Symmetric and Quadratic Complexes
with Geometric Control

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Dedicated to Professor Frank Raymond on his 60th birthday

Introduction.

The quadratic L-groups $L_n(A)$ $(n \geq 0)$ of Wall [10] are defined for any ring with involution A, and are 4-periodic. An n-dimensional normal map $(f,b) : M \longrightarrow X$ determines its quadratic signature

$$\sigma_*(f,b) \in L_n(\mathbf{Z}[\pi])$$

for any oriented covering $\widetilde{X}$ with group of covering translations π. If $\widetilde{X}$ is the universal cover, $\sigma_*(f,b)$ is the surgery obstruction, and $\sigma_*(f,b) = 0$ if (and for $n \geq 5$ only if) (f,b) is normally bordant to a homotopy equivalence. The n-dimensional quadratic L-group $L_n(A)$ was expressed in Ranicki [4,6] as the cobordism group of n-dimensional quadratic Poincaré complexes over A, which are chain complexes C of finitely generated free A-modules with an n-dimensional quadratic structure ψ inducing Poincaré duality isomorphisms $(1+T)\psi_0 : H^{n-*}(C) \cong H_*(C)$.

The symmetric L-groups $L^n(A)$ $(n \geq 0)$ were introduced by Mishchenko [1] to describe the symmetric part of the surgery obstruction, and are not 4-periodic in general. The n-dimensional symmetric L-group $L^n(A)$ is the cobordism group of n-dimensional symmetric Poincaré complexes over A, which are chain complexes C of finitely generated free A-modules with an n-dimensional symmetric structure ϕ inducing Poincaré duality isomorphisms $\phi_0 : H^{n-*}(C) \cong H_*(C)$. A geometric Poincaré complex X determines its symmetric signature (or "higher signature")

$$\sigma^*(X) \in L^n(\mathbf{Z}[\pi])$$

for any oriented covering $\widetilde{X}$ with group of covering translations π. If n is a multiple of 4 and if we do not take the cover, then

$$\sigma^*(X) \in L^n(\mathbf{Z}) = \mathbf{Z}$$

is the usual signature of X.

In this note we show that the symmetric signatures of PL manifolds and the quadratic signatures of normal maps between PL manifolds are equipped with geometric control.

More precisely we construct controlled symmetric/quadratic signatures, which are geometric module symmetric/quadratic Poincaré complexes. They give rise to the ordinary symmetric and quadratic signatures, respectively, by the forget-control assembly process. We can get a better control by using finer triangulations of the manifolds.

We closely follow the original construction, making some necessary modifications to get control. The systematic approach to controlled algebraic Poincaré complexes of Ranicki [7, Chapter 5] is more powerful when more general spaces than PL manifolds are concerned, but the method in this note is considerably simpler.

The controlled signatures are closely related to the rational Pontrjagin classes. In [9] we shall prove that if the control is good enough then the controlled symmetric signature determines the L-theoretic orientation of a manifold, which is both topologically invariant and rationally equivalent to the Hirzebruch $\mathcal{L}$-genus. This will involve a refinement of the splitting construction of [11]. The controlled signatures will thus give a new proof of the topological invariance of rational Pontrjagin classes originally due to Novikov [2]. (See [7, Appendix C] for a proof using bounded surgery theory.)

1. Block systems and diagonal subcomplexes.

The symmetric and the quadratic constructions of Ranicki [5] are both based on the method of acyclic models, so they have an implicit geometric control. For our purposes, we need to make the geometric control more explicit. In this section we introduce the notion of block systems and diagonal subcomplexes to give the desired explicit control in the constructions. For the convenience of the reader, basic definitions concerning chain complexes are summarized in the appendix A1.

In geometric topology, "controls" are defined using covers of spaces. The "block systems" defined below have the corresponding role in a combinatorial setting.

