.*

These were originally proved by Chapman [3] and West [24], respectively. Actually, for these applications the controlled algebra of [16], [17] suffices, with geometric morphisms defined without using paths.

Proof of 10.1: Let $f : K \to L$ be a homeomorphism between finite CW complexes. We shall show that the Whitehead torsion $\tau(f)$ is 0. Without loss of generality, we may assume that L is embedded in the Hilbert cube and that K and L are transverse. Since the Whitehead torsion is combinatorially invariant (Whitehead [25], Milnor [15], Cohen [7]), we may replace K and L by their subdivisions K' and L' respectively. Approximate f by a t-cellular map f'. As in §9 for some $\delta > 0$, $\mathcal{C}(f'_\%)$ defines an element in $Wh(L', 1_{L'}, \delta) = Wh(L, 1_L, \delta)$ whose image in $Wh(\pi_1 L)$ via the 'forget-control' assembly map is the torsion $\tau(f)$. One can make δ arbitrarily small by choosing fine subdivisions and a close approximation f'. Therefore, $\tau(f)$ is 0 by 8.3. □

Proof of 10.2: Without loss of generality, we may assume that X is a subspace of the Hilbert cube. X has a neighbourhood V with a retraction $r : V \to X$. If N is sufficiently large, we can find a smaller neighbourhood $U \subset V$ of the form $K \times I^{\infty - N}$, where K is a codimension 0 PL submanifold of I^N. Let $j : X \to K$ denote the composition:

$$j \ : \ X \xrightarrow{\text{inclusion}} U \xrightarrow{\text{projection}} K \ ,$$

and $f : K \to X$ denote the composition:

$$f \ : \ K \ = \ K \times 0 \xrightarrow{\text{inclusion}} U \xrightarrow{r} X \ .$$

Then $f : K \to X$ is a finite domination of X: there is a homotopy $k_t : 1_X \ \simeq \ fj$. Let $p : K \to K$ be a t-cellular approximation of jf and let $h_t : jf \ \simeq \ p$ be a homotopy. Define $p_* : f_\sharp C(K) \to f_\sharp C(K)$ by the composition

$$f_\sharp C(K) \xrightarrow{\cong} f_\sharp p_\sharp C(K) \xrightarrow{f_\sharp(p_\%)} f_\sharp C(K)$$

where the first map is the geometric isomorphism induced by the homotopy

$$H_t \ : \ f \ \overset{k_t f}{\simeq} \ fjf \ \overset{f h_t}{\simeq} \ fp \ .$$

There is a homotopy $K_t : p^2 \simeq jfjf \simeq jf \simeq p$, and there is induced a chain homotopy between p_*^2 and $p_*\lambda$, where $\lambda : f_\sharp C(K) \to f_\sharp C(K)$ is the geometric isomorphism induced by the homotopy:

$$f \ \simeq \ fp \ \overset{H_t p}{\simeq} \ fpp \ \overset{fK_t}{\simeq} \ fp \ \simeq \ f \ .$$

If N is very large (i.e., $I^{\infty - N}$ is very thin), then the homotopy k_t is very small. Also the homotopy h_t can be assumed to be arbitrarily small. As X is locally contractible, λ is homotopic ($\sim$) to the identity for sufficiently large N. Thus we may assume that p_* is a chain homotopy projection. As in §9 this situation determines an element of $\widetilde{K}_0(X, 1_X, \delta)$ for some $\delta > 0$. Its image in $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ via the 'forget-control' assembly map is the ordinary Wall finiteness obstruction of X. Since one can make δ arbitrarily small, the finiteness obstruction of X must vanish, by 8.3. $\square$

Appendix. Controlled lower K-theory.

The stably exact sequences in §§5 and 6 can be extended to the right by introducing controlled lower K-groups.

Definition. For a control map $p_X : M \to X$ and an integer $i \geq 0$, define

$$\begin{aligned}
\widetilde{K}_{-i}(X, p_X, n, \epsilon) \ &= \ \widetilde{K}_0^M(X \times \mathbb{R}^i, p_X \times 1_{\mathbb{R}^i}, n, \epsilon) \ \ (n \geq 0) \\
Wh_{1-i}(X, Y, p_X, n, \epsilon) \ &= \ Wh^M(X \times \mathbb{R}^i, Y \times \mathbb{R}^i, p_X \times 1_{\mathbb{R}^i}, n, \epsilon) \ \ (n > 0) \ ,
\end{aligned}$$

using M-locally finite chain complexes (§§3,4). When $i = 0$, these are equal to the original controlled $\widetilde{K}_0$- and Wh-groups. As in §7, we use the maximum metric for product metric spaces (including $\mathbb{R}^i$).

