Non-Kähler Complex Geometric Structures on Homogeneous Spaces

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Introduction.

We first recall the definition of Kähler and pseudo-Kähler structure.

On a C^∞manifold M, let us consider a C^∞triple structure $\{J, g, \omega\}$ defined on $V = T_p(M)$ for each $p \in M$, where J is a complex structure (a linear automorphism such that $J^2 = -I$), g is a pseudo-Riemannian metric (non-degenerate symmetric bilinear form), and ω is a symplectic form (a non-degenerate skew-symmetric bilinear form), satisfying the compatibility condition

$$\omega(JX, JY) = \omega(X, Y),\ \omega(X, Y) = g(JX, Y)$$

for all $X, Y \in V$. Note that $\{J, g, \omega\}$ also satisfy

$$g(JX, JY) = g(X, Y),\ g(X, Y) = \omega(X, JY).$$

Since g and ω are non-degenerate bilinear form, we have linear isomorphisms $\phi_g, \phi_\omega : V \to V^*$. We can express compatibility condition of $\{J, g, \omega\}$ as the following commutative diagrams.

$$\begin{array}{ccc} V & \xrightarrow{\phi_\omega} & V^* \\ J\downarrow & \nearrow \phi_g & \\ V & & \end{array} \qquad\qquad \begin{array}{ccc} V^* & \xleftarrow{\phi_\omega} & V \\ J^*\downarrow & \swarrow \phi_g & \\ V^* & & \end{array}$$

In particular, a triple $\{J, g, \omega\}$ is determined by two of J, g, ω.

Remark. For any symplectic form ω, there exists a complex structure J such that $g(X, Y) = \omega(X, JY)$ is positive definite.

We impose the integrability condition for complex structure J and symplectic structure ω.

- For a C^∞complex structure J on M, J defines a complex structure on M, making M a complex manifold. For instance, the

Nijenhuis tensor

$$N_J(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]$$

vanishes for all vector fields X, Y on M.

- For a C^∞symplectic structure ω on M, ω is closed:

$$d\omega = 0$$

A C^∞triple $\{J, g, \omega\}$ on M satisfying the compatibility condition and the above integrability conditions is a *pseudo-Kähler structure*; and if in addition g is positive definite, it is a *Kähler structure*. If we impose only the first integrability condition, then it is a *pseudo-Hermitian structure*; and a *Hermitian structure* respectively.

Remark. For a fixed Riemannian metric (pseudo-Riemannian metric) g, a triple $\{J, g, \omega\}$ is Kähler (pseudo-Kählller) if and only if either one of the following conditions issatisfied.

$$(1)\ \nabla_g J = 0, \quad (2)\ \nabla_g \omega = 0,$$

where ∇_g is a Riemannian (pseudo-Riemannian) connection.

Examples. A complex projective space $\mathbf{C}P^1$ is a quotient manifold of $W = \mathbf{C}^2 - \{O\}$ by the action of $\mathbf{C}^*$,

$$\phi_\lambda : (z_1, z_2) \to (\lambda z_1, \lambda z_2)\ (\lambda \in \mathbf{C}^*).$$

On the other hand, a Hopf surface S is a quotient manifold of W by the action of $\mathbf{Z}$

$$\psi_t : (z_1, z_2) \to (\mu^t z_1, \mu^t z_2)\ (t \in \mathbf{Z}),$$

for some $\mu \in \mathbf{C}^* \ (|\mu| > 1)$.

Since $\Gamma = \{\mu^t \mid t \in \mathbf{Z}\}$ is a discrete subgroup of $\mathbf{C}^*$ and $\mathbf{C}^*/\Gamma$ is a complex torus ${T_\mathbf{C}}^1$, S is a ${T_\mathbf{C}}^1$ bundle over $\mathbf{C}P^1$.

Consider a $(1,1)$-form on W,

$$\omega = -i\,(dz_1 \wedge d\overline{z_1} + dz_2 \wedge \overline{z_2}),$$

and put

$$\Omega = \frac{1}{|z_1|^2 + |z_2|^2}\,\omega,$$

then Ω defines a real 2-form on S. Ω is not closed, but satisfies

$$d\,\Omega = \theta \wedge \Omega,$$

with

$$\theta = -\frac{1}{|z_1|^2 + |z_2|^2}\,(z_1 d\overline{z_1} + z_2 d\overline{z_2} + \overline{z_1} dz_1 + \overline{z_2} dz_2).$$

For $\psi(X) = \theta(JX)$, if we defines $\varpi = d\psi$, then

$$\varpi = \frac{-i}{(|z_1|^2 + |z_2|^2)^2}\,(|z_2|^2 dz_1 \wedge d\overline{z_1} + |z_1|^2 dz_2 \wedge d\overline{z_2} - \overline{z_1} z_2\, dz_1 \wedge d\overline{z_2} - \overline{z_2} z_1\, dz_2 \wedge d\overline{z_1})$$

which is so called Fubini-Study form. In the affine coordinates $z = \dfrac{z_2}{z_1}$, ϖ is expressed as

$$\varpi = \frac{-i}{(1 + |z|^2)^2}\, dz \wedge d\overline{z}$$

We can express $\mathbf{C}P^1$ and Hopf surface S as homogeneous complex manifolds.

Since $G = \mathrm{SL}_2(\mathbf{C})$ acts on $\mathbf{C}P^1$ transitively, we have

$$\mathbf{C}P^1 = G/B,$$

where B is a Borel subgroup of G:

$$B = \left\{ \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} | \alpha, \beta \in \mathbf{C}^*, \alpha\beta = 1, \gamma \in \mathbf{C} \right\}$$

Consider a subgroup B_μ of B

$$B_\mu = \left\{ \begin{pmatrix} \mu^t & \gamma \\ 0 & \mu^{-t} \end{pmatrix} | \, \mu, \gamma \in \mathbf{C}, |\mu| > 1, t \in \mathbf{Z} \right\}.$$

Then we have $S = G/B_\mu$, and B/B_μ is a complex torus $T^1_{\mathbf{C}}$. S is a hilomorphic fiber bundle over $\mathbf{C}P^1$ with fiber $T^1_{\mathbf{C}}$.

Homogeneous structures.

Let M be a *homogeneous space* of Lie group G. We can express M as G/H, where G is a simply connected Lie group, H a closed subgroup of G. Let H_0 be the identity component of H.

Then, $\widetilde{M} = G/H_0$ is simply connected and a principal bundle over $M = G/H$ with structure group $\Gamma = H/H_0$ (the fundamental group of M) acting on $\widetilde{M}$ on the right.

We also consider the case when a discrete subgroup Γ of G is acting freely and properly discontinuously on $\widetilde{M}$ on the left. In this case M can be considered as $\Gamma \backslash G/H_0$ (double coset space), which defines a *locally homogeneous space*.

Definitions.

- A *homogeneous complex structure* on $\widetilde{M} = G/H_0$ is defined by an integrable complex structure J on $\mathfrak{g}/\mathfrak{h}$, which satisfies the condition $Jad(X) = ad(X)J$ for $X \in \mathfrak{h}$.

- A homoegenous complex structure J on M is a homogeneous complex structure on $\widetilde{M}$ which is invariant by the right action of Γ. It may be defined as an integrable complex structure on J on $\mathfrak{g}/\mathfrak{h}$ satisfying the condition $JAd(h) = Ad(h)J$ for $h \in H$.