Definition. A *block system* for a *CW* complex X is a covering $\kappa = \{K_\alpha\}_{\alpha\in A}$ of X by subcomplexes. Each K_α is called a *block*. A block system $\kappa = \{K_\alpha\}$ is said to *satisfy the contractibility condition* if, whenever K_α and K_β have non-empty intersection, their union $K_\alpha \cup K_\beta$ is contractible. (In particular, each non-empty block has to be contractible.)

When we use the contractibility condition on block systems in the next section, only the contractibility of the blocks are used. The contractibility of unions will be used in §3 to prove the well-definedness of a certain chain map.

Examples. (1) Let X be a polyhedron with a triangulation K. Then $\kappa = \{\sigma\}_{\sigma\in K}$ is a block system for K and it satisfies the contractibility condition. This block system will be called the *standard block system*.

(2) Suppose the triangulation K of a polyhedron X is the second derived subdivision L'' of another simplicial complex L. For each simplex $\Delta \in L$, let $N(\Delta, K)$ be the simplicial neighbourhood of Δ with respect to K, i.e., the union of all the simplices of K that intersect Δ. Then $\{N(\Delta, K)\}_{\Delta \in L}$ is a block system for X, and it satisfies the contractibility condition. This block system will be called the *regular block system* for X with respect to L.
(3) Let X be a *CW* complex with a block system $\kappa = \{K_\alpha\}$. Adjoin a point to X to get a pointed *CW* complex $X_+ = X \sqcup \{pt.\}$. Let Σ denote the reduced suspension operation on pointed spaces. Define the suspension $\Sigma\kappa$ by $\{\Sigma(K_\alpha)_+\}$, then it is a block system for ΣX_+ with respect to the *CW* structure induced from that of X.

Next we consider chain complexes. Let C be a based free $\mathbf{Z}$-module chain complex.

Definition. A subcomplex D of $C \otimes_{\mathbf{Z}} C$ is *diagonal* if $x \otimes x$ belongs to D for every basis element x of C. Such a D is *symmetric* if $x \otimes y \in D$ implies $y \otimes x \in D$.

Examples. (1) Let $\kappa = \{K_\alpha\}$ be a block system for a *CW* complex X, and let $C(X)$ denote the cellular $\mathbf{Z}$-module chain complex of X. $C(X)$ is freely generated by the cells of K. Then the set

$$\{\ \sigma \otimes \tau \in C(X) \otimes_{\mathbf{Z}} C(X) \mid \sigma,\, \tau \text{ are the cells of } K_\alpha \text{ for some } \alpha\ \}$$

generates a symmetric diagonal subcomplex of $C(X) \otimes_{\mathbf{Z}} C(X)$. This subcomplex will be denoted $D^\kappa(C(X))$. If X is pointed, the set

$$\{\ \sigma \otimes \tau \in \widetilde{C}(X) \otimes_{\mathbf{Z}} \widetilde{C}(X) \mid \sigma,\, \tau \text{ are the cells of } K_\alpha \text{ for some } \alpha\ \}$$

generates a symmetric diagonal subcomplex of $\widetilde{C}(X) \otimes_{\mathbf{Z}} \widetilde{C}(X)$. This subcomplex will be denoted $D^\kappa(\widetilde{C}(X))$. Here $\widetilde{C}(X)$ denotes the cellular $\mathbf{Z}$-module chain complex $C(X, pt.)$.
(2) If D is a diagonal subcomplex of $C \otimes_{\mathbf{Z}} C$, then $2p$-th suspension $S^{2p}D$ can be viewed as a diagonal subcomplex of $S^pC \otimes_{\mathbf{Z}} S^pC$ for any positive integer p. See A1 for suspensions of chain complexes. If $D = D^\kappa(C(X))$, then $S^{2p}D = D^{\Sigma^p\kappa}(\widetilde{C}(\Sigma X_+))$.

The chain complex $D^\kappa(C(X))$ given above is isomorphic to the chain complex $\mathrm{Hom}^\kappa_{\mathbf{Z}}(C(X)^*, C(X))$ defined below.

