SURGERY AND C^*-ALGEBRAS

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A C^*-algebra is a complex Banach algebra A with an involution $*$, which satisfies the identity

$$\|x^*x\| = \|x\|^2 \quad \forall x \in A.$$

Key examples

- The algebra $C(X)$ of continuous complex-valued functions on a compact Hausdorff space X.

- The algebra $\mathfrak{B}(H)$ of bounded linear operators on a Hilbert space H.

Gelfand and Naimark (about 1950) proved: Any commutative C^*-algebra with unit is of the form $C(X)$; any C^*-algebra is a subalgebra of some $\mathfrak{B}(H)$.

The spectral theorem

Let $x \in A$ be normal, that is $xx^* = x^*x$. Then x generates a commutative C^*-subalgebra of A which must be of the form $C(X)$. In fact can identify X as the spectrum

$$X = \sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ has no inverse}\}$$

with x itself corresponding to the canonical $X \to \mathbb{C}$.

Hence the Spectral Theorem: for any $\phi \in C(\sigma(x))$ we can define $\phi(x) \in A$ so that the assignment $\phi \mapsto \phi(x)$ is a ring homomorphism.

If x is self adjoint ($x = x^*$), then $\sigma(x) \subseteq \mathbb{R}$.

K-Theory

One can define K-theory groups for C^*-algebras. For A unital

- $K_0(A) =$ Grothendieck group of f.g. projective A-modules

- $K_1(A) = \pi_0 GL_\infty(A)$

with a simple modification for non-unital A.

Reference N.E. Wegge-Olsen, 'K-theory and C^*-algebras' (Oxford University Press)

For any integer i define $K_i = K_{i\pm 2}$. Then to any short exact sequence of C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

there is a long exact K-theory sequence

$$\ldots K_i(J) \to K_i(A) \to K_i(A/J) \to K_{i-1}(J) \ldots$$

The 2-periodicity is a version of the Bott periodicity theorem.

Remark Algebraic K-theory does not satisfy Bott periodicity; analysis is essential here.

Example: The Fredholm index

An operator T on a Hilbert space H is called Fredholm if it has finite-dimensional kernel and cokernel. Then the index of T is the difference of the dimensions of the kernel and cokernel.

Definition The algebra of compact operators, $\mathfrak{K}(H)$, is the C^*-algebra generated by the operators with finite-dimensional range.

Atkinson's Theorem $T \in \mathfrak{B}(H)$ is compact if and only if its image in $\mathfrak{B}(H)/\mathfrak{K}(H)$ is invertible.

Thus a Fredholm operator T defines a class $[T]$ in $K_1(\mathfrak{B}/\mathfrak{K})$. Under the connecting map this passes to $\partial[T] \in K_0(\mathfrak{K}) = \mathbb{Z}$; this is the index.

Abstract signatures

Recall isomorphisms $L^0(\mathbb{Z}) \to L^0(\mathbb{R}) \to \mathbb{Z}$. The second map associates to a nonsingular real symmetric matrix its signature = (Number of positive eigenvalues) $-$ (Number of negative eigenvalues).

Can we generalize to other rings?

If M is a nonsingular symmetric matrix over a C^*-algebra A we can use the spectral theorem to define projections p_+ and p_- corresponding to the positive and negative parts of the spectrum. Their difference is a class in $K_0(A)$.

This procedure defines an isomorphism $L^p_0(A) \to K_0(A)$ for every C^*-algebra A.

Reference J. Rosenberg, 'Analytic Novikov for topologists'.

The group C^*-algebra

Let Γ be a discrete group. The group ring $\mathbb{Z}\Gamma$ acts faithfully by convolution on the Hilbert space $\ell^2\Gamma$. The C^*-subalgebra of $\mathfrak{B}(\ell^2\Gamma)$ generated by $\mathbb{Z}\Gamma$ acting in this way is called the group C^*-algebra, $C_r^*\Gamma$.

We have a map $L^0(\mathbb{Z}\Gamma) \to K_0(C_r^*\Gamma)$.

Gelfand and Mischenko observed that this map is an isomorphism for Γ free abelian. (Then $C_r^*\Gamma = C(T^k)$ by Fourier analysis.)

Remark There is no map from L^0 to K_0 for general rings; the C^* properties are vital.

The signature operator

Let M be a complete oriented Riemannian manifold. Define the operator $F = D(1 + D^2)^{-\frac{1}{2}}$ on L^2 differential forms, where $D = d + d^*$, d =exterior derivative, d^* = its adjoint.

F is graded by an involution $\epsilon = i^? *$. Thus graded it is called the signature operator.