- If a discrete subgroup Γ of G is acting freely and properly discontinuously on $\widetilde{M}$ on the left, a homogeneous complex structure J on $\widetilde{M}$ defines a complex structure on $M = \Gamma\backslash G/H_0$, which is called a *locally homogeneous complex structure* on M.

- M is a *homogeneous complex Kähler manifold*, if M is a homogeneous complex manifold G/H which admits a Kähler structure.

- M is a *homogeneous Kähler manifold*, if it is a homogeneous complex Kähler manifold G/H and the Kähler structure is invariant by the action of G on the left.

- If a discrete subgroup Γ of G acts freely and properly discontinuously on a simply connected homogeneous Kähler manifold G/K on the left, it defines a *locally homogeneous (or left-invariant) Kähler structure* on $M = \Gamma\backslash G/K$, where K is a compact subgroup of G.

Compact homogeneous and locally homogeneous Kähler manifolds.

Theorem (Matsushima, Borel-Remmert). *A compact homogeneous complex Kähler manifold is biholomorphic to a product of a complex torus and a flag manifold.*

Remark. We have a class of compact locally homogeneous Kähler manifolds which do not admit any homogeneous Kähler structures: $G =: \mathbf{C}^l \rtimes \mathbf{R}^{2k}$, where the action $\phi : \mathbf{R}^{2k} \to \mathrm{Aut}(\mathbf{C}^l)$ is defined by

$$\phi(\bar{t}_i)((z_1, z_2, \ldots, z_l)) = (e^{\sqrt{-1}\,\eta_1^i\,t_i} z_1, e^{\sqrt{-1}\,\eta_2^i\,t_i} z_2, \ldots, e^{\sqrt{-1}\,\eta_l^i\,t_i} z_l),$$

where $\bar{t}_i = t_i e_i$ (e_i: the i-th unit vector in $\mathbf{R}^{2k}$), and $e^{\sqrt{-1}\,\eta_j^i}$ is the s_i-th root of unity, $i = 1, \ldots, 2k, j = 1, \ldots, l$.

If an abelian lattice $\mathbf{Z}^{2l}$ of $\mathbf{C}^l$ is preserved by the action ϕ on $\mathbf{Z}^{2k}$, then $M = \Gamma\backslash G$ defines a solvmanifold, where $\Gamma = \mathbf{Z}^{2l} \rtimes \mathbf{Z}^{2k}$ is a lattice of G.

The Lie algebra $\mathfrak{g}$ of G is the following:

$$\mathfrak{g} = \{X_1, X_2, \ldots, X_{2l}, X_{2l+1}, \ldots, X_{2l+2k}\}_{\mathbf{R}},$$

where the bracket multiplications are defined by

$$[X_{2l+2i}, X_{2j-1}] = -X_{2j}, [X_{2l+2i}, X_{2j}] = X_{2j-1}$$

for $i = 1, \ldots, k, j = 1, \ldots, l$, and all other brackets vanish.

The canonical left-invariant complex structure is defined by

$$JX_{2j-1} = X_{2j}, JX_{2j} = -X_{2j-1},$$

$$JX_{2l+2i-1} = X_{2l+2i}, JX_{2l+2i} = -X_{2l+2i-1}$$

for $i = 1, \ldots, k, j = 1, \ldots, l$.

Notes.

- The class of complex surfaces with $l = k = 1$ in the above example coincides with the class of hyperelliptic surfaces.
- A compact solvmanifold admits a Kähler structure if and only if it belongs to the above class of compact locally homogeneous Kähler solvmanifolds.
- It is well known that a simply connected homogeneous Kähler manifold is biholomorphic to $\mathbf{C}^k \times S \times D$, where S is a flag manifold, which is a projective manifold, D is a bounded homogeneous domain.
- We conjecture that a compact locally homogeneous Kähler manifold is, up to finite covering, biholomorphic to $T^k_{\mathbf{C}} \times S \times \Gamma \backslash D$, where D is a symmetric bounded domain.

Compact homogeneous and locally homogeneous pseudo-Kähler manifolds.

Theorem (Dorfmeister-Guan). *A compact homogeneous pseudo-Kähler manifold is biholomorphic to a product of a complex torus and a flag manifold.*

Remark. There is an example of a compact locally homogeneous pseudo-Kähler manifold which do not admit any homogeneous pseudo-Kähler structures: $G = N_3 \times \mathbf{R}$, where N_3 is the Heisenberg Lie group of dimension 3. The Lie algebra $\mathfrak{g}$ is generated by X, Y, Z, W with only non-zero bracket multiplication $[X, Y] = -Z$. An integrable complex structure J is defined by $JX = Y, JZ = W$. $\Omega = y \wedge z + w \wedge x$ defines a pseudo-Kähler structure on $S = \Gamma \backslash G$ for a suitable lattice Γ (Kodaira surface).

Hermitian and pseudo-Hermitian manifolds.

Definition. A Hermitian manifold M is *Hermitian symmetic* if each point $p \in M$ is an isolated fixed point of an involutive holomorphic isometry s_p of M.

- A *Hermitian symmetic space* M is a Riemannian symmetric space $\{M; g\}$ with its compatible complex structure J, defining a Kähler structure on M. It is a simply connected homogeneous Kähler manifold.
- A Hermitian symmetic space M is *irreducible* if it is irreducible as a Riemannian symmetric space (i.e. the holonomy representation is irreducible).

There are two types, *non-compact type* and *compact type*, of irreducible Hermitian symmetric spaces.

- If M is *of non-compact type*, then it can be written as G/H (effectively), where G is a connected non-compact simple Lie group with center $\{e\}$ and H is a maximal compact subgroup of G which has non-discrete center Z_H.
- If M is *of compact type*, then it can be written as G/H (effectively), where G is a connected compact simple Lie group with center $\{e\}$ and H is a maximal connected proper subgroup of G which has non-discrete center Z_H.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ that of H. Then we have the standard decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \text{ (as a vector space)},$$

where $\mathfrak{h} = \{X \in \mathfrak{g} | \sigma X = X\}$, $\mathfrak{m} = \{X \in \mathfrak{g} | \sigma X = -X\}$, and $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$ is isomorphic to the holonomy algebra $ad_{\mathfrak{m}}[\mathfrak{m}, \mathfrak{m}]$.

Any G-invariant complex structure on M is considered as $J \in GL(\mathfrak{m})$, satisfying the following conditions:

(1) $J^2 = -1$.

(2) $J \cdot ad_{\mathfrak{m}} X = ad_{\mathfrak{m}} X \cdot J$ for every $X \in \mathfrak{h}$.

(3) $[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$ for all $X, Y \in \mathfrak{m}$.

We know that for an irreducible Hermitian symmetric space, the complex structure $J \in GL(\mathfrak{m})$ is of the form $J = ad_{\mathfrak{m}}Z$ for some $Z \in \mathfrak{z}_{\mathfrak{h}}$. Since Z_H is actually a cyclic group with the Lie algebra $\mathfrak{z}_{\mathfrak{h}}$ of dimension 1, *we have only two G-invariant complex structure J and $-J$, which are compactible with the Riemannian metric.*

Conjecture (Burstall-Rawnsley). *An irreducible Hermitian symmetric space $\{M, g, J\}$ admit no compatible complex structures other than the original complex structure J and $-J$.*

They showed that the conjecture holds for Hermitian symmetric spaces of compact type. The proof is based on Twistor theory of symmetic spaces they have developed. We have a counter-example to the conjecture for Hermitian symmetric spaces of non-compact type.