Definition. Let D be a diagonal subcomplex of a based free chain complex C. Let x and y be basis elements of C_p and C_q respectively. We define a $\mathbf{Z}$-module homomorphism $f_{x,y} : C^p \to C_q$ by:

$$f_{x,y}(z^*) = \begin{cases} y & \text{if } z = x, \\ 0 & \text{if } z \neq x, \end{cases}$$

for a basis element z of C_p. Here z^* denotes the dual of z. Define $\mathrm{Hom}^D_{\mathbf{Z}}(C^p, C_q)$ to be the submodule of $\mathrm{Hom}_{\mathbf{Z}}(C^p, C_q)$ generated by $f_{x,y}$'s, where $x \in C_p$ and $y \in C_q$ are the basis elements such that $x \otimes y \in D$. These form a subcomplex of $\mathrm{Hom}_{\mathbf{Z}}(C^*, C)$, and it will be denoted $\mathrm{Hom}^D_{\mathbf{Z}}(C^*, C)$. If κ is a block system for a polyhedron X, $\mathrm{Hom}^{D^\kappa(C(X))}_{\mathbf{Z}}(C(X)^*, C(X))$ is denoted $\mathrm{Hom}^\kappa_{\mathbf{Z}}(C(X)^*, C(X))$, and $\mathrm{Hom}^{S^pD^\kappa(C(X))}_{\mathbf{Z}}((S^pC(X))^*, S^pC(X))$ is denoted $\mathrm{Hom}^\kappa_{\mathbf{Z}}((S^pC(X))^*, S^pC(X))$.

An isomorphism $D \to \mathrm{Hom}^D_{\mathbf{Z}}(C^*, C)$ is given by the slant product chain map

$$x \otimes y \longmapsto (f \mapsto f(x)y).$$

If D is symmetric, the generator $T \in \mathbf{Z}_2$ acts on D by the transposition involution $x \otimes y \mapsto (-)^{pq} y \otimes x$ ($x \in C^p$, $y \in C_q$, $x \otimes y \in D_{p+q}$), and it acts on $\mathrm{Hom}^D_{\mathbf{Z}}(C^*, C)$ by the duality involution $\phi \mapsto (-)^{pq}\phi^*$ ($\phi \in \mathrm{Hom}^D_{\mathbf{Z}}(C^p, C_q)$). With respect to these actions, the map above is an isomorphism of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes.

Let X and Y be CW complexes, $f : X \to Y$ be a cellular map, and κ and λ be block systems for X and Y, respectively. Assume that f preserves the block system in the sense that, for any block $K_\alpha \in \kappa$, f maps K_α to some block $L_\beta \in \lambda$. (This is the case if f is a simplicial map between polyhedra and κ and λ are both standard.) Let $g : C(X) \to C(Y)$ be the induced $\mathbf{Z}$-module chain map. Then g induces $\mathbf{Z}[\mathbf{Z}_2]$-module chain maps:

$$\begin{aligned} g \otimes g \ &: \ D^\kappa(C(X)) \to D^\lambda(C(Y)) \\ \mathrm{Hom}(g^*, g) \ &: \ \mathrm{Hom}^\kappa_{\mathbf{Z}}(C(X)^*, C(X)) \longrightarrow \mathrm{Hom}^\lambda_{\mathbf{Z}}(C(Y)^*, C(Y)). \end{aligned}$$

2. Symmetric and quadratic constructions.

In this section we review the symmetric and quadratic constructions of Ranicki. We restrict ourselves to rather simple situations. See [5] for a fuller treatment. On the other hand we incorporate explicit controls into the constructions, using the tools from the previous section.

