# Parabolic isometries of hyperbolic spaces and discreteness

**John R Parker**
Durham University, UK
j.r.parker@durham.ac.uk

# Shige

It is a great privilege for me to speak at the conference in honour of Shigeyasu Kamiya on the occasion of his retirement.

I have known Shige for many years.

These are pictures of us

- At the 2007 conference for Alan Beardon's retirement
- In my home in Durham
- At the conference Shige organised in Okayama in 1998

# Shimizu's lemma

## Lemma (Shimizu 1963)

*Let* $T$ *and* $S$ *be the following matrices in* $\mathrm{SL}(2,\mathbb{R})$

$$T = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

*Suppose* $c \neq 0$. *If the group* $\Gamma = \langle T, S \rangle$ *is discrete then* $|ct| \geq 1$.

# Shimizu's lemma

## Lemma (Shimizu 1963)

*Let $T$ and $S$ be the following matrices in $\mathrm{SL}(2,\mathbb{R})$*

$$T = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

*Suppose $c \neq 0$. If the group $\Gamma = \langle T, S \rangle$ is discrete then $|ct| \geq 1$.*

The proof has three steps.

**Step 1**

Consider the sequence in $\Gamma$ defined by $S_0 = S$ and $S_{j+1} = S_j T S_j^{-1}$.

Write $S_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$. Then $S_{j+1}$ is given by

$$\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_j & -b_j \\ -c_j & a_j \end{pmatrix} = \begin{pmatrix} 1 - a_j c_j t & a_j^2 t \\ -c_j^2 t & 1 + a_j c_j t \end{pmatrix}.$$

In particular $c_{j+1} = -c_j^2 t$

# Proof of Shimizu's lemma continued

**Step 2**

From the sequence $\{S_j\}$ construct a dynamical system:
$|c_{j+1}t| = |c_j t|^2$ and so $|c_j t| = |c_0 t|^{2^j} = |ct|^{2^j}$.
Find a condition that ensures we lie in a finite basin of attraction:
If $|ct| = |c_0 t| < 1$ then $|c_j t|$, and hence $c_j$, tends to $0$.
Moreover, since $c \neq 0$ then $c_j \neq 0$.

# Proof of Shimizu's lemma continued

Step 2

From the sequence $\{S_j\}$ construct a dynamical system:

$|c_{j+1}t| = |c_j t|^2$ and so $|c_j t| = |c_0 t|^{2^j} = |ct|^{2^j}$.

Find a condition that ensures we lie in a finite basin of attraction:

If $|ct| = |c_0 t| < 1$ then $|c_j t|$, and hence $c_j$, tends to $0$.

Moreover, since $c \neq 0$ then $c_j \neq 0$.

Step 3

We use $c_j t \longrightarrow 0$ to show $S_j$ tends to $T$:

We have $a_{j+1} - 1 = -a_j c_j t = -(a_j - 1)c_j t - c_j t$, so for $j \geq 1$:

$|a_j - 1| \leq |a_0 - 1|\,|c_0 t|^{2^j} + j|c_0 t|^{2^j}$

Hence $a_j$ tends to $1$.

Using our expression for $S_{j+1}$, this shows $S_j$ tends to $T$.

Since $c_j \neq 0$ then $S_j \neq T$ and $\Gamma = \langle T, S \rangle$ is not discrete.

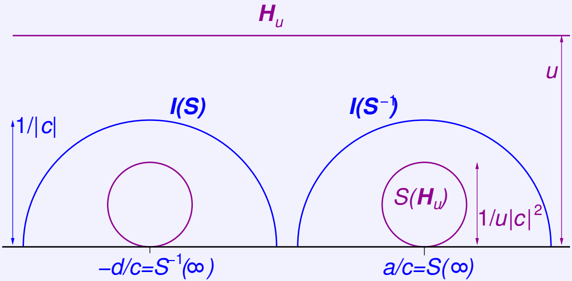
Hence if $\Gamma$ is discrete, we must have $|ct| \geq 1$. □

# Some hyperbolic geometry

- A matrix $S$ in $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane as a Möbius transformation $S(z)$ in $\mathrm{PSL}(2,\mathbb{R})$
  $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to $S(z) = \frac{az+b}{cz+d}$.
- $S(z)$ is an isometry of the hyperbolic plane $\mathbf{H}^2 = \mathbf{H}^2_{\mathbb{R}}$
- A discrete subgroup $\Gamma$ of $\mathrm{SL}(2,\mathbb{R})$ acts properly discontinuously on $\mathbf{H}^2_{\mathbb{R}}$.
- The quotient $M = \mathbf{H}^2_{\mathbb{R}}/\Gamma$ is an orbifold.
- The matrix $T$ corresponds to the Möbius transformation $T(z) = z + t$.
  This has (Euclidean) translation length $\ell_T = |t|$
- For $u > 0$ the horoball $H_u$ of height $u$ at $\infty$ is
  $H_u = \{z = x + iy \in \mathbf{H}^2_{\mathbb{R}} \ : \ y > u\}$
- A horoball at a point $x \in \mathbb{R}$ is an open disc tangent to $\mathbb{R}$ at $x$

# Isometric spheres

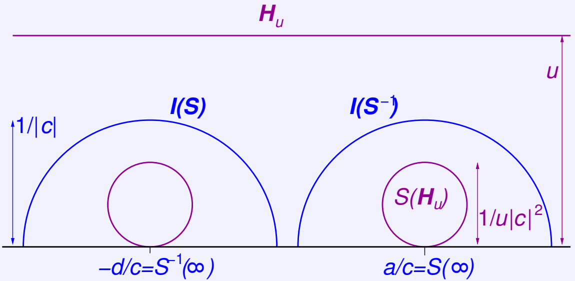
Let $S(z) = (az + b)/(cz + d) \in \mathrm{PSL}(2, \mathbb{R})$ not fixing $\infty$
So $c \neq 0$. The isometric sphere $I(S)$ of $S$ is the Euclidean semi-circle with centre $S^{-1}(\infty) = -d/c$ and radius $r_s = 1/|c|$




$S$ sends the outside of $I(S)$ to the inside of $I(S^{-1})$.