For a non-compact simple Lie group G, we have Iwasawa decomposition: $G = SH$, where S is a simply connected solvable Lie group (called the *Iwasawa group*).

S acts simply-transitively on the Hermitian symmetric space $M = G/H$. Hence, M can be considered as a homogeneous Kähler solvable Lie group.

Let $\mathfrak{s}$ be the Lie algebra of S. Then $\mathfrak{s}$ is a *non-unimodular* and *split* solvable Lie algebra, and has a so-called *normal J-algebra* structure, which is defined as follows:

Definition. A *normal J-algebra* is a solvable Lie algebra with an inner product $<,>$ and a complex structure $J \in GL(\mathfrak{s})$ $(J^2 = -1)$, satisfying the following conditions:

(i) $< JX, JY > = < X, Y >$ for all $X, Y \in \mathfrak{s}$.

(ii) $< [X, Y], JZ > + < [Y, Z], JX > + < [Z, X], JY > = 0$ for all $X, Y, Z \in \mathfrak{s}$.

(iii) $[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$ for all $X, Y, Z \in \mathfrak{s}$.

(iv) $ad_{\mathfrak{s}} X$ has only real eigenvalues for all $X \in \mathfrak{s}$.

(v) there is a linear form ω such that $< X, Y > = \omega[JX, Y]$.

A solvable Lie algebra satisfying (i), (ii), (iii) is called a *solvable Kähler algebra*. A solvable Lie algebra satisfying (iv) is of *split* (or *completely solvable*) type.

Theorem. (due to Gindikin-Vinberg, Pyatetskii-Shariro) A split solvable Kähler algebra $\mathfrak{s}$ is decomposed into the semi-direct sum of an abelian J-invariant ideal and a normal J-algebra.

The corresponding Lie group S is a homogeneous Kähler solv-manifold which is biholomorphic to a direct product of $\mathbf{C}^k$ and a bounded homogeneous domain D.

Definition. J-algebras $\{\mathfrak{s}; J\}$ and $\{\mathfrak{s}'; J'\}$ are *isomorphic* if there exists a Lie algebra isomorphism $\phi : \mathfrak{s} \to \mathfrak{s}'$ such that $\phi J = J'\phi$.

Notes.

- It is known (due to Pyatetskii-Shapiro) that there exists one to one correspondence between isomorphism classes of normal J-algebras and biholomorphic equivalence classes of bounded homogeneous domains.

- It is known (due to Dotti-Miatello) that irreducible normal J-algebras $\{\mathfrak{s}; J\}$ and $\{\mathfrak{s}'; J'\}$ are *isomorphic* up to sign if and only if solvable Lie algebras $\mathfrak{s}$ and $\mathfrak{s}'$ are isomorphic as Lie algebras.

Observation. There exists one to one correspondence between complex structures J on a solvable Lie algebra $\mathfrak{g}$ and complex Lie subalgebras $\mathfrak{h}$ which satisfy $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h} \oplus \overline{\mathfrak{h}}$, given by $J \to \mathfrak{h}_J$ and $\mathfrak{h} \to J_{\mathfrak{h}}$, where $\mathfrak{h} = \{X + \sqrt{-}JX | X \in \mathfrak{g}\}$.

For a complex structure J, the complex Lie subgroup H_J of $G_{\mathbf{C}}$ corresponding to $\mathfrak{h}_J$ is closed, simply connected, and $G_{\mathbf{C}}/H_J$ is biholomorphic to $\mathbf{C}^m$.

The canonical inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbf{C}}$ induces an inclusion $G \hookrightarrow G_{\mathbf{C}}$, and $\Gamma = G \cap H_J$ is a discrete subgroup of G. We have the following canonical map $g = i \circ \pi$:

$$G \xrightarrow{\pi} G/\Gamma \overset{i}{\hookrightarrow} G_{\mathbf{C}}/H_J,$$

where π is a covering map, and i is an inclusion. The left-invariant complex structure J on G is the one induced by g from an open set $U = \mathrm{Im}\, g \subset \mathbf{C}^m$.

Example. Let $\mathfrak{s}_{m+1}$ be a solvable Lie algebra of dimension $2m+2$ with a basis $\beta = \{X_i, Y_j, Z, W\}$ for which the bracket multiplications are defined by

$$[X_i, Y_i] = -Z, \ [W, X_j] = \frac{1}{2} X_j, \ [W, Y_k] = \frac{1}{2} Y_k, \ [W, Z] = Z,$$

where $i, j, k = 1, \ldots, m$, and all other brackets are 0.

We can express $\mathfrak{s}_{m+1}$ as the semi-direct sum of a nilpotent ideal $\mathfrak{n}_m$ generated by $X_i, Y_j, Z, i, j = 1, ..., m$ and an abelian Lie algebra $\mathfrak{w}$ generated by $\{W\}$.

The inner product $<, >$ is defined with respect to which β is an orthonormal basis.

The complex structure J is defined by

$$JW = Z, JZ = -W, JX_i = Y_i, JY_j = -X_j,$$

where $i, j = 1, ..., m$.

It is easy to check that J is integrable, and a linear form ω defined by

$$\omega(Z) = 1, \omega(X_i) = \omega(Y_j) = \omega(W) = 0,$$

satisfies $< A, B >= \omega([JA, B])$ for any $A, B \in \mathfrak{s}_{m+1}$; and thus $\{\mathfrak{s}_{m+1}; J\}$ is a (irreducible) normal J-algebra.

We now take another complex structure J_k on $\mathfrak{s}_{m+1}$. The complex structure J_k, $k = 1, 2, ..., m$ is defined by

$$J_k W = Z, J_k Z = -W, J_k X_i = Y_i, J_k Y_i = -X_i, i = 1, 2, ..., k$$

and

$$J_k X_j = -Y_j, J_k Y_j = X_j, j = k+1, 2, ..., m,$$

then J_k is compatible with the inner product and integrable, but the condition (ii) of normal J-algebra does not hold (Kähler form is not closed).

We see that the complex subalgebra $\mathfrak{h}$ and $\mathfrak{h}_k$ of $\mathfrak{s}_{\mathbf{C}}$ corresponding to J and J_k is given by,

$$\mathfrak{h} = \{W + \sqrt{-1}Z, X_1 + \sqrt{-1}Y_1, X_2 + \sqrt{-1}Y_2, ..., X_m + \sqrt{-1}Y_m\}_{\mathbf{C}},$$

$$\mathfrak{h}_k = \{W + \sqrt{-1}Z, .., X_k + \sqrt{-1}Y_k, X_{k+1} - \sqrt{-1}Y_{k+1}, .., X_m - \sqrt{-1}Y_m\}_{\mathbf{C}}$$

where $[W+\sqrt{-1}Z, X_i \pm \sqrt{-1}Y_i] = \frac{1}{2}(X_i \pm \sqrt{-1}Y_i)$, $i = 1, 2, ..., m$.