Let X be a polyhedron. Suppose κ is a block system for X. Then the method of acyclic models gives a $\mathbf{Z}$-module chain map ("diagonal approximation")

$$\Delta_X : C(X) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, D^\kappa(C(X))),$$

where W denotes the standard free $\mathbf{Z}[\mathbf{Z}_2]$-resolution of $\mathbf{Z}$

$$W \ : \ \ldots \longrightarrow \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1-T} \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1+T} \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1-T} \mathbf{Z}[\mathbf{Z}_2] \longrightarrow 0.$$

Using the identification of $D^\kappa(C(X))$ and $\mathrm{Hom}^\kappa_{\mathbf{Z}}(C(X)^*, C(X))$, we obtain a $\mathbf{Z}$-module chain map

$$\phi_X : C(X) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \mathrm{Hom}^\kappa_{\mathbf{Z}}(C(X)^*, C(X))).$$

This is the *symmetric construction* on X. If we assume that κ satisfies the contractibility condition, then ϕ_X is well defined up to chain homotopy.

There are also symmetric constructions on the iterated suspensions $\Sigma^p X_+$, with the CW structure induced from the triangulation of X. First we have the diagonal approximation:

$$\widetilde{\Delta}_{\Sigma^p X_+} \ : \ S^p C(X) = \widetilde{C}(\Sigma^p X_+) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, S^{2p} D^\kappa(C(X))),$$

for each $p \geq 0$. Using the identification $S^{2p}D^\kappa(C(X)) = \mathrm{Hom}^\kappa_{\mathbf{Z}}((S^pC(X))^*, S^pC(X))$, we obtain the symmetric construction on $\Sigma^p X_+$:

$$\widetilde{\phi}_{\Sigma^p X_+} : S^p C(X) = \widetilde{C}(\Sigma^p X_+) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \mathrm{Hom}^\kappa_{\mathbf{Z}}((S^pC(X))^*, S^pC(X))).$$

This is well-defined if κ satisfies the contractibility condition.

Let us use the following notation:

$$W^\%_\kappa(S^pC(X)) = \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \mathrm{Hom}^\kappa_{\mathbf{Z}}((S^pC(X))^*, S^pC(X))).$$

$W^\%_\kappa(S^pC(X))_n$ is isomorphic to $\sum_{s\geq 0} \mathrm{Hom}^\kappa_{\mathbf{Z}}((S^pC(X))^*, S^pC(X))_{n+s}$, and an n-chain $\phi \in W^\%_\kappa(S^pC(X))_n$ is a collection $(\phi_s \in \mathrm{Hom}^\kappa_{\mathbf{Z}}((S^pC(X))^*, S^pC(X))_{n+s})_{s\geq 0}$.

There are suspension maps between these:

$$S : SW^\%_\kappa(S^pC(X)) \longrightarrow W^\%_\kappa(S^{p+1}C(X)).$$

To avoid sign complications we define only the double suspension S^2 here. For an n-chain $\phi = (\phi_s)_{s\geq 0}$ of $S^2W^\%_\kappa(S^pC(K))$, define its double suspension $S^2\phi \in W^\%_\kappa(S^{p+2}C(X))_n$ by $(S^2\phi)_s = \phi_{s-2}$. (Here $\phi_s = 0$ if $s < 0$.) If κ satisfies the contractibility condition, the method of acyclic models gives a chain homotopy

$$\widetilde{\phi}_{\Sigma^{2p} X_+} \simeq S^{2p}\phi_X \ : \ S^{2p}C(X) \to W^\%_\kappa(S^{2p}C(X))$$

for each $p \geq 0$, as we mentioned before.

Now we discuss the quadratic construction. Let X, Y be polyhedra equipped with block systems κ, λ respectively, and let $G : \Sigma^p X_+ \to \Sigma^p Y_+$ be a cellular map which preserves the block system, with $p(\gg \dim Y)$ even.