# Isometric spheres

Let $S(z) = (az+b)/(cz+d) \in \mathrm{PSL}(2,\mathbb{R})$ not fixing $\infty$
So $c \neq 0$. The isometric sphere $I(S)$ of $S$ is the Euclidean semi-circle with centre $S^{-1}(\infty) = -d/c$ and radius $r_s = 1/|c|$




$S$ sends the outside of $I(S)$ to the inside of $I(S^{-1})$.

$S$ sends the horoball $H_u$ of height $u$ centred at $\infty$ to a horoball $S(H_u)$ of diameter $1/u|c|^2$ at $S(\infty)$.

So if $u \geq r_S = 1/|c|$ then $H_u$ and $S(H_u)$ are disjoint.

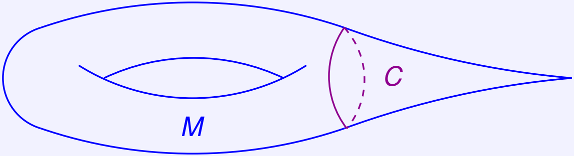
# Geometric interpretation of Shimizu's lemma

Let $\Gamma$ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ containing $T(z) = z + t$ where $t > 0$.

(1) If $S$ is any element of $\Gamma$ not fixing $\infty$
then the radius $r_S$ of the isometric sphere of $S$ satisfies
$r_S \leq \ell_T$
(Note this is an inequality between two Euclidean quantities)
Proof: $r_s = 1/|c|$, $\ell_T = |t|$ and Shimizu says $|ct| \geq 1$.

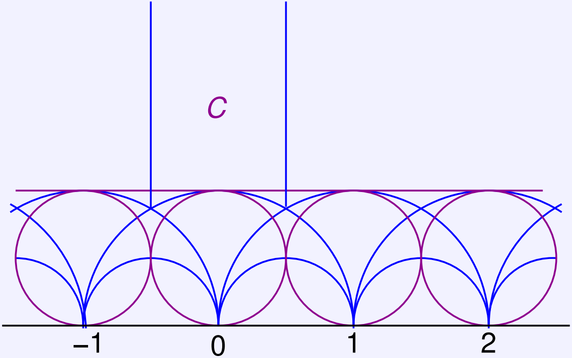
(2) Horoball $H_t$ of height $u = t$ is precisely invariant under $\Gamma$
That is, for any $S \in \Gamma$ either $S(H_t) = H_t$ or $S(H_t) \cap H_t = \emptyset$.

On the orbifold $M = \mathbf{H}^2_{\mathbb{R}}/\Gamma$
$C = H_t/\Gamma_\infty$ is a cusp neighbourhood of hyperbolic area $1$.



# The modular group

Shimizu's lemma is sharp for the modular group $\mathrm{PSL}(2,\mathbb{Z})$
In this case, $t = 1$ and $S(z) = -1/z$ has $r_S = 1$.
There is a cusp neighbourhood $C$ which cannot be enlarged.
It has area is $1$, which is large compared to the
area of $\mathbf{H}^2_{\mathbb{R}}/\mathrm{PSL}(2,\mathbb{Z})$, which is $\pi/3$



# Generalisations of Shimizu's lemma

- Leutbecher 1967 Subgroups of $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{Isom}_0(\mathbf{H}^3_{\mathbb{R}})$ containing a translation.
- Wielenberg 1977 Subgroups of $\mathrm{PO}_0(n, 1) = \mathrm{Isom}_0(\mathbf{H}^n_{\mathbb{R}})$ containing a translation.
- Kamiya 1983 Subgroups of $\mathrm{PU}(n, 1) = \mathrm{Isom}_0(\mathbf{H}^n_{\mathbb{C}})$ or $\mathrm{PSp}(n, 1) = \mathrm{Isom}_0(\mathbf{H}^n_{\mathbb{H}})$ containing a vertical Heisenberg translation.
- Apanasov 1985, Ohtake 1985 Subgroups of $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}})$ with $n \geq 4$ containing a screw parabolic map: there is no uniform bound on radii of isometric spheres.
- JRP 1992 Subgroups of $\mathrm{PU}(n, 1)$ containing a non-vertical Heisenberg translation: there is no uniform bound on radii of isometric spheres.
- Waterman 1993 Subgroups of $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}})$ with $n \geq 4$ containing a screw parabolic map: radii of isometric spheres bounded in terms of parabolic translation length at centres.