The corresponding Lie group S_{m+1} is expressed as

$$S_{m+1} = H_m \rtimes \mathbf{R},$$

where H_m is the Heisenberg group and the action $\phi : \mathbf{R} \to \mathrm{Aut}\,(H_k)$ is defined by

$$\phi(s) : \begin{pmatrix} 1 & \mathbf{x} & z \\ 0 & \mathbf{I}_m & \mathbf{y}^t \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & e^{\frac{1}{2}s}\mathbf{x} & e^s z \\ 0 & I_m & e^{\frac{1}{2}s}\mathbf{y}^t \\ 0 & 0 & 1 \end{pmatrix}.$$

The complex subgroup $\mathscr{H}_k$ of $S_{\mathbf{C}}$ corresponding to $\mathfrak{h}_k$ is expressed as a semi-direct product $\mathscr{H}_k = \mathscr{U}_k \rtimes \mathscr{V}$, where

$$\mathscr{U}_k = \begin{pmatrix} 1 & \mathbf{u} & \frac{1}{2}\sqrt{-1}\|\mathbf{u}\|_k \\ 0 & \mathbf{I}_m & \sqrt{-1}\varepsilon_k \mathbf{u}^t \\ 0 & 0 & 1 \end{pmatrix}, k = 1, 2, ..., m,$$

$$\mathscr{V} = (\begin{pmatrix} 1 & 0 & \sqrt{-1}(e^s - 1) \\ 0 & \mathbf{I}_m & 0 \\ 0 & 0 & 1 \end{pmatrix}, s),$$

$\mathbf{u} \in \mathbf{C}^m, s \in \mathbf{C}$, $\|\mathbf{u}\|_k = \mathbf{u}\epsilon_k\mathbf{u}^t$ ($\epsilon_k = \begin{pmatrix} \mathbf{I}_{m-k} & 0 \\ 0 & -\mathbf{I}_k \end{pmatrix}$). Note that $\mathscr{U}_k$ is an abelian subgroup of $S_{\mathbf{C}}$ and $\mathscr{V}$ is a 1-parameter subgroup of $S_{\mathbf{C}}$ corresponding to $W + \sqrt{-1}V$.

Define $\phi_k : S_{\mathbf{C}} \to \mathbf{C}^{m+1}$ by

$$\left(\begin{pmatrix} 1 & \mathbf{u} & z \\ 0 & \mathbf{I}_m & \mathbf{v}^t \\ 0 & 0 & 1 \end{pmatrix}, s\right) \to (\mathbf{u} + \sqrt{-1}\epsilon_k \mathbf{v}, (< \mathbf{u}, \mathbf{v} > -2z) +$$

$$\sqrt{-1}\,(\frac{1}{2}(\|\mathbf{u}\|_k^2 + \|\mathbf{v}\|_k^2) + 2e^s)).$$

Then, ϕ_k induces a biholomorphic map $\overline{\phi}_k : S_{\mathbf{C}}/\mathscr{H}_k \to \mathbf{C}^{m+1}$, and the image of S_{m+1} is the open subset of $\mathbf{C}^{m+1}$:

$$\mathscr{S}_k = \overline{\phi}_k(S_{m+1}) = \{(\mathbf{z}, w) \in \mathbf{C}^{m+1} \mid \mathbf{Im}\, w > \frac{1}{2}\|\mathbf{z}\|_k^2\}.$$

We know that $\mathscr{S}_0$ is biholomorphic to $D_{m+1} = \{(\mathbf{z}, w) | \; \|\mathbf{z}\|^2 + |w|^2 < 1\}$, which is a complex hyperbolic $(m+1)$-space (or a Siegel domain of type II). And we can see that $\mathscr{S}_m$ is biholomorphic to $D'_{m+1} = \{(\mathbf{z}, w) \in \mathbf{C}^{m+1} \mid \mathbf{Im}\, w < \frac{1}{2}\|\mathbf{z}\|^2\}$, which can be considered as $\mathbf{CP}^{m+1} - \overline{D}_{m+1} \cup \mathscr{P}$, where $\mathscr{P}$ is a projective m-plane tangent to the boundary of D_{m+1}.

Remark. The homogeneous complex solvmanifold $\mathscr{S}_k = \{S_{m+1}; J_k\}$ is non-Kähler in any S_{m+1}-invariant metric: Suppose it admits a S_{m+1}-invariant Kähler metric. Then $\{s_{m+1}; J_k\}$ defines an irreducible split solvable Kähler algebra. Since $\mathfrak{s}_{m+1}$ has no J_k-invariant abelian ideal, it is an irreducible normal J-algebra. But then, according to the above result of Dotti-Miatello, we must have $J_k = J, \mathrm{or} - J$. In particular, $\mathscr{S}_k$ is not biholomorphic to $\mathscr{S}_0 = \{S_{m+1}; \pm J\}$.

Strongly KT structure.

Definition. A *strongly Kähler with torsion structure* (or shortly *SKT structure* on a differentiable manifold M is a Hermitian structure $\{h, J\}$ on M with its associated fundamental form Ω satisfying $\partial\overline{\partial}\Omega = 0$ or equivalently $d\, d^c\Omega = 0$, where $d^c = \sqrt{-1}(\partial - \overline{\partial})$. In terms of the *Bismut connection* (the unique metric connection ∇ with respect to which J is parallel, $\nabla J = 0$ and its torsion 3-form $c(X, Y, Z) = g(X, T^{\nabla}(Y, Z))$ is skew-symmetric), the condition $\partial\overline{\partial}\Omega = 0$ is equivalent to $dc = 0$ where c is actually given by $c = -Jd\Omega$.

Note.

- It is known (due to Gauduchon) that any compact Hermitian manifold of dimension 4 admits a SKT structure in the conformal class of the given Hermitian metric.

- For a compact (non-Kähler) Hermitian manifold of dimension greater than 6, SKT structure and LCK structure (which will be defined next) are mutually exclusive (due to Alexandrov and Ivanov).

- For a bi-Hermitian manifold $\{M, h, J_\pm\}$ with its associated fundamental forms Ω_+, Ω_- satisfying that $d^c_+\Omega_+ = -d^c_-\Omega_- = 0$ is d-closed, both $\{h, J_+\}$ and $\{h, J_-\}$ define STK structures on M.

- Any compact Lie group of even dimension admits a homogeneous SKT structure (due to Spindel et al).

Locally conformally Kähler structure.

Definition. A *locally conformally Kähler structure* (or shortly *LCK structure*) on a differentiable manifold M is a Hermitian structure (h, J) on M with its associated fundamental form Ω satisfying $d\Omega = \theta \wedge \Omega$ for some closed 1-form θ (which is called *Lee form*).

Note.

- A LCK structure Ω is *locally conformally Kähler*, in the sense that there is a open covering $\{U_i\}$ of M such that $\Omega_i = e^{-\sigma_i}\Omega$ is Kähler form on U_i for some functions σ_i, that is, $d\Omega_i = 0$. The condition $d\Omega = \theta \wedge \Omega$ is equivalent to the existence of a global close 1-form θ such that $\theta|U_i = d\sigma_i$.

- A LCK structure Ω is globally conformally Kähler (or Kähler) if and only if θ is exact (or 0 respectively).

Definition. A *homogeneous locally conformally Kähler* (or *homogeneous l.c.K*) *manifold* M is a homogeneous Hermitian manifold with its homogeneous Hermitian structure h, defining a locally conformally Kähler structure Ω on M.

