KNOTS, LINKS AND 4-DIMENSIONAL SURGERY

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Abstract. Let K be a knot or a non-split link in S^3, and let $M(K)$ denote the 4-manifold $\partial(E(K) \times D^2)$, where $E(K)$ is the exterior of K. We show that the TOP surgery obstruction theory works for $M(K)$, *i.e.* the TOP surgery sequence $\mathcal{S}(M(K)) \longrightarrow [M(K), G/TOP] \longrightarrow L_4(\pi_1(M(K)))$ is exact.

1. Introduction

In [3], Hegenbarth and Repovš used the controlled surgery exact sequence of Pedersen-Quinn-Ranicki [6] to show that the surgery obstruction theory works for certain 4-manifolds without assuming that the fundamental groups are good. Among their examples are 4-manifolds whose fundamental groups are knot groups. Let K be a knot, or more generally a link, in S^3, and let $E(K)$ dcenote its exterior. Let $M(K)$ denote the 4-manifold $\partial(E(K) \times D^2)$. The fundamental group of $M(K)$ is isomorphic to the fundamental group of $E(K)$, *i.e.* the knot (or link) group of K. Hegenbarth and Repovš showed that the surgery obstruction theory works in the topological category when K is a torus knot. The aim of this article is to show that their strategy works when K is any knot. Actually it also works when K is a non-split link, *i.e.* when no locally-flat sphere in $S^3 - K$ separates K.

Theorem 1. *The TOP-surgery sequence*

$$\mathcal{S}(M(K)) \longrightarrow [M(K), G/TOP] \longrightarrow L_4(\pi_1(M(K)))$$

is exact when K is a knot or a non-split link.

The key ingredients of the proof are (1) the construction of a UV^1 control map $p : M(K) \to B$ to a spine B of $E(K)$, (2) the controlled surgery exact sequence for UV^1 control maps, and (3) the existence of a non-positively curved Riemannian metric on the knot complement $S^3 - K$ together with the topological rigidity results of Farrell and Jones [2].

2. Construction of a UV^1 control map

A proper surjection $f : X \to Y$ is said to be UV^1 if, for any $y \in Y$ and for any neighborhood U of $f^{-1}(y)$ in X, there exists a smaller neighborhood V of $f^{-1}(y)$ such that any map $K \to V$ from a complex of dimension ≤ 1 to V is homotopic to a constant map as a map $K \to U$. A UV^1 map induces an isomorphism on fundamental groups. See [4] for the detail.

Let K be a knot or a link in S^3. In this section, we construct a UV^1 map $p : M(K) \to B$, where B is a spine of $E(K)$.

Let us recall that a topological ideal triangulation of a space is a method of glueing ideal tetrahedra (= tetrahedra whose vertices are removed) via topological identifications of faces and edges to obtain the given space. For example, the figure eight knot complement can be obtained by glueing two ideal tetrahedra [11,12]. The following seems to be a folklore:

Key words and phrases. Knot; Link; Surgery sequence.

Theorem 2. *The complement $S^3 - K$ of any knot/link K in S^3 has a topological ideal triangulation.*

An illustration of a proof can be found in [9, §2]. Different arguments can be found in [10,14].

We use the dual spine B of such a topological ideal triangulation to construct a UV^1 map $M(K) \to B$.

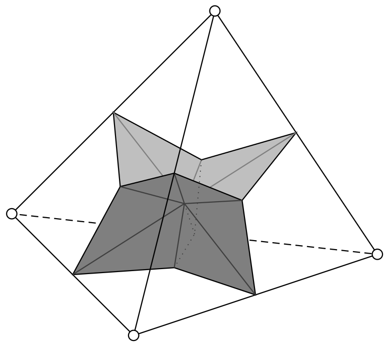

FIGURE 1. The dual spine of a tetrahedra.

Fix a topological ideal triangulation of $S^3 - K$. We can construct a dual spine B inductively as follows: Take one point from each edge; the union of these points is the dual spine of the 1-skeleton and there is a collapsing map from the 1-skeleton to the spine. Next, take one point from the interior of each face, and, after unidentifying the edges if necessary, take the join of the point and the the spine of the boundary. The union of these joins is the spine of the 2-skeleton. The collapsing map of the 1-skeleton extends to the collapsing map of the 2-skeleton to the spine. Finally, take one point from the interior of each ideal tetrahedra, and, after unidentifying the faces if necessary, take the join of the point and the spine of the boundary (Fig. 1). The union of these joins is the desired spine B, and the collapsing map of the 2-skeleton extends to a collapsing map $q : S^3 - K \to B$.

B is also a spine of $E(K)$, and $q : S^3 - K \to B$ restricts to a collapsing map $q : E(K) \to B$. We may assume that, for each point $x \in B$, $q^{-1}(x)$ is the join of x and a finite set $A(x) \subset \partial E(K)$.