The M-locally finite version of Mayer-Vietoris sequence for the triad $X \times (\mathbb{R}^{i+1}; \mathbb{R}^i \times (-\infty, 0], \mathbb{R}^i \times [0, \infty))$ reduces to the stable isomorphism :

$$0 \longrightarrow Wh^M(X \times \mathbb{R}^{i+1}, p_X \times 1, n, \epsilon) \xrightarrow{\partial_+} \widetilde{K}_0^M(X \times \mathbb{R}^i \times J, p_X \times 1 \times 1, n, \kappa_n \epsilon) \longrightarrow 0$$

where $\kappa_n = 1081n + 6600$ (as in §6) and J is some interval $[-s, s]$. This is because the terms concerning half infinite intervals vanish (Eilenberg swindle). For example, let $[D]$ be an element of $Wh^M(X \times \mathbb{R}^i \times [0, \infty), p_X \times 1 \times 1, n, \epsilon)$, $\bar{D}$ be a complex representing the additive inverse of $[D]$, and t be the translation of $M \times \mathbb{R}^i \times [0, \infty) \times \mathbb{R}^{-i}$ by ϵ in the positive direction of $[0, \infty)$, then

$$D \oplus t_\sharp \bar{D} \oplus t_\sharp^2 D \oplus t_\sharp^3 \bar{D} \oplus \cdots$$

is M-locally finite and represents $[D]$ as well as 0. By 7.2, the projection $M \times \mathbb{R}^i \times J \to M \times \mathbb{R}^i$ induces an isomorphism

$$\widetilde{K}_0^M(X \times \mathbb{R}^i \times J, p_X \times 1 \times 1, n, \kappa_n \epsilon) \ \cong\ \widetilde{K}_{-i}(X, p_X, n, \kappa_n \epsilon).$$

Thus there is a stable isomorphism

$$\alpha \ :\ Wh_{-i}(X, p_X, n, \epsilon) \longrightarrow \widetilde{K}_{-i}(X, p_X, n, \kappa_n \epsilon)$$

with an inverse

$$\beta \ :\ \widetilde{K}_{-i}(X, p_X, n, \epsilon) \longrightarrow Wh_{-i}(X, p_X, n, \lambda_n \epsilon)$$

where $\lambda_n = M_n^2 \kappa_n^2$. (Note that we have already encountered the map α with $i = 0$ in §7.)

This observation permits us to extend the sequences of §§5 and 6 to the right as follows :

$$\begin{aligned} & Wh_{1-i}(Y, p_Y, n, \epsilon) \to Wh_{1-i}(X, p_X, n, \epsilon) \to Wh_{1-i}(X, Y, p_X, n, \epsilon) \\ & \qquad \xrightarrow{\partial} Wh_{-i}(W, p_W, n, \epsilon') \to Wh_{-i}(X, p_X, n, \epsilon') \\ & \qquad (i \geq 0,\ n > 0,\ \epsilon > 0,\ W \supset Y^{K_n \epsilon}\ ,\ \epsilon' \geq K_n \lambda_n \epsilon) \\ & Wh_{1-i}(X_0, p, n, \epsilon) \to Wh_{1-i}(X_-, p, n, \epsilon) \oplus Wh_{1-i}(X_+, p, n, \epsilon) \to Wh_{1-i}(X, p, n, \epsilon) \\ & \qquad \xrightarrow{\partial_+} Wh_{-i}(W, p, n, \epsilon') \to Wh_{-i}(X_- \cup W, p, n, \epsilon') \oplus Wh_{-i}(X_+ \cup W, p, n, \epsilon') \\ & \qquad (i \geq 0,\ n > 0,\ \epsilon > 0,\ W \supset Y^{\kappa_n \epsilon},\ \epsilon' \geq \kappa_n \lambda_n \epsilon)\ . \end{aligned}$$

These are stably exact, but the details will be omitted.

When X is a point $\{*\}$ and M is connected and locally 1-connected, $\widetilde{K}_{-i}(\{*\}, M \to \{*\}, n, \epsilon)$ and $Wh_{1-i}(\{*\}, M \to \{*\}, n, \epsilon)$ are isomorphic to the ordinary reduced lower K-group $\widetilde{K}_{-i}(\mathbb{Z}[\pi_1(M)])$.

We can use these controlled lower K-groups to do the stable calculation of §8.

Let $p_K : M \to K$ be as in 8.1, a control map of M to a compact polyhedron K with an iterated mapping cylinder structure such that

$$Wh_{1-i}(\pi_1(p_K^{-1}(v))) \;=\; 0 \;\; (v \in K, i \geq 0) \; .$$

Theorem A1. *For any $n > 0$ and $\epsilon > 0$, there exists a $\delta > 0$ such that the stabilization map*

$$Wh_{1-i}(K, p_K, n, \delta) \longrightarrow Wh_{1-i}(K, p_K, n, \epsilon)$$

is zero for every $i \geq 0$.

See Ranicki [22] for an algebraic treatment of lower K-theory using the bounded algebra of Pedersen and Weibel.

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