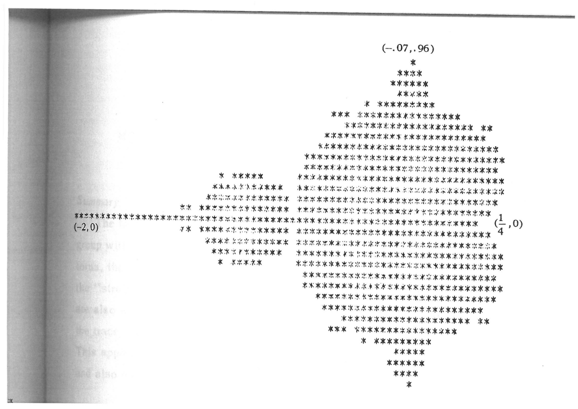
# Jørgensen's inequality

Jørgensen 1976 If $\langle T, S\rangle$ subgroup of $\mathrm{SL}(2,\mathbb{C})$ discrete, then elementary or $|\mathrm{tr}^2(S)-4|+|\mathrm{tr}[S,T]-2|\geq 1$
(Same structure of proof as Shimizu's lemma).

# Jørgensen's inequality

Jørgensen 1976 If $\langle T, S\rangle$ subgroup of $\mathrm{SL}(2,\mathbb{C})$ discrete, then elementary or $|\mathrm{tr}^2(S) - 4| + |\mathrm{tr}[S, T] - 2| \geq 1$
(Same structure of proof as Shimizu's lemma).

While working on basin of attraction part of this problem Brooks & Matelski 1978/1979 produced




Possibly the first picture of the Mandelbrot set

# Shimizu's lemma for real hyperbolic space

A parabolic isometry $T$ of $\mathbf{H}^n_{\mathbb{R}}$ fixing $\infty$ acts on $\mathbb{R}^{n-1}$ as $T(x) = Ux + t$ where $U \in \mathrm{O}(n-1)$, $t \in \mathbb{R}^{n-1}$ and $Ut = t$.

Let $N_U = \max\{\|(U-I)x\| \ : \ \|x\| = 1\}$
The Euclidean translation length of $T$ at $x$ is
$\ell_T(x) = \|T(x) - x\| = \|(U-I)x + t\| = \sqrt{\|(U-I)x\|^2 + \|t\|^2}$

# Shimizu's lemma for real hyperbolic space

A parabolic isometry $T$ of $\mathbf{H}^n_{\mathbb{R}}$ fixing $\infty$ acts on $\mathbb{R}^{n-1}$ as $T(x) = Ux + t$ where $U \in \mathrm{O}(n-1)$, $t \in \mathbb{R}^{n-1}$ and $Ut = t$.

Let $N_U = \max\{\|(U-I)x\| \ : \ \|x\| = 1\}$
The Euclidean translation length of $T$ at $x$ is
$\ell_T(x) = \|T(x) - x\| = \|(U-I)x + t\| = \sqrt{\|(U-I)x\|^2 + \|t\|^2}$

## Theorem (Waterman 1993)

*Let* $\Gamma < \mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}})$ *be discrete and contain* $T(x) = Ux + t$.
*Suppose* $N_U < 1/4$ *and write* $K = \frac{1}{2}(1 + \sqrt{1 - 4N_U})$.
*Let* $S \in \Gamma$ *not fixing* $\infty$ *have isometric sphere of radius* $r_S$.
*Then* $r_S^2 \leq \dfrac{\ell_T(S^{-1}(\infty))\ell_T(S(\infty))}{K^2}$.

# Shimizu's lemma for real hyperbolic space

A parabolic isometry $T$ of $\mathbf{H}^n_{\mathbb{R}}$ fixing $\infty$ acts on $\mathbb{R}^{n-1}$ as $T(x) = Ux + t$ where $U \in \mathrm{O}(n-1)$, $t \in \mathbb{R}^{n-1}$ and $Ut = t$.

Let $N_U = \max\{\|(U-I)x\| \ : \ \|x\| = 1\}$
The Euclidean translation length of $T$ at $x$ is
$\ell_T(x) = \|T(x) - x\| = \|(U-I)x + t\| = \sqrt{\|(U-I)x\|^2 + \|t\|^2}$

## Theorem (Waterman 1993)

*Let* $\Gamma < \mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}})$ *be discrete and contain* $T(x) = Ux + t$*.*
*Suppose* $N_U < 1/4$ *and write* $K = \frac{1}{2}(1 + \sqrt{1 - 4N_U})$*.*
*Let* $S \in \Gamma$ *not fixing* $\infty$ *have isometric sphere of radius* $r_S$*.*
*Then* $r_S^2 \leq \dfrac{\ell_T(S^{-1}(\infty))\ell_T(S(\infty))}{K^2}$.

- If $U = I$ then $N_U = 0$ and $K = 1$.
  Also, $\ell_T(x) = \|t\|$ and Waterman gives $r_S \leq \|t\|$
  This is Wielenberg's version of Shimizu's lemma.

We want to generalise this to other hyperbolic spaces

# Hyperbolic spaces

Let $\mathbb{F}$ be one of

- the real numbers $\mathbb{R}$,
- the complex numbers $\mathbb{C}$,
- the quaternions $\mathbb{H}$

Let $\mathbb{F}^{n,1}$ be the $n+1$ dimensional $\mathbb{F}$-vector space (with scalars in $\mathbb{F}$ acting on the right) equipped with $\langle \cdot, \cdot \rangle$ Hermitian form (bilinear for $\mathbb{R}^{n,1}$) of signature $(n, 1)$
Let $V_- = \{\mathbf{z} \in \mathbb{F}^{n,1} \ : \ \langle \mathbf{z}, \mathbf{z} \rangle < 0\}$
and $V_0 = \{\mathbf{z} \in \mathbb{F}^{n,1} - \{\mathbf{0}\} \ : \ \langle \mathbf{z}, \mathbf{z} \rangle = 0\}$
Let $\mathbb{P} : \mathbb{F}^{n,1} - \{\mathbf{0}\} \longrightarrow \mathbb{FP}^n$ be the (right) projection map.