Definition. If a simply connected homogeneous LCK manifold $M = G/H$, where G is a connected Lie group and H a closed subgroup of G, admits a free action of a discrete subgroup Γ of G on the left, then we call a double coset space $\Gamma\backslash G/H$ a *locally homogeneous LCK manifold*.

Observation. Classification of non-Kähler complex surfaces with $b_2 = 0$ is known: *Kodaira surfaces, Inoue surfaces, properly elliptic surfaces of odd type or Hopf surfaces*. Except for the class of Hopf surfaces with eigenvalues λ_1, λ_2 $(|\lambda_1| \neq |\lambda_2|)$, all of these non-Kähler complex surfaces, up to small deformations, admit either homogeneous or locally homogeneous LCK structures.

In fact, we can express each of these LCK complex surfaces S as $\Gamma\backslash G$ (up to finite covering), where G is a 4-dimensional Lie group with lattice Γ which admits homogeneous l.c.K structures.

It is known (due to Brunella) that Kato surfaces, which are non-Kähler complex surfaces with $b_2 > 0$, also admit LCK structures. There is a conjecture that Kato surfaces exhaust all non-Kähler complex surfaces with $b_2 > 0$.

The following is a list of all 4-dimensional unimodular Lie algebras $\mathfrak{g}$ with LCK structure, defining LCK complex surfaces, where the Lie algebra $\mathfrak{g}$ is generated by X, Y, Z, W with only non-zero bracket multiplication specified.

(1) Primary Kodaira surface:
$[X, Y] = -Z$

(2) Secondary Kodaira surface:
$[X, Y] = -Z$, $[W, X] = -Y$, $[W, Y] = X$

(3) Inoue surface $S^{\pm}$:
$[Y, Z] = -X$, $[W, Y] = Y$, $[W, Z] = -Z$

(4) Inoue surface S^0:
$[W, X] = -\frac{1}{2}X - bY$, $[W, Y] = bX - \frac{1}{2}Y$, $[W, Z] = Z$

(5) Properly elliptic surface:

$[X,Y]=-Z,\ [Z,X]=Y,\ [Z,Y]=-X$

(6) Hopf surface:

$[X,Y]=-Z,\ [Z,X]=-Y,\ [Z,Y]=X$

For all cases, we have a complex structure defined by

$$JX=-Y, JY=X, JZ=-W, JW=Z,$$

and its compatible LCK form $\Omega = x\wedge y + z\wedge w$ with the Lee form $\theta = w$, where x,y,z,w are the Maurer-Cartan forms corresponding to X,Y,Z,W respectively.

Notes.

- For Inoue surfaces of type S^+, we have other complex structures on $\mathfrak{g}$:

$$JX = Y, JY = -X, JZ = W - qY, JW = -Z - qX,$$

with no-zero real number q, defining a complex structure on S^+ for which there exist no compatible LCK structures (due to Belgun).

- For Hopf surfaces, we have other complex structures on $\mathfrak{g}$

$$JX = Y, JY = -X, JZ = W + dZ, J(W + dZ) = -Z,$$

with no-zero real number d, defining a homogeneous LCK structure on Hopf surface, as we will discuss in detail later.

Generalization of some of the above LCK complex surfaces to the higher dimension.

(i) Let $\mathfrak{h}_{2n+1}$ be the Heisenberg Lie algebra of dimension $2n+1$, which is a nilpotent Lie algebra generated by $X_1, X_2, ..., X_n$, $Y_1, Y_2, ..., Y_n, Z$ with non-zero bracket multiplication:

$$[X_i, Y_i] = -Z, i = 1, 2, ..., n.$$

A nilpotent Lie algebra $\mathfrak{g} = \mathbf{R}^1 \times \mathfrak{h}_{2n+1}$ admits a LCK structure Ω:

$$\Omega = z \wedge w + \sum_{i=1}^{n} x_i \wedge y_i$$

with the Lee form $\theta = w$, where x_i, y_j, z, w are the Maure-Cartan forms corresponding to X_i, Y_j, Z, W respectively; and a complex struture J:

$$JZ = W, JW = -Z, JX_i = Y_i, JY_i = -X_i, i = 1, 2, ..., n.$$

The corresponding Lie group G admits a lattice Γ, defining a locally homogeneous LCK structure on its compact quotient space $\Gamma \backslash G$. This is a generalization of primary Kodaira surface.

(ii) Let $\mathfrak{g}$ be a solvable Lie algebra of dimension $2n+2$, generated by $X, Y, Z_1, Z_2, ..., Z_n, W_1, W_2, ..., W_n$ with non-zero bracket multiplication:

$$[W_i, X] = -\frac{1}{2}X - b_i Y, [W_i, Y] = b_i X - \frac{1}{2}Y, [W_i, Z_j] = \frac{1}{n}Z_j,$$

where $i = 1, 2, ..., n, j = 1, 2, ..., n$.

The solvable Lie algebra $\mathfrak{g}$ admits a LCK structure Ω:

$$\Omega = x \wedge y + n \sum_{i,j=1}^{n} z_i \wedge w_j,$$

with the Lee form $\theta = \frac{1}{n} \Sigma_{i=1}^{n} w_i$, where x, y, z_i, w_j are the Maure-Cartan forms corresponding to X, Y, Z_i, W_j respectively; and a complex structure J:

$$JX = Y, JW = -Z, JZ_i = W_i, JW_i = -Z_i, i = 1, 2, ..., n.$$

The corresponding Lie group G admits a lattice Γ (due to Oeljeklaus-Toma), defining a locally homogeneous LCK structure on its compact quotient space $M = \Gamma \backslash G$. This is a generalization of Inoue surface S^0. We have $b_1(M) = \dim H^1(\mathfrak{g}) = \dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = n$.

Definition. A LCK manifold M is of *Vaisman type* if its Lee form θ is parallel w.r.t. the Levi-Civita connection of h; or equivalently, the Lee field $\xi = h^{-1}\theta$ is parallel.

Definition. We define an exterior differential d_θ on the de Rham compex $\Lambda^*(M)$ of a LCK manifold M as

$$d_\theta : w \to -\theta \wedge w + dw,$$

which satisfies $d_\theta^2 = 0$ for $w \in \Lambda^*(M)$. We call $H_\theta^k(M)$ the k-th *twisted cohomology group* with respect to θ.

- For a LCK manifold M of Vaisman type, all $H_\theta^k(M)$ vanish (due to de León-López-Marrero-Pardón)
- For a reductive or nilpotent Lie algebra $\mathfrak{g}$, all $H_\theta^k(\mathfrak{g})$ vanish. (due to Hochschild-Serre, Diximier respectively)

Notes.

- For locally homogeneous LCK manifold $\Gamma\backslash G$, we can check whether the Lee filed ξ is parallel or not, by using the formula:

$$h(\nabla_X\xi, Y) = h([X,\xi],Y) - h([\xi,Y],X) + h([Y,X],\xi)$$

for any $X, Y \in \mathfrak{g}$. Since $d\theta(Y,X) = h([Y,X],\xi) = 0$, the Lee filed ξ is parallel if and only if it is Killing.