Define a $\mathbf{Z}$-module chain complex $W_{\%}^{\lambda}(C(Y))$ by:

$$\begin{aligned} W_{\%}^{\lambda}(C(Y)) &= W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \mathrm{Hom}_{\mathbf{Z}}^{\lambda}(C(Y)^*, C(Y)) \\ &= \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W^*, \mathrm{Hom}_{\mathbf{Z}}^{\lambda}(C(Y)^*, C(Y))). \end{aligned}$$

$W_{\%}^{\lambda}(C(Y))_n$ is isomorphic to $\sum_{s\geq 0} \mathrm{Hom}_{\mathbf{Z}}^{\lambda}(C(Y)^*, C(Y))_{n-s}$, and an n-chain $\psi \in W_{\%}^{\kappa}(C(Y))_n$ is a collection $(\psi_s \in \mathrm{Hom}_{\mathbf{Z}}^{\lambda}(C(Y)^*, C(Y))_{n-s})_{s\geq 0}$. There is a $\mathbf{Z}$-module chain map $1+T \ : \ W_{\%}^{\lambda}(C(Y)) \to W_{\lambda}^{\%}((C(Y))$ defined by:

$$((1+T)\psi)_s = \begin{cases} (1+T)\psi_0 & \text{if } s = 0, \\ 0 & \text{if } s \geq 1. \end{cases}$$

Assume that λ satisfies the contractibility condition. The quadratic construction associates to G a $\mathbf{Z}$-module chain map

$$\psi_G \ : \ C(X) \longrightarrow W_{\%}^{\lambda}(C(Y))$$

such that

$$(1+T)\psi_G \simeq \phi_Y g - g^{\%}\phi_X \ : \ C(X) \longrightarrow W_{\lambda}^{\%}(C(Y))$$

with $g : C(X) = \Omega^p \widetilde{C}(\Sigma^p X_+) \longrightarrow \Omega^p \widetilde{C}(\Sigma^p Y_+) = C(Y)$ the $\mathbf{Z}$-module chain map induced by G and $g^{\%} : W_{\kappa}^{\%}(C(X)) \to W_{\lambda}^{\%}(C(Y))$ the $\mathbf{Z}$-module chain map induced by $\mathrm{Hom}(g^*, g)$.

The composite $\mathbf{Z}$-module chain map

$$S^p(\phi_Y g - g^{\%}\phi_X) : C(X) \xrightarrow{\phi_Y g - g^{\%}\phi_X} W_{\lambda}^{\%}(C(Y)) \xrightarrow{S^p} \Omega^p W_{\lambda}^{\%}(C(\Sigma^p Y_+))$$

is chain homotopic to $\widetilde{\phi}_{\Sigma^p Y_+} g - g^{\%}\widetilde{\phi}_{\Sigma^p X_+}$;

$$S^p(\phi_Y g - g^{\%}\phi_X) = S^p\phi_Y g - g^{\%} S^p \phi_X \simeq \widetilde{\phi}_{\Sigma^p Y_+} g - g^{\%}\widetilde{\phi}_{\Sigma^p X_+}.$$

And there is a chain homotopy

$$\widetilde{\phi}_{\Sigma^p Y_+} g - g^{\%}\widetilde{\phi}_{\Sigma^p X_+} \simeq 0 \ : C(X) = \Omega^p \widetilde{C}(\Sigma^p X_+) \longrightarrow \Omega^p W_{\lambda}^{\%}(C(\Sigma^p Y_+))$$

by the naturality of symmetric construction applied to the map $G : \Sigma^p X_+ \to \Sigma^p Y_+$. (The contractibility condition on λ is used here.) Thus we obtain a $\mathbf{Z}$-module chain homotopy

$$\begin{aligned} &\widehat{\psi}_G : S^p(\phi_Y g - g^{\%}\phi_X) \simeq 0 \ : \\ &\qquad C(X) \longrightarrow \Omega^p W_{\lambda}^{\%}(C(\Sigma^p Y_+)) = \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(\Omega^p W, \mathrm{Hom}_{\mathbf{Z}}^{\lambda}(C(Y)^*, C(Y))). \end{aligned}$$