Next consider the composite map

$$E(K) \times D^2 \xrightarrow{\text{proj.}} E(K) \xrightarrow{q} B$$

and define $p : M(K) \to B$ to be its restriction to the boundary, as was done in [3]. For each point $x \in B$, $p^{-1}(x)$ is the union of finitely many copies of 2-discs along the boundary and is simply-connected:

$$p^{-1}(x) = A(x) \times D^2 \cup q^{-1}(x) \times S^1 \subset \partial(E(K)) \times D^2 \cup E(K) \times S^1 = M(K) \ .$$

Therefore, p is UV^1, and p induces an isomorphism between $\pi_1(M(K))$ and $\pi_1(B) = \pi_1(E(K))$.

Remark. We used an ideal triangulation of $S^3 - K$ to construct a spine. Actually we only need to have an ideal cell decomposition, which is easier to construct. This construction will be discussed in §5

3. Surgery sequences and assembly maps

The strategy of Hegenbarth and Repovš [3] is to use the controlled surgery theory. Since the map $p : M(K) \to B$ is UV^1, there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{S}_{\epsilon,\delta}(M(K);p) & \longrightarrow & [M(K), G/TOP] & \longrightarrow & L_4^c(B;p) \\ {\scriptstyle F}\downarrow & & \| & & \downarrow{\scriptstyle F} \\ \mathcal{S}(M(K)) & \longrightarrow & [M(K), G/TOP] & \longrightarrow & L_4(\pi_1(M(K))) \end{array}$$

for sufficiently small $\epsilon \gg \delta > 0$. The first row is the controlled surgery sequence with trivial local fundamental groups and is exact [6]. The second row is the ordinary surgery sequence we are interested in. The two vertical maps labeled F are the forget-control maps. $L_4^c(B;p)$ is the controlled L-group of $p : M(K) \to B$. It is defined to be the limit of (ϵ,δ)-controlled L-groups [7]. Note that, if $F : L_4^c(B;p) \to L_4(\pi_1(M(K)))$ is injective, a simple diagram chase shows that the second row is exact.

We will identify $F : L_4^c(B;p) \to L_4(\pi_1(M(K)))$ with the assembly map $A : H_4(E(K);\mathbb{L}) \to L_4(\pi_1(E(K)))$ [8,13]. Here $\mathbb{L}$ is the simply-connected surgery spectrum $\mathbb{L}_\bullet(\mathbb{Z})$, whose homotopy groups $\pi_i(\mathbb{L})$ are the 4-periodic surgery obstruction groups $L_i(1)$ of the trivial group for $i \in \mathbb{Z}$. Actually the 0-periodic one $\overline{\mathbb{L}}_\bullet = \mathbb{L}_\bullet\langle 0\rangle(\mathbb{Z})$ and the 1-periodic one $\overline{\mathbb{L}}_\bullet = \mathbb{L}_\bullet\langle 1\rangle(\mathbb{Z})$ both give the same homology group in this dimension, because $\dim E(K) < 4$.

Since this assembly map is an isomorphism when K is a knot or a non-split link [1], we can obtain the main theorem. The argument given in [1] will be reviewed in the next section.

Recall that there is actually a functor $\mathbb{L}$ from spaces to spectra so that $\pi_i(\mathbb{L}(X)) \cong L_i(\pi_1(X))$, and the simply-connected surgery spectrum $\mathbb{L}$ can be thought of as $\mathbb{L}(X)$, for any simply-connected space X; *e.g.* $\mathbb{L} = \mathbb{L}(\{*\})$.

We can apply this functor $\mathbb{L}(\ )$ 'fiberwise' to p, and define sheaf homology groups $H_*(B;\mathbb{L}(p))$. Since p is UV^1, the controlled L-group $L_4^c(B;p)$ is isomorphic to the homology group $H_4(B;\mathbb{L}(p))$. Under this identification, the forget-control map $F : L_4^c(B;p) \to L_4(\pi_1(M(K)))$ can be identified with the assembly map $A : H_4(B;\mathbb{L}(p)) \to L_4(\pi_1(M(K)))$. Now the commutative diagram

$$\begin{array}{ccc} M(K) & \xrightarrow{\ p\ } & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle 1} \\ B & \xrightarrow[1]{} & B \end{array}$$

induces a homomorphism

$$H_4(B;\mathbb{L}(p)) \to H_4(B;\mathbb{L}(1 : B \to B)) = H_4(B;\mathbb{L}).$$

This is an isomorphism, since p is UV^1. On the other hand p induces an isomorphism

$$L_4(\pi_1(M(K))) = \pi_4(\mathbb{L}(M(K))) \to \pi_4(\mathbb{L}(B)) = L_4(\pi_1(B)),$$

and we can identify the assembly map $A : H_4(B;\mathbb{L}(p)) \to L_4(\pi_1(M(K)))$ with the assembly map $A : H_4(B;\mathbb{L}) \to L_4(\pi_1(B))$. Finally the homotopy equivalence $q : E(X) \to B$ induces an identification of this assembly map with the assembly map $A : H_4(E(X);\mathbb{L}) \to L_4(\pi_1(E(X)))$.