Then $\mathbf{H}^n_{\mathbb{F}} = \mathbb{P}V_-$ and $\partial\mathbf{H}^n_{\mathbb{F}} = \mathbb{P}V_0$; metric on $\mathbf{H}^n_{\mathbb{F}}$ is given by:

$$ds^2 = \frac{-4}{\langle \mathbf{z}, \mathbf{z} \rangle^2} \det \begin{pmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle d\mathbf{z}, \mathbf{z} \rangle \\ \langle \mathbf{z}, d\mathbf{z} \rangle & \langle d\mathbf{z}, d\mathbf{z} \rangle \end{pmatrix}$$

The hyperbolic spaces are $\mathbf{H}^n_{\mathbb{R}}$, $\mathbf{H}^n_{\mathbb{C}}$, $\mathbf{H}^n_{\mathbb{H}}$ together with $\mathbf{H}^2_{\mathbb{O}}$ where $\mathbb{O}$ are the octonions (see Chen & Greenberg 1974).

# More about hyperbolic spaces

Let $\mathrm{O}(n,1)$, $\mathrm{U}(n,1)$, $\mathrm{Sp}(n,1)$ be the group preserving $\langle\cdot,\cdot\rangle$ when $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ respectively (acting on $\mathbb{F}^{n,1}$ on the left).
This group acts (projectively) by isometries on $\mathbf{H}^n_{\mathbb{F}}$
(For $\mathbb{F}=\mathbb{O}$ there is no vector space $\mathbb{O}^{2,1}$
but there is an analogous isometry group $F_{4(-20)}$ of $\mathbf{H}^2_{\mathbb{O}}$.
We will not consider this case here.)
We pass between matrix groups and isometries without comment.

An isometry $S$ of a hyperbolic space $\mathbf{H}$ is

- loxodromic (or hyperbolic) if it has two fixed points, both on $\partial\mathbf{H}$
- parabolic if it has a unique fixed point, lying on $\partial\mathbf{H}$
- elliptic if it fixes (at least) one point of $\mathbf{H}$

There are finer classifications of these types.
We will mainly be interested in parabolic maps, which are:
Either (Heisenberg)-translations or screw parabolic maps.

# Shimizu's lemma for other hyperbolic spaces

- Kamiya 1983 Subgroups of $\mathrm{SU}(n,1)$ or $\mathrm{Sp}(n,1)$ containing a vertical Heisenberg translation.
- Hersonsky-Paulin 1996 Subgroups of $\mathrm{SU}(n,1)$ containing a non-vertical Heisenberg translation (not given geometrically).
- JRP 1997 Subgroups of $\mathrm{SU}(n,1)$ containing a non-vertical Heisenberg translation.
- I. Kim & JRP 2003 Subgroups of $\mathrm{Sp}(n,1)$ containing a non-vertical Heisenberg translation.
- Jiang & JRP 2003 Subgroups of $\mathrm{SU}(2,1)$ containing a screw parabolic map (not given geometrically).
- D. Kim 2004 Subgroups of $\mathrm{Sp}(2,1)$ containing (certain types of) screw parabolic maps (not given geometrically).
- Kamiya & JRP 2008 Subgroups of $\mathrm{SU}(2,1)$ containing a positively oriented screw parabolic map.
- Cao & JRP 2014 Subgroups of $\mathrm{SU}(n,1)$ or $\mathrm{Sp}(n,1)$ containing any parabolic map.

# Other generalisations

The stable basin theorem

- Basmajian & Miner 1998 Stable basin theorem – stronger hypothesis than Shimizu/Jørgensen. Includes version of Shimizu's lemma for $\mathrm{SU}(2,1)$
- Kamiya 2000, Kamiya & JRP 2002 SBT for Heisenberg translations follows from Shimizu's lemma.

Generalisations of Jørgensen's inequality:

- Jiang, Kamiya & JRP 2003 Jørgensen's inequality for subgroups of $\mathrm{SU}(2,1)$
- Markham 2003 Jørgensen's inequality for subgroups of $\mathrm{PSp}(2,1)$ and (some) subgroups of $F_{4(-20)}$
- D. Kim 2004 Jørgensen's inequality for subgroups of $\mathrm{PSp}(2,1)$
- Cao & JRP 2011 Jørgensen's inequality for subgroups of $\mathrm{SU}(n,1)$ or $\mathrm{Sp}(n,1)$.

# Heisenberg groups and the boundary of hyperbolic spaces

We can identify

- $\partial\mathbf{H}^n_{\mathbb{R}}$ with $\mathbb{R}^{n-1}\cup\{\infty\}$,
- $\partial\mathbf{H}^n_{\mathbb{C}}$ with $\mathfrak{N}_{2n-1}\cup\{\infty\}$,
- $\partial\mathbf{H}^n_{\mathbb{H}}$ with $\mathfrak{N}_{4n-1}\cup\{\infty\}$.