- For locally homogeneous LCK manifold $\Gamma\backslash G$, where G is simply connected solvable Lie group, there is a canonical injection

$$H^k_\theta(\mathfrak{g}) \hookrightarrow H^k_\theta(\Gamma\backslash G).$$

(cf. Raghunathan; *Discrete subgroups of Lie groups*)

- In the above examples, (i) is of Vaisman type, and (ii) is not.

Examples.

- For secondary Kodaira surface, the Lee filed $\xi = W$, and the bracket multiplication is given by $[X, Y] = -Z$, $[W, X] = -Y$, $[W, Y] = X$. We get by simple calculation,

$$h(\nabla_U W, V) = h([W, U], Y) + h(U, [W, V]) = 0$$

for any $U, V \in \mathfrak{g}$. It is also easy to check $\Omega = -w \wedge z + dz$.

- For Inoue surface $S^{\pm}$, the Lee filed $\xi = W$, and the bracket multiplication is given by $[Y, Z] = -X$, $[W, Y] = Y$, $[W, Z] = -Z$. The Lee field $\xi = W$ is not Killing:

$$h(\nabla_Z W, Z) = h([W, Z], Z) + h(Z, [W, Z]) = -2h(Z, Z) \neq 0.$$

It is also easy to check that there is no invariant 1-form v such that $\Omega = -w \wedge v + dv$; and thus no such 1-form v on $S^{\pm}$.

Definitions.

- A *contact metric structure* $\{\phi, \eta, \widetilde{J}, g\}$ on M^{2n+1} is a contact structure $\phi\,, \phi \wedge (d\phi)^n \neq 0$ with the *Reeb field* $\eta\,, i(\eta)\phi = 1, i(\eta)d\phi = 0$, a $(1,1)$-tensor $\widetilde{J}\,, \widetilde{J}^2 = -I + \phi \otimes \eta$ and a Riemannian metric $g\,, g(X,Y) = \phi(X)\phi(Y) + d\,\phi(X, \widetilde{J}Y)$.

- A *Sasaki structure* on M^{2n+1} is a contact metric structure $\{\phi, \eta, \psi, g\}$ satisfying $\mathscr{L}_\eta g = 0$ (Killing field) and the integrability of $J = \widetilde{J}|\mathscr{D}$ on $\mathscr{D} = \ker\phi$ (CR-structure).

- For any Sasaki manifold N, its *Kähler cone* $C(N)$ is defined as $C(N) = \mathbf{R}_+ \times N$ with the Kähler form $\omega = r dr \wedge \phi + \frac{r^2}{2} d\phi$, where a compatible complex structure $\widehat{J}$ is defined by $\widehat{J}\eta = \frac{1}{r}\partial_r$ and $\widehat{J}|\mathscr{D} = J$.

Note. For any Sasaki manifold N with contact form ϕ, we can define a LCK form $\Omega = \frac{2}{r^2}\omega = \frac{2}{r}dr \wedge \phi + d\phi$; or taking $t = -2\log r$, $\Omega = -dt \wedge \phi + d\phi$ on $M = \mathbf{R} \times N$ or $S^1 \times N$, which is of Vaisman type. We can define a family of complex structures J compatible with Ω by

$$J\,\partial_t = b\,\partial_t + (1 + b^2)\,\eta, J\eta = -\partial_t - b\,\eta,$$

where $b \in \mathbf{R}$ and the Lee field is $J\eta$. Conversely, any simply connected complete Vaisman manifold is of the form $\mathbf{R} \times N$ with LCK structure as above, where N is a simply connected complete Sasaki manifold.

Remark. It is known (due to Ornea and Verbitsky) that a compact Vaisman manifold is a fiber bundle over S^1 with fiber a compact Sasaki manifold.

Homogeneous and locally homogeneous LCK structures on Hopf surfaces.

Let $\mathfrak{g} = \mathfrak{u}(2) = \mathbf{R} + \mathfrak{su}(2)$ be a reductive Lie algebra with basis $\{T, X, Y, Z\}$ of $\mathfrak{g}$, where T is a generator of the center $\mathbf{R}$ of $\mathfrak{g}$, and

$$X = \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},\ Y = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},\ Z = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

satisfying the bracket multiplications

$$[X, Y] = Z,\ [Y, Z] = X,\ [Z, X] = Y.$$

Then $\mathfrak{g}$ admits a family of complex structures $J_\delta, \delta = c + \sqrt{-1}\, d$ $(c \neq 0)$ defined by

$$J_\delta(T - dX) = cX,\ J_\delta(cX) = -(T - dX),\ J_\delta Y = \pm Z,\ J_\delta Z = \mp Y.$$

Homogeneous Hopf surfaces. Let $G = S^1 \times SU(2)$ (which is diffeomorphic to $S^1 \times S^3$). Then all homogeneous complex structures on G admit their compatible homogeneous LCK structures, defining a primary Hopf surfaces S_λ which are compact quotient spaces of the form W/Γ_λ, where $W = \mathbf{C}^2 \backslash \{\mathbf{0}\}$ and Γ_λ is a cyclic group of holomorphic automorphisms on W generated by a contraction $f : (z_1, z_2) \to (\lambda z_1, \lambda z_2)$ with $|\lambda| \neq 0, 1$.

Proof. We consider a canonical diffeomorphism Φ_δ:

$$\Phi_\delta : \mathbf{R} \times SU(2) \longrightarrow W$$

defined by

$$(t, z_1, z_2) \longrightarrow (\lambda_\delta^t z_1, \lambda_\delta^t z_2),$$

where $\lambda_\delta = e^{c+\sqrt{-1}\,d}$ and $SU(2)$ is identified with

$S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ by the correspondence:

$$\begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \longleftrightarrow (z_1, z_2).$$

Then we see that Φ_δ is a biholomorphic map. It is now clear that Φ_δ induces a biholomorphism between $G = S^1 \times SU(2)$ with homogeneous complex structure J_δ and a primary Hopf surface $S_{\lambda_\delta} = W/\Gamma_{\lambda_\delta}$. Q.E.D.

Remark. We have the Lee field $\xi = T - \frac{d}{c}X$, which is irregular for an irrational $\frac{d}{c}$, and the Reeb field $\eta = cX$, which is always regular.

Note. $U(2)$ is a quotient Lie group of G by the central subgroup $\mathbf{Z}_2 = \{(1, I), (-1, -I)\}$.