Remark. In [6], the homology group $H_4(B;\mathbb{L})$ is used as the controlled surgery obstruction group. The description above makes it easier to check the commutativity of the diagram given at the beginning of this section.

4. Geometry of $S^3 - K$ and assembly maps

In [1], Aravinda, Farrell and Roushon calculated the L-groups of knot and link groups. In the process, they proved the following:

Theorem 3. *If K is a knot or a non-split link, then the assembly map $A : H_i(E(X);\mathbb{L}) \to L_i(\pi_1(E(X)))$ is an isomorphism for every $i \in \mathbb{Z}$.*

As was mentioned in the previous section, this theorem finishes the proof of our main theorem. We will review their argument in the rest of this section.

By a work of Leeb [5], $S^3 - K$ has a complete Riemannian metric of nonpositive curvature when K is a knot or a non-split link. The double $D(K)$ of $E(K)$ inherits a metric of non-positive curvature. Then the topological rigidity result of Farrell and Jones [2] can be applied to $D(K)$, and we obtain

$$\mathcal{S}(D(K) \times D^n \text{ rel } \partial) = \{*\} \quad (n \geq 2).$$

This implies the vanishing of the algebric structure groups [8]:

$$\mathcal{S}_{n+4}(D(K)) \cong \mathcal{S}_{n+4}(D(K) \times D^n) \cong \mathcal{S}(D(K) \times D^n \text{ rel } \partial)$$

for $n \geq 2$. Since $E(K)$ is a retract of $D(K)$, the algebraic structure groups $\mathcal{S}_i(E(K))$ are all trivial for $i \geq 6$. In these dimensions, $\mathcal{S}_i(E(K))$ are equal to the 4-periodic algebraic structure groups $\mathcal{S}_i(\mathbb{Z}, E(K))$. By the 4-periodicity, the structure groups $\mathcal{S}_i(\mathbb{Z}, E(K))$ are trivial for all $i \in \mathbb{Z}$. These 4-periodic algebraic structure groups fit into the algebraic surgery exact sequence

$$\cdots \to \mathcal{S}_{i+1}(\mathbb{Z}, E(K)) \to H_i(E(K);\mathbb{L}) \xrightarrow{A} L_i(\pi_1(E(K))) \to \mathcal{S}_i(\mathbb{Z}, E(K)) \to \dots$$

of Ranicki, where the map A is the assembly map. Therefore the assembly maps are isomorphisms for all $i \in \mathbb{Z}$.

5. Ideal cell decomposition of $S^3 - K$

In §2, we used an ideal triangulation of $S^3 - K$ to construct a spine B. In this section we discuss an alternative method that uses an ideal cell decomposition.

Identify S^3 with $S^2 \times (-\infty, \infty) \cup \{\pm\infty\}$, and consider a knot projection to $S^2 \times 0$, with n crossings. We assume that K stays in $S^2 \times 0$ except at the overcrossings as shown in Fig. 2.

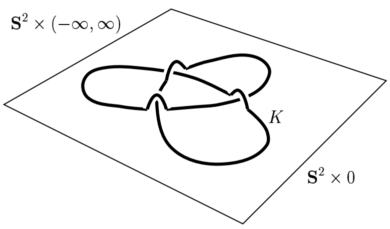

Figure 2. The knot projection.

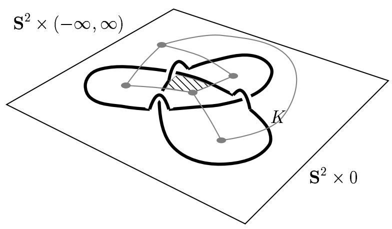

Figure 3. The dual graph.

Consider the dual graph of the knot/link diagram (Fig. 3). The dual graph and the knot/link diagram together decompose $S^2 \times 0$ into $4n$-many quadrangles R_i. One such quadrangle is indicated in Fig. 3. Roughly speaking, $R_i \times (-\infty, \infty) - K$ (Fig. 4) are the desired ideal 3-cells.

Unfortunately their union is not $S^3 - K$, but $S^3 - \{\pm\infty\} - K$. So pick an intersection point of K and the dual graph, and dig tunnels from that point to $\pm\infty$ along the edges. This affects four of the 3-cells (Fig. 5), and gives a decomposition of $S^3 - K$ into ideal cells. The dual spine of this ideal cell decomposition can be

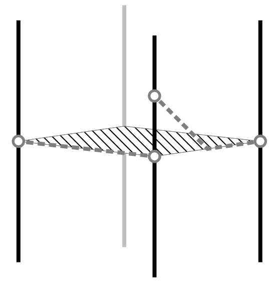

Figure 4. An ideal 3-cell.

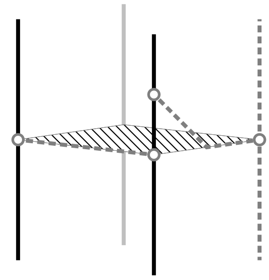

Figure 5. A modified ideal 3-cell.

defined in the same way as in §2, and this can be used to construct a desired UV^1 map $M(K) \to B$.

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