$\mathfrak{N}_{2n-1}=\mathbb{C}^{n-1}\times\mathbb{R}$ is the $(2n-1)$-dimensional Heisenberg group,
and $\mathfrak{N}_{4n-1}=\mathbb{H}^{n-1}\times\mathbb{R}^3=\mathbb{H}^{n-1}\times\Im\mathbb{H}$ is the
$(4n-1)$-dimensional generalised Heisenberg group
both with the group law
$(\zeta_1,v_1)\cdot(\zeta_2,v_2)=(\zeta_1+\zeta_2,v_1+v_2+2\Im(\zeta_2^*\zeta_1))$
(where $z^*$ is the conjugate transpose)
We will write $\mathfrak{N}$ for both cases

The Cygan metric on $\mathfrak{N}$ is the metric associated to the norm
$\|(\zeta,v)\|=\left(\|\zeta\|^4+|v|^2\right)^{1/4}$
It generalises the Euclidean metric on $\mathbb{R}^{n-1}$ for $\mathbf{H}^n_{\mathbb{R}}$ and
the square root of the Euclidean metric on $\mathbb{R}$ for $\mathbf{H}^1_{\mathbb{C}}\approx\mathbf{H}^2_{\mathbb{R}}$.

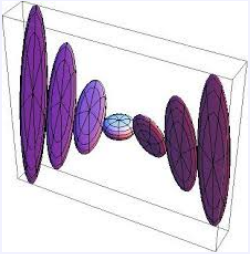
# Action of $\mathrm{PU}(n,1)$ and $\mathrm{PSp}(n,1)$ on $\mathfrak{N}$

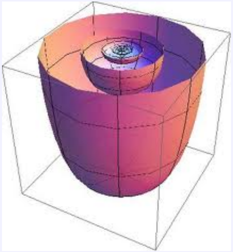
We write an element $S$ of $\mathrm{PU}(n,1)$ or $\mathrm{PSp}(n,1)$ and its inverse as

$$S = \begin{pmatrix} a & \gamma^* & b \\ \alpha & A & \beta \\ c & \delta^* & d \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} \overline{d} & \beta^* & \overline{b} \\ \delta & A^* & \gamma \\ \overline{c} & \alpha^* & \overline{a} \end{pmatrix}$$

where $a,b,c,d \in \mathbb{F}$, $\alpha,\beta,\gamma,\delta \in \mathbb{F}^{n-1}$, $A \in \mathrm{U}(n-1)$ or $\mathrm{Sp}(n-1)$.
If $c = 0$ then $S$ fixes $\infty$; if $c \neq 0$ we define isometric spheres.

The isometric sphere $I(S)$ of $S$ is the Cygan sphere of radius $r_S = 1/|c|^{1/2}$ with centre $S^{-1}(\infty) = \left(\delta\overline{c}^{-1}/\sqrt{2}, \Im(\overline{d}\overline{c}^{-1})\right) \in \mathfrak{N}$.
$S$ sends the outside of $I(S)$ to the inside of $I(S^{-1})$.





Pictures of Cygan spheres and hemispheres by Anton Lukyanenko.

# Heisenberg translations in $\mathrm{PU}(n,1)$ and $\mathrm{PSp}(n,1)$

The simplest parabolic maps in $\mathrm{PU}(n,1)$ and $\mathrm{PSp}(n,1)$ are Heisenberg translations:

The (generalised) Heisenberg group $\mathfrak{N}$ acts on itself by left translation: $T_{(\tau,t)} : (\zeta, v) \longmapsto \big(\zeta + \tau, v + t + 2\Im(\zeta^*\tau)\big)$

As a matrix in $\mathrm{PU}(n,1)$ or $\mathrm{PSp}(n,1)$ it is

$$T_{(\tau,t)} = T = \begin{pmatrix} 1 & -\sqrt{2}\tau^* & -\|\tau\|^2 + t \\ 0 & I & \sqrt{2}\tau \\ 0 & 0 & 1 \end{pmatrix}.$$

Note: $t$ in top right hand entry is pure imaginary so is $it$ in complex case.

A Heisenberg translation by $(0,t)$ is called a vertical translation and lies in the centre of $\mathfrak{N}$.

The Cygan translation length of $T$ at $(\zeta, v) \in \mathfrak{N}$ is $\ell_T((\zeta,v)) = \big(\|\tau\|^4 + |t + 4\Im(\zeta^*\tau)|^2\big)^{1/4}$.

# Shimizu's lemma for Heisenberg translations

**Theorem (JRP 1997, Kim-JRP 2003)**

*Let $\Gamma < \mathrm{PU}(n,1)$ or $\mathrm{PSp}(n,1)$ be discrete and contain Heisenberg translation $T$ by $(\tau, t)$.*
*Let $S \in \Gamma$ not fixing $\infty$ have isometric sphere of radius $r_S$.*
*Then $r_S^2 \leq \ell_T(S^{-1}(\infty))\ell_T(S(\infty)) + 4\|\tau\|^2$.*

- When $\tau = 0$ then $\ell_T((\zeta, v)) = |t|^{1/2}$ get $r_S^2 \leq \ell_T^2 = |t|$ (that is $|ct| \geq 1$), due to Kamiya 1983.

# Shimizu's lemma for Heisenberg translations

Theorem (JRP 1997, Kim-JRP 2003)

*Let* $\Gamma < \mathrm{PU}(n,1)$ *or* $\mathrm{PSp}(n,1)$ *be discrete and contain Heisenberg translation* $T$ *by* $(\tau, t)$.
*Let* $S \in \Gamma$ *not fixing* $\infty$ *have isometric sphere of radius* $r_S$.
*Then* $r_S^2 \leq \ell_T(S^{-1}(\infty))\ell_T(S(\infty)) + 4\|\tau\|^2$.

- When $\tau = 0$ then $\ell_T((\zeta, v)) = |t|^{1/2}$ get $r_S^2 \leq \ell_T^2 = |t|$ (that is $|ct| \geq 1$), due to Kamiya 1983.

Structure of proof same as for classical Shimizu:

- Consider the sequence $S_0 = S$, $S_{j+1} = S_j T S_j^{-1}$
- Show in finite basin of attraction of dynamical system
- Deduce $S_j$ tends to $T$.