We can also consider $= S^1 \times S^3$ as a compact homogeneous space $\tilde{G}/H$, where $\tilde{G} = S^1 \times U(2)$ with its Lie algebra $\tilde{\mathfrak{g}} = \mathbf{R} + \mathfrak{u}(2)$ and $H = U(1)$ with its Lie algebra $\mathfrak{h}$. Then, we have a decomposition $\tilde{\mathfrak{g}} = \mathfrak{m} + \mathfrak{h}$ for the subspace $\mathfrak{m}$ of $\tilde{\mathfrak{g}}$ generated by S, T, Y, Z and $\mathfrak{h}$ generated by W, where

$$S = \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix},\ W = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.$$

Locally homogeneous Hopf surfaces. Let $\hat{G} = \mathbf{R} \times U(2)$, and let $\Gamma_{p,q}\,(p, q \neq 0)$ be a discrete subgroup of $\hat{G}$ defined by

$$\Gamma_{p,q} = \{(k, \begin{pmatrix} e^{\sqrt{-1}\,pk} & 0 \\ 0 & e^{\sqrt{-1}\,qk} \end{pmatrix}) \in \mathbf{R} \times U(2) \mid k \in \mathbf{Z}\}.$$

Then $\Gamma_{p,q}\backslash\hat{G}/H$ is biholomorphic to a Hopf surface $S_{p,q} = W/\Gamma_{\lambda_1,\lambda_2}$, where $\Gamma_{\lambda_1,\lambda_2}$ is the cyclic group of automorphisms on W generated by

$$\phi : (z_1, z_2) \longrightarrow (\lambda_1 z_1, \lambda_2 z_2)$$

with $\lambda_1 = e^{r+\sqrt{-1}\,p}, \lambda_2 = e^{r+\sqrt{-1}\,q}, r \neq 0.$

In fact, if we take a homogeneous complex structure J_r on $\hat{G}/H$ induced from the diffeomorphism

$$\Phi_r : \hat{G}/H \to W$$

defined by

$$(t, z_1, z_2) \longrightarrow (e^{rt} z_1, e^{rt} z_2),$$

Φ_r induces a biholomorphism between $\Gamma_{p,q}\backslash\hat{G}/H$ and $S_{p,q}$.

Homogeneous Hopf manifolds.

Let $M = G/H$, where $G = S^1 \times SU(n)$ and $H = SU(n-1)$, which is diffeomorphic to $S^1 \times S^{2n+1}$. Then M admits a homogeneous LCK structure. The Lie algebra $\mathfrak{g} = \mathbf{R} + \mathfrak{s}u(n)$ has a decomposition:

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h},$$

satisfying $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, where $\mathfrak{h} = \mathfrak{s}u(n-1)$, and $\mathfrak{m}$ is generated by T, X, Y_i, Z_j $(i, j = 1, 2, ..., n-1)$ with a generator T of the center $\mathbf{R}$, and non-zero bracket multiplications:

$$[Y_i, Z_i] = -X \bmod \mathfrak{h} \ (i = 1, 2, ..., n-1).$$

We have a LCK form Ω and the Lee form θ:

$$\Omega = t \wedge x + \sum_{i=1}^{n} y_i \wedge z_i, \ \theta = t.$$

As in the case $n = 1$, $\mathfrak{g}$ admits a family of complex structures $J_\delta, \delta = c + \sqrt{-1}\, d$ defined by

$J_\delta(T - dX) = cX,\ J_\delta(cX) = -(T - dX),\ J_\delta Y_i = Z_i,\ J_\delta Z_i = -Y_i,$

where $c \neq 0$, $i = 1, 2, ..., n-1$, defining a homogeneous LCK structure of Vaisman type on M.

Note. $S^{2n+1} = SU(n)/SU(n-1)$ admits a homogeneous *Sasaki structure*: we have a Hopf fibration $S^{2n+1} \to \mathbf{CP}^n$ with fiber $S^1 = U(n-1)/SU(n-1)$ and the base space $\mathbf{CP}^n = SU(n)/U(n-1)$. It has a homogeneous contact form x, defining a Kähler structure $\omega = dx$ on $\mathbf{CP}^n$ defined by

$$\omega = \sum_{i=1}^{n} y_i \wedge z_i \, .$$

Structure of compact homogeneous LCK manifolds

Theorem. A compact homogeneous LCK manifold M is biholomorphic to a holomorphic principal fiber bundle over a flag manifold with fiber a 1-dimensional complex torus $T^1_{\mathbf{C}}$. And its LCK structure is of Vaisman type.

To be more precise, we can express M as a homogeneous space form G/H, where G is a compact connected Lie group of holomorphic isometries on M which is of the form

$$G = S^1 \times S,$$

where S is a compact semi-simple Lie group, including a closed subgroup H of G.

S/H is a compact homogeneous Sasaki manifold, which is a principal fiber bundle over a flag manifold S/Q with fiber $S^1 =$

Q/H for some parabolic subgroup Q of S including H.

Sketch of Proof. Since G is a compact Lie group, it is reductive; and its Lie algebra $\mathfrak{g}$ is of the form:

$$\mathfrak{g} = \mathfrak{t} + \mathfrak{s},$$

where $\mathfrak{t}$ is the center of $\mathfrak{g}$ and $\mathfrak{s}$ a semi-simple Lie algebra with $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{s}$. Since the Lee form θ is closed but not 0, we must have $\theta \in \mathfrak{t}^*$. Let ξ be the Lee field with $\theta(\xi) = 1$, and $\eta = J\xi$ (the Reeb field) for the complex structure J with its Maerer-Cartan form ϕ. Then we can express $\mathfrak{g}$ as

$$\mathfrak{g} =< \xi > + \mathfrak{g}', \ \mathfrak{g}' =< \eta > + \mathfrak{k},$$

where $< \xi >$ is the 1-dimensional subspace of $\mathfrak{g}$ generated by ξ, $\mathfrak{k} = \ker \phi|_{\mathfrak{g}'}$ with $\mathfrak{k} \supset \mathfrak{h}$, and both of these sums are orthogonal direct sums with respect to the Hermtian metric h.

We can see

- $1 \leq \dim \mathfrak{t} \leq 2$, *and* ξ, η *are infinitesimal automorphisms of* J *and infinitesimal isometries (Killing fields) with respect to* h.
- The case $\dim \mathfrak{t} = 2$ can be reduced to the case $\dim \mathfrak{t} = 1$.

Let $\mathfrak{q} =< \eta > +\mathfrak{h}$, then $\mathfrak{q}$ is a Lie subalgebra of $\mathfrak{g}'$; in fact we have $\mathfrak{q} = \{X \in \mathfrak{g}' \,|\, d\phi(X, \mathfrak{g}') = 0\}$. Then, $\mathfrak{h}$ is an ideal of $\mathfrak{q}$.

Let S and Q be the corresponding Lie subgroup of G, then Q is a closed subgroup of S since we have $Q = \{x \in S \,|\, ad(x)^*\phi = \phi\}$; in particular, H is a normal subgroup of Q with $Q/H = S^1$, and η generates an S^1 action on S.

Since $d\phi$ defines a homogeneous symplectic structure on $\mathfrak{k} \bmod \mathfrak{h}$, S/Q admits a homogeneous symplectic structure compatible with J, defining a Kähler structure on S/Q (due to Borel).

We can see that the Lie subalgebra $< \xi > + < \eta >$ of $\mathfrak{g}$ corresponds to a 2-dimensional torus T^2 of G; $\xi - \sqrt{-1}\eta$ defines a 1-dimensional complex torus action on $M = G/H$ on the right which is holomorphic and isometric. We have $M = S^1 \times S/H$, where $S/H \to S/Q$ is a principal S^1-bundle over the flag manifold S/Q; and $M \to S/Q$ is a holomorphic principal fiber bundle over the flag manifold S/Q with fiber $T^1_{\mathbf{C}}$. Q.E.D.

Corollary There exist no compact complex homogeneous LCK manifolds; in particular, no compact complex paralellizable manifolds admit their compatible LCK structures.

Proof. Only compact complex Lie groups are complex tori, which can not act transitively on a compact LCK manifold. Q.E.D.