If M is compact, then F is Fredholm. Moreover the index of F is the signature of M. This is a simple consequence of Hodge theory.

Remark The choice of normalizing function ϕ in $F = \phi(D)$ does not matter as long as it has the right asymptotic behaviour.

The higher index

Consider signature operator on the universal cover $\tilde{M}$ of a compact manifold M. F belongs to the algebra A of $\Gamma = \pi_1 M$ equivariant operators. Moreover it is invertible modulo the ideal J of Γ equivariant locally compact operators.

Thus via the connecting map ∂: $K_1(A/J) \to K_0(J)$ we get an 'index' in $K_0(J)$.

Lemma $J \simeq C_r^*\Gamma \otimes \mathfrak{K}$. Consequently $K_0(J) = K_0(C_r^*\Gamma)$.

We have defined the analytic signature of M as an element of this group. In general it can be defined in $K_i(M)$ where i is the dimension of M mod 2.

Prop The analytic signature is the image of the Mischenko-Ranicki symmetric signature under the map $L^0 \to K_0$.

Corollary The analytic signature is invariant under orientation-preserving homotopy equivalence. (See J. Kaminker and J. Miller, 'Homotopy invariance of the index of signature operators over C^*-algebras', J Operator Theory 14(1985) 113-127)

We can define an 'analytic surgery obstruction' (= difference of analytic signatures) for a degree one normal map.

Can we mimic the rest of the surgery exact sequence?

K-Homology

Let A be a C^*-algebra. A Fredholm module for A is made up of the following things.

- A representation $\rho\colon A \to \mathfrak{B}(H)$ of A on a Hilbert space

- An operator $F \in \mathfrak{B}(H)$ such that for all $a \in A$ the operators

  $$F\rho(a)-\rho(a)F, \quad (F^2-1)\rho(a), \quad (F-F^*)\rho(a)$$

  belong to $\mathfrak{K}(H)$.

The signature operator is an example with $A = C_0(M)$.

One can define both 'graded' and 'ungraded' Fredholm modules.

Form Grothendieck groups of such objects to obtain Kasparov's K-homology groups $K^i(A)$.

Remark That $[F, \rho(a)] \in \mathfrak{K}$ is a continuous control condition.

Reference G.G. Kasparov, 'Topological invariants of elliptic operators I', Math USSR Iszvestija 9(1975) 513-572.

Paschke duality

For a C^*-algebra A and ideal J define the algebra $\Psi(A//J)$ to consist of those $T \in \mathfrak{B}(H)$ such that

- $[T, \rho(a)] \in \mathfrak{K}\ \forall a \in A$, and
- $T\rho(j) \in \mathfrak{K}\ \forall j \in J$

where ρ is a good representation of A on H.

Proposition $K^i(A) = K_{i+1}(\Psi(A//0)/\Psi(A//A))$.

Notation for locally compact X,

$$\Psi^0(X) = \Psi(C_0(X)//0), \quad \text{(pseudolocal)}$$

$$\Psi^{-1}(X) = \Psi(C_0(X)//C_0(X)).$$

Analytic assembly

For $\tilde{M}$ as above consider exact sequence

$$0 \to \Psi^{-1}(\tilde{M})^{\Gamma} \to \Psi^{0}(\tilde{M})^{\Gamma} \to \Psi^{0}(M)/\Psi^{-1}(M) \to 0$$

Note Isomorphism of quotients given by localization.

Note that $\Psi^{-1}(\tilde{M}) =$ locally compact operators. We get a boundary map

$$A\colon K^{i}(M) = K_{i+1}(\Psi^{0}(M)/\Psi^{-1}(M)) \to K_{i}(C_{r}^{*}\Gamma)$$

.

This analytic assembly map takes the homology class of the signature operator F, to the analytic signature.

On the Novikov conjecture

Assume $B\Gamma$ is compact and let $f\colon M \to B\Gamma$. Consider the diagram

$$\begin{array}{ccccc} H_*(M;\mathbb{Q}) & \xleftarrow{ch} & K^*(M) & \longrightarrow & K_*(C_r^*\Gamma) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \nearrow & \\ H_*(B\Gamma;\mathbb{Q}) & \xleftarrow{ch} & K_*(B\Gamma) & & \end{array}$$

By the Atiyah-Singer index theorem $f_*(ch[F])$ is Novikov's higher signature (the push forward of the Poincaré dual of the L-class). So, if $A\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is injective, the Novikov conjecture is true for Γ.

Analytic approaches to the Novikov conjecture

- Cyclic cohomology — pair $K_*(C_r^*\Gamma)$ with $H^*(B\Gamma;\mathbb{R})$. Need suitable dense subalgebras — very delicate.