# Shimizu's lemma for Heisenberg translations

### Theorem (JRP 1997, Kim-JRP 2003)

*Let* $\Gamma < \mathrm{PU}(n,1)$ *or* $\mathrm{PSp}(n,1)$ *be discrete and contain Heisenberg translation* $T$ *by* $(\tau, t)$.
*Let* $S \in \Gamma$ *not fixing* $\infty$ *have isometric sphere of radius* $r_S$.
*Then* $r_S^2 \leq \ell_T(S^{-1}(\infty))\ell_T(S(\infty)) + 4\|\tau\|^2$.

- When $\tau = 0$ then $\ell_T((\zeta, v)) = |t|^{1/2}$ get $r_S^2 \leq \ell_T^2 = |t|$ (that is $|ct| \geq 1$), due to Kamiya 1983.

Structure of proof same as for classical Shimizu:

- Consider the sequence $S_0 = S$, $S_{j+1} = S_j T S_j^{-1}$
- Show in finite basin of attraction of dynamical system
- Deduce $S_j$ tends to $T$.

The variables in the dynamical system are:
$X_j = \left(\max\{\ell_T(S_j^{-1}(\infty)), \ell_T(S_j(\infty))\}/r_{S_j}\right)^2, \quad Y_j = \left(\|\tau\|/r_{S_j}\right)^2$
They satisfy $X_{j+1} \leq X_j^2 + 4Y_j, \quad Y_{j+1} \leq X_j Y_j$

# Shimizu's lemma for Heisenberg translations

### Theorem (JRP 1997, Kim-JRP 2003)

*Let* $\Gamma < \mathrm{PU}(n,1)$ *or* $\mathrm{PSp}(n,1)$ *be discrete and contain Heisenberg translation* $T$ *by* $(\tau, t)$.
*Let* $S \in \Gamma$ *not fixing* $\infty$ *have isometric sphere of radius* $r_S$.
*Then* $r_S^2 \le \ell_T(S^{-1}(\infty))\ell_T(S(\infty)) + 4\|\tau\|^2$.

- When $\tau = 0$ then $\ell_T((\zeta, v)) = |t|^{1/2}$ get $r_S^2 \le \ell_T^2 = |t|$ (that is $|ct| \ge 1$), due to Kamiya 1983.

Structure of proof same as for classical Shimizu:

- Consider the sequence $S_0 = S$, $S_{j+1} = S_j T S_j^{-1}$
- Show in finite basin of attraction of dynamical system
- Deduce $S_j$ tends to $T$.

The variables in the dynamical system are:
$X_j = \left(\max\{\ell_T(S_j^{-1}(\infty)), \ell_T(S_j(\infty))\}/r_{S_j}\right)^2, \quad Y_j = \left(\|\tau\|/r_{S_j}\right)^2$
They satisfy $X_{j+1} \le X_j^2 + 4Y_j, \quad Y_{j+1} \le X_j Y_j$
If bound on $r_S$ is not true then we show $X_j + 4Y_j < 1$
and $X_j$, $Y_j$ tend to $0$ as $j \to \infty$

# Invariant horoballs

We can give $\mathbf{H}^n_{\mathbb{F}}$ the structure $\mathfrak{N} \times \mathbb{R}_+$
A horoball $H_u$ of height $u$ at $\infty$ is $\mathfrak{N} \times (u, \infty)$.

- For vertical translations by $(0, t)$ the horoball $H_{|t|}$ is precisely invariant.
- For non-vertical translations by $(\tau, t)$ with $\tau \neq 0$ there is a precisely invariant sub-horospherical region. We will not go into details about these.
- There is a sharp version of Shimizu's lemma for $\mathrm{PU}(2,1)$ yielding a cusp neighbourhood of volume $1/4$
  This cusp neighbourhood is as maximal for the Eisenstein-Picard lattice $\mathrm{PU}(2,1;\mathbb{Z}[\frac{1+i\sqrt{3}}{2}])$ and its sister – which have covolume $\pi^2/27$

# Positively oriented screw parabolic maps in $\mathrm{PU}(2,1)$

Consider $T = \begin{pmatrix} 1 & 0 & it \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{PU}(2,1)$

$T$ is positively oriented if $t\sin(\theta) > 0$.

$T$ acts on $\mathfrak{N}_3$ as $T : (\zeta, v) \longmapsto (e^{i\theta}\zeta, v + t)$

Its Cygan translation length $\ell_T((\zeta, v))$ at $(\zeta, v)$ is:

$$\left|2|\zeta|^2(e^{i\theta} - 1) + it\right|^{1/2} = \left(|\zeta|^4|e^{i\theta} - 1|^4 + (2|\zeta|^2\sin(\theta) + t)^2\right)^{1/4}$$

We have the following version of Shimizu's lemma for groups with positively oriented screw parabolic maps (cf Waterman's theorem):

### Theorem (Kamiya-JRP 2008)

*Let $\Gamma < \mathrm{PU}(2,1)$ be discrete and contain positively oriented $T$. Suppose $|e^{i\theta} - 1| < 1/4$ and write $K = \frac{1}{2}\left(1 + \sqrt{1 - 4|e^{i\theta} - 1|}\right)$. Let $S \in \Gamma$ not fixing $\infty$ have isometric sphere of radius $r_S$. Then $r_S^2 \le \dfrac{\ell_T(S^{-1}(\infty))\ell_T(S(\infty))}{K^2}$.*

- Note that if $\theta = 0$ we obtain Kamiya's 1983 result.