Example. There exists a LCK structure on $\mathfrak{g} = \mathbf{R} \oplus \mathfrak{sl}(2, \mathbf{R})$, which is not of Vaisman type. Take a basis $\{W, X, Y, Z\}$ for $\mathfrak{g}$ with bracket multiplication defined by

$$[X, Y] = -Z, \ [Z, X] = Y, \ [Z, Y] = -X,$$

and all other brackets vanish. We have a homogeneous complex structure defined by

$$JY = X, JX = -Y, JW = Z, JZ = -W,$$

and its compatible LCK form Ω on $\mathfrak{g}$ defined by

$$\Omega = z \wedge w + x \wedge y,$$

with the Lee form $\theta = w$, where x, y, z, w are the Maurer-Cartan forms corresponding to X, Y, Z, W respectively. We can take another LCK form

$$\Omega_\psi = \psi \wedge w + d\psi,$$

where $\psi = by + cz\,(b, c \in \mathbf{R})$ with $0 < b < c$ and $c^2 - b^2 = c$, making the corresponding metric h_ψ positive definite. The Lee field ξ is given as

$$\xi = \frac{1}{c^2 - b^2}(cW + bX).$$

It is easy to check that $h([\xi, X], Y) + h(X, [\xi, Y]) \not\equiv 0$; and thus ξ is not a Killing field.

For any lattice Γ of $G = \mathbf{R} \times \widetilde{SL}(2, \mathbf{R})$ with the above homogeneous l.c.K. structure, we get a complex surface $\Gamma\backslash G$ (properly elliptic surface) with locally homogeneous non-Vaisman l.c.K. structure.

Generalized Hopf manifolds and their Deformation.

A *generalized Hopf manifold* is, a compact complex manifold of which the universal covering is $W = \mathbf{C}^n - \{\mathbf{0}\}$. We call it here simply a *Hopf manifold*.

Let $M = W/G$ be a Hopf manifold, where G is the covering transformation group of M consisting of analytic automorhisms over $\mathbf{C}^n$ which fixes the origin $\mathbf{0}$. G acts on W properly discontinuously and fixed point free. We can express G as

$$G = H \rtimes Z,$$

where Z is an infinite cyclic group generated by a contraction ρ on W, and H is a finite normal subgroup of G. There exists $m \in \mathbf{N}$ such that for $Z' =< \rho' >$, $\rho' = \rho^m$, $G' = H \times Z'$ is a normal subgroup of finite index in G. We write G, Z in place of G', Z'.

We can see that W/G is diffeomorphic to $S^1 \times S^{2n-1}/H$, where H is a finite unitary group acting freely on S^{2n-1}. In fact, we can construct a complex analytic family $\{M(t), t \in \mathbf{C}\}$ which deforms W/G to $W/l(G)$, where $l(G)$ is the linear transformation group on W consisting of linear terms of $g \in G$.

Let $T_t, (t \neq 0)$ be an analytic automorphism over W defined by

$$T_t(z_1, z_2, ..., z_n) = (tz_1, tz_2, ..., tz_n),$$

and set $g_t = T_t^{-1} g T_t$, $G(t) = \{g_t \mid g \in G'\}$ and $G(0) = l(G)$.

We can see by *Cartan's uniqueness theorem* that the canonical map $G \to G(0)$ is a group isomorphism, and $G(0)$ acts on W properly discontinously and fixed-point free. It follows that $\{M(t) = W/G(t), t \in \mathbf{C}\}$ defines a complex analytic family.

We can further deform a Hopf manifold $M = W/G$ to $W/l_0(G)$ with $l_0(G) = l_0(Z) \times l_0(H)$, where $l_0(Z)$ is generated by a diagonal matrices $d(\alpha_1, \alpha_2, ..., \alpha_n)$ with eigenvalues of $\alpha_1, \alpha_2, ..., \alpha_n$ of the linear term of the generator ρ of Z and $l_0(H) \subset U(n)$.

In fact, we can assume that ρ is of Jordan form $J(\alpha, n)$. Let $T_t, (t \neq 0)$ be an analytic automorphism over W defined by

$$T_t(z_1, z_2, ..., z_n) = (t^{n-1}z_1, t^{n-2}z_2, ..., z_n),$$

and set $g_t = T_t^{-1} g T_t$, $G(t) = \{g_t \mid g \in G\}$, which defines a complex analytic family with $G(0) = l_0(G)$.

As a consequence, a Hopf manifold $M = W/G$ has a primary Hopf manifold $\widehat{M} = W/Z$ as a finite normal covering, which can be deformed to a *diagonal Hopf manifold* $\widehat{M}_0 = W/d(\alpha_1, \alpha_2, ..., \alpha_n)$. (cf. K.H., Illinois J. Math. 1993)

Kähler potential and LCK structures

Observation. A LCK structure on M may be defined as a Kähler structure $\tilde{\omega}$ on the universal covering $\tilde{M}$ on which the the fundamental group Γ acts homothetically; that is, for every $\gamma \in \Gamma$, $\gamma^*\tilde{\omega} = \rho(\gamma)\tilde{\omega}$ holds for some positive constant $\rho(\gamma)$.

Let $M = G/H$ be a homogeneous LCK manifold. Then its universal covering $\tilde{M} = \tilde{G}/\tilde{H}_0$ is also a homogeneous LCK manifold. Since the Lee form $\tilde{\theta}$ is exact, $\tilde{\Omega}$ is globally conformal to a Kähler structure $\tilde{\omega}$. The Lie group $\tilde{G}$ acts homothetically on $\tilde{M}$ on the left, and the fundamental group $\Gamma = \tilde{H}/\tilde{H}_0$ acts on $\tilde{M}$ homothetically on the right. Conversely, a Kähler structure $\tilde{\omega}$ on $\tilde{M}$ with homothetic action of $\tilde{G}$ on the left and Γ from the right on $\tilde{M}$ defines a LCK structure on M.

Definition. Let M be a LCK manifold. Suppose that the universal covering $\tilde{M}$ admits a *Kähler potential* ϕ, which is a real positive function on $\tilde{M}$ such that $\tilde{\omega} = -\sqrt{-1}\partial\overline{\partial}\phi$ defines a Kähler structure on $\tilde{M}$. If the fundamental group Γ acts homothetically on ϕ, then we call ϕ a *LCK potential* for M. $\tilde{\omega}$ clearly defines a LCK structure on M.

Example. A diagonal Hopf surfaces $S_\lambda = W/\Gamma_\lambda$, where Γ_λ is generated by a contraction $f : (z_1, z_2) \to (\lambda z_1, \lambda z_2)$ with $|\lambda| \neq 0, 1$ on W, admits a LCK potential

$$\phi(z_1, z_2) = |z_1|^2 + |z_2|^2.$$

We have a Kähler structure $\tilde{\omega} = -\sqrt{-1}\,(d\,z_1 \wedge d\,\overline{z_1} + d\,z_2 \wedge d\,\overline{z_2})$ on W for which $\tilde{\omega} = -\sqrt{-1}\partial\overline{\partial}\phi$ holds.

Generalized Hopf manifold and their LCK structures

We know (due to Ornea-Verbitsky) that a small deformation of a compact LCK manifold with potential is also a LCK manifold with potential. In other words, LCK structure with potential is preserved under small deformations.

We have seen that any primary Hopf manifold can be deformed to a diagonal Hopf manifold, which admits a LCK potential. Hence we see that any Hopf manifold admits a LCK structure.