- Kasparov KK-theory — sometimes allows one to construct an inverse of the assembly map as an 'analytic generalized transfer'.

- Controlled C^*-algebra theory — parallel development to controlled topology, see later.

The analytic structure set

Recall exact sequence

$$K_{i+1}(C_r^*\Gamma) \to K_i(\Psi^0(\tilde{M})^\Gamma) \to K^i(M) \to K_i(C_r^*\Gamma)$$

We should think of $K_*(\Psi^0(\tilde{M})^\Gamma)$ as the 'analytic structure set' of M.

Example Suppose M is spin. Then one has the Dirac operator D and one can normalize as before to get a homology class $[F]$, $F = \phi(D)$.

If M has a metric of positive scalar curvature, then by Lichnerowicz there is a gap in the spectrum of D near zero. Thus we can choose the normalizing function ϕ so that $F^2 = 1$ exactly. Then $[F] \in K_*(\Psi^0(\tilde{M})^\Gamma)$ gives the structure invariant of the positive scalar curvature metric.

Manifold structures and analytic structures

Recall Pedersen's description of the structure set, as the L-theory of the category

$$\mathcal{B}(\tilde{M} \times I, \tilde{M} \times 1; \mathbb{Z})$$

Replacing $\mathbb{Z}$ by $\mathbb{C}$ we have a category

- whose objects can be completed to Hilbert spaces with $C_0(\widetilde{M})$-action
- whose morphisms are pseudolocal

Using Voiculescu's theorem we should get a map from the structure set to $K^*(\Psi^0(\tilde{M})^\Gamma)$.

Problem Are the morphisms bounded operators?

Controlled C^*-algebras

Let W be a metric space (noncompact) and suppose ρ: $C_0(W) \to \mathfrak{B}(H)$ as usual.

An operator T on H is boundedly controlled if there is $R = R(T)$ such that $\rho(\phi)T\rho(\psi) = 0$ whenever distance from Support ϕ to Support ψ is greater than R.

Example If D is a Dirac-type operator on complete Riemannian M, and ϕ has compactly supported Fourier transform, then $\phi(D)$ is boundedly controlled.

Define $\Psi^j_{bc}(W)$, $j = 0, -1$, to be the C^* subalgebras generated by boundedly controlled elements.

Open manifold index theory

From the above one has that a Dirac type operator on a complete Riemannian manifold W has a 'boundedly controlled index' in $K_*(\Psi_{bc}^{-1}(W))$.

In fact in full generality one has bounded assembly $A\colon K_{lf}^*(W) \to K_*(\Psi_{bc}^{-1}(W))$.

The assembly of the signature operator is the bounded analytic signature.

This can also be defined for suitable ('bounded, bounded') Poincaré complexes.

Controlled C^*-algebra and K-homology

Consider a metrizable pair (X, Y), let $W = X \setminus Y$. We can define continuously controlled C^*-algebras (following Pedersen), $\Psi^j_{cc}(W)$.

Proposition We have

- $\Psi^0_{cc}(W) = \Psi^0(X) = \Psi(C(X)//0)$
- $\Psi^{-1}_{cc}(W) = \Psi(C(X)//C_0(W))$

The result for $\Psi^{-1}_{cc}(W)$ is an analytic counterpart to the theorem 'control means homology at infinity'.

A geometric structure invariant

Given homotopy equivalence $f\colon M' \to M$. Assume $\Gamma = 1$ for simplicity. Form 'double trumpet space' W - a 'bounded, bounded' Poincaré space with a map to $M \times \mathbb{R}$, continuously controlled by $M \times S^0$.

Thus have analytic signature in $K_*(\Psi_{cc}^{-1}(X \times \mathbb{R}))$. Map by composite

$$\Psi_{cc}^{-1}(X\times\mathbb{R}) \to \Psi_{cc}^{0}(X\times\mathbb{R}) = \Psi^{0}(X\times I) \to \Psi^{0}(X).$$

The image is the desired structure invariant.

The various maps we have defined fit into a diagram relating the geometric and C^* surgery exact sequences.

The diagram commutes up to some factors of 2.

Final remarks

C^*-surgery can produce some information in a wide range of problems.

Surjectivity of C^*-assembly maps is related to representation theory.

Some techniques for Novikov are only available in the C^*-world

But We don't understand how to do analysis on topological manifolds.

Topologists construct; analysts only obstruct.

One reference 'Index theory, coarse geometry, and topology of manifolds' — CBMS lecture notes volume 90, American Math Society, 1996.