# General parabolic maps in $\mathrm{PU}(n,1)$

- A general parabolic map in $\mathrm{PU}(n,1)$ has the form
  $$T = \begin{pmatrix} 1 & -\sqrt{2}\tau^* & -\|\tau\|^2 + it \\ 0 & U & \sqrt{2}\tau \\ 0 & 0 & 1 \end{pmatrix}$$
  where $U \in \mathrm{U}(n-1)$ with $U\tau = \tau$
  (so if $n = 2$ and $U \neq I$ then $\tau = 0$).
- If $U \neq I$ then this is a screw parabolic map.
- $T$ acts on $\mathfrak{N}_{2n-1}$ as
  $T : (\zeta, v) \longmapsto (U\zeta + \tau, v + t + 2\Im(\zeta^*\tau))$
- Its Cygan translation length at $(\zeta, v)$ is
  $\ell_T((\zeta, v)) = \left(\|(U-I)\zeta+\tau\|^4+|t+2\Im((\zeta^*-\tau^*)(U\zeta+\tau))|^2\right)^{1/4}.$
- If $U = I$ then this map is a Heisenberg translation.
  Action on $\mathfrak{N}_{2n-1}$ and Cygan translation length are as before.

# General parabolic maps in $\mathrm{PSp}(n,1)$

Cao-JRP 2014

A general parabolic map in $\mathrm{PSp}(n,1)$ has the form

$$T = \begin{pmatrix} \mu & -\sqrt{2}\tau^*\mu & (-\|\tau\|^2+t)\mu \\ 0 & U & \sqrt{2}\tau\mu \\ 0 & 0 & \mu \end{pmatrix}$$

where $U \in \mathrm{Sp}(n-1)$ and $\mu \in \mathbb{H}$, $|\mu| = 1$

$$\text{with } \begin{cases} U\tau = \mu\tau,\ U^*\tau = \overline{\mu}\tau,\ \mu\tau \neq \tau\overline{\mu} & \text{if } \tau \neq 0 \text{ and } \mu \neq \pm 1, \\ U\tau = \tau,\ U^*\tau = \tau & \text{if } \tau \neq 0 \text{ and } \mu = \pm 1, \\ \mu t \neq t\overline{\mu} & \text{if } \tau = 0 \text{ and } \mu \neq \pm 1, \\ t \neq 0 & \text{if } \tau = 0 \text{ and } \mu = \pm 1. \end{cases}$$

This acts on $\mathfrak{N}_{4n-1}$ as

$$T : (\zeta, v) \longmapsto (U\zeta\overline{\mu} + \tau, \mu v\overline{\mu} + t + 2\Im(\mu\zeta^*\overline{\mu}\tau))$$

Its Cygan translation length $\ell_T((\zeta, v))$ is

$$\left(\|U\zeta\overline{\mu} - \zeta + \tau\|^4 + |\mu v\overline{\mu} - v + t + 2\Im((\zeta^* - \tau^*)(U\zeta\overline{\mu} + \tau))|^2\right)^{1/4}.$$

# Vertical projection

Before discussing the generalised Shimizu's lemma, there is one more ingredient.

- Define vertical projection
  $\Pi : \mathfrak{N}_{2n-1} = \mathbb{C}^{n-1} \times \mathbb{R} \longrightarrow \mathbb{C}^{n-1}$
  $\Pi : \mathfrak{N}_{4n-1} = \mathbb{H}^{n-1} \times \mathbb{R}^3 \longrightarrow \mathbb{H}^{n-1}$
  by $\Pi : (\zeta, v) \longmapsto \zeta$.

- If $T$ is one of the parabolic maps defined above, its vertical projection acts on $\mathbb{C}^{n-1}$ or $\mathbb{H}^{n-1}$ respectively as
  $T_\Pi : \zeta \longmapsto U\zeta\overline{\mu} + \tau$ (where $\mu = 1$ in the complex case)
- The Euclidean translation length of the vertical projection of $T$ at $\zeta \in \mathbb{F}^{n-1}$ is $\ell_T^\Pi(\zeta) = \|U\zeta\overline{\mu} - \zeta + \tau\|$

# The generalised Shimizu's lemma

Let $T$ be a general parabolic map, $U$, $\mu$ as before.
Define $N_{U,\mu} = \max\{\|U\zeta\overline{\mu} - \zeta\| \ : \ \|\zeta\| = 1\}$
$N_{\mu} = \max\{\|\mu\zeta\overline{\mu} - \zeta\| \ : \ \|\zeta\| = 1\} = |\Im(\mu)|$

# The generalised Shimizu's lemma

Let $T$ be a general parabolic map, $U$, $\mu$ as before.
Define $N_{U,\mu} = \max\{\|U\zeta\overline{\mu} - \zeta\| \ : \ \|\zeta\| = 1\}$
$N_\mu = \max\{\|\mu\zeta\overline{\mu} - \zeta\| \ : \ \|\zeta\| = 1\} = |\Im(\mu)|$

## Theorem (Cao-JRP 2014)

*Let* $\Gamma \in \mathrm{PSp}(n,1)$ *be discrete and contain parabolic* $T$ *as above.*
*Suppose* $N_\mu < 1/4$ *and* $N_{U,\mu} < (3 - 2\sqrt{2 + N_\mu})/2$
*Define* $K = \frac{1}{2}\left(1 + 2N_{U,\mu} + \sqrt{1 - 12N_{U,\mu} + 4N_{U,\mu}^2 - 4N_\mu}\right)$
*Let* $S \in \Gamma$ *not fixing* $\infty$ *have isometric sphere of radius* $r_S$. *Then*

$$r_S^2 \leq \frac{\ell_T(S^{-1}(\infty))\ell_T(S(\infty))}{K} + \frac{4\ell_T^\Pi(\Pi S^{-1}(\infty))\ell_T^\Pi(\Pi S(\infty))}{K(K - 2N_{U,\mu})}$$

# The generalised Shimizu's lemma

Let $T$ be a general parabolic map, $U$, $\mu$ as before.
Define $N_{U,\mu} = \max\{\|U\zeta\overline{\mu} - \zeta\| \ : \ \|\zeta\| = 1\}$
$N_\mu = \max\{\|\mu\zeta\overline{\mu} - \zeta\| \ : \ \|\zeta\| = 1\} = |\Im(\mu)|$

## Theorem (Cao-JRP 2014)

*Let $\Gamma \in \mathrm{PSp}(n,1)$ be discrete and contain parabolic $T$ as above.*
*Suppose $N_\mu < 1/4$ and $N_{U,\mu} < (3 - 2\sqrt{2 + N_\mu})/2$*
*Define $K = \frac{1}{2}\Big(1 + 2N_{U,\mu} + \sqrt{1 - 12N_{U,\mu} + 4N_{U,\mu}^2 - 4N_\mu}\Big)$*
*Let $S \in \Gamma$ not fixing $\infty$ have isometric sphere of radius $r_S$. Then*

$$r_S^2 \leq \frac{\ell_T(S^{-1}(\infty))\ell_T(S(\infty))}{K} + \frac{4\ell_T^\Pi(\Pi S^{-1}(\infty))\ell_T^\Pi(\Pi S(\infty))}{K(K - 2N_{U,\mu})}$$

- When $\mu = 1$ (including case of $\mathrm{PU}(n,1)$) hypotheses simplify:
  $N_U = N_{U,1} < (\sqrt{2} - 1)^2/2$ and
  $K = \frac{1}{2}\Big(1 + 2N_U + \sqrt{1 - 12N_U + 4N_U^2}\Big)$
  The conclusion remains the same.
- When $U = I$, $\mu = 1$ get version for Heisenberg translations.

# Sketch of the proof when $U \neq I$ (so $N_{U,\mu} \neq 0$)

- Consider the sequence $S_0 = S$, $S_{j+1} = S_j T S_j^{-1}$
- Show in finite basin of attraction of dynamical system
- Deduce $S_j$ tends to $T$.

## Sketch of the proof when $U \neq I$ (so $N_{U,\mu} \neq 0$)

- Consider the sequence $S_0 = S$, $S_{j+1} = S_j T S_j^{-1}$
- Show in finite basin of attraction of dynamical system
- Deduce $S_j$ tends to $T$.

The variables in the dynamical system are:
$X_j = \left(\max\{\ell_T(S_j^{-1}(\infty)), \ell_T(S_j(\infty))\}/r_{S_j}\right)^2,$
$Y_j = \left(\max\{\ell_T^\Pi(\Pi S_j^{-1}(\infty)), \ell_T^\Pi(\Pi S_j(\infty))\}/r_{S_j}\right)^2$
They satisfy recursion inequalities:
$X_{j+1} \leq X_j^2 + 4Y_j + 2N_{U,\mu} + N_\mu, \quad Y_{j+1} \leq X_j Y_j + 2N_{U,\mu} Y_j + N_{U,\mu}^2$

# Sketch of the proof when $U \neq I$ (so $N_{U,\mu} \neq 0$)

- Consider the sequence $S_0 = S$, $S_{j+1} = S_j T S_j^{-1}$
- Show in finite basin of attraction of dynamical system
- Deduce $S_j$ tends to $T$.

The variables in the dynamical system are:
$X_j = \left(\max\{\ell_T(S_j^{-1}(\infty)), \ell_T(S_j(\infty))\}/r_{S_j}\right)^2$,
$Y_j = \left(\max\{\ell_T^{\Pi}(\Pi S_j^{-1}(\infty)), \ell_T^{\Pi}(\Pi S_j(\infty))\}/r_{S_j}\right)^2$
They satisfy recursion inequalities:
$X_{j+1} \leq X_j^2 + 4Y_j + 2N_{U,\mu} + N_\mu, \quad Y_{j+1} \leq X_j Y_j + 2N_{U,\mu} Y_j + N_{U,\mu}^2$

If the bound on $r_S$ in the theorem is not true then:

- $X_j + 4Y_j/(K - 2N_{U,\mu}) < K$
- for large enough $j$ we have
  $X_j < K - 2N_{U,\mu}$ and $Y_j < (K - 2N_{U,\mu})N_{U,\mu}/2$
- for all $\varepsilon > 0$ there exists $J_\varepsilon$ so for all $j \geq J_\varepsilon$:
  $X_j < 1 - K + \varepsilon$ and $Y_j < (1 - K)N_{U,\mu}/2 + \varepsilon$

# Where do we go from here?

- For screw parabolic maps $T$ where $U$ has infinite order, the asymptotic growth (in terms of distance from axis) of bounds on $r_S$ are worse than in examples.
- Erlandsson & Zakeri: In $\mathrm{PO}(4,1)$ use same bounds for carefully chosen powers of $T$ to improve asymptotics. 'Carefully chosen' means use Diophantine approximation of rotation angle.
- Erlandsson-Zakeri's idea also works in $\mathrm{PU}(2,1)$.
- Try to generalise these results to $F_{4(-20)}$