For each $\sigma \in C(X)_n$, $\widehat{\psi}_G(\sigma)$ is a collection

$$(\ \widehat{\psi}_G(\sigma)_s \in \mathrm{Hom}_{\mathbf{Z}}^{\lambda}(C(Y)^*, C(Y))_{n+1+s}\ )_{s \geq -p}.$$

When p is sufficiently large, the negative part $(\widehat{\psi}_G(\text{-})_s)_{-p\leq s\leq -1}$ can be interpreted as a $\mathbf{Z}$-module chain map $\psi_G \ : \ C(X) \longrightarrow W_{\%}^{\lambda}(C(Y))$, and the non-negative part $(\widehat{\psi}_G(\text{-})_s)_{s\geq 0}$ can be interpreted as the desired chain homotopy.

Exercise. Work out the relative versions of these constructions.

3. Geometric module chain complexes.

In this section we relate the "block system" control used in the previous sections with a different type of control which uses the geometric algebra of Quinn [3]. As an analogue of $\mathrm{Hom}^{\kappa}_{\mathbf{Z}}(C^*, C)$, we shall introduce a $\mathbf{Z}[\mathbf{Z}_2]$-module chain complex $\overline{\mathrm{GMor}}_\epsilon(C^*, C)$, where ϵ is a positive number and C is a geometric module chain complex on a metric space. A good reference for geometric algebra is [8]. See A2 for a summary.

Suppose X is a topological space equipped with a control map $p : X \to Z$ to a metric space Z. Let C and D be geometric module chain complexes on X, and ϵ be a positive number.

Definition. $\mathrm{GMor}_{p^{-1}(\epsilon)}(C, D)(r, s)$ is the set of all geometric morphisms $f : C_r \longrightarrow D_s$ that satisfy

1. $d^i_D f d^j_C$ has radius ϵ for every $i, j \geq 0$, and
2. $d^i_D f d^j_C \sim_\epsilon 0$ for every $i, j \geq 2$.

Note that this is not an empty set, because it contains the zero geometric morphism. Also note that it depends not only on C_r and D_s but also on the boundary morphisms of C and D. This is the reason why we do not employ the notation $\mathrm{GMor}_{p^{-1}(\epsilon)}(C_r, D_s)$. The ϵ homotopy $\sim_\epsilon$ is certainly an equivalence relation on the set $\mathrm{GMor}_{p^{-1}(\epsilon)}(C, D)(r, s)$. The following is a finer relation.

Definition. Let f, f' be two geometric morphisms in $\mathrm{GMor}_{p^{-1}(\epsilon)}(C, D)(r, s)$. f, f' are said to be *equivalent* if $d^i_D f d^j_C \sim_\epsilon d^i_D f' d^j_C$ for every $i, j \geq 0$. This is an equivalence relation, and the set of the equivalence classes is denoted by $\overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C, D)(r, s)$. The equivalence class of f is denoted $[f]$.

$\overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C, D)(r, s)$ is an abelian group with respect to the sum $[f] + [g] = [f + g]$.

Definition. $\overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C, D)$ is a $\mathbf{Z}$-module chain complex defined by

$$d : \overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C, D)_n = \sum_{s-r=n} \overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C, D)(r, s) \longrightarrow \overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C, D)_{n-1};$$

$$[f] \in \overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C, D)(r, s) \longmapsto d[f] = [d_D f] + (-)^s [f d_C]$$

(If we use $\sim_\epsilon$ as the equivalence relation, then d is not necessarily well-defined.) When $p : X \to Z$ is the identity map, we use the notation $\overline{\mathrm{GMor}}_\epsilon(C, D)$.

$T \in \mathbf{Z}_2$ acts on $\overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C^*, C)_n$ by

$$T[f] = (-)^{rs}[f^*] \qquad \text{for} \quad [f] \in \overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C^*, C)(r, s).$$

With respect to this action, $\overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(C^*, C)$ is a $\mathbf{Z}[\mathbf{Z}_2]$-module chain complex.

The following relates the block-system control with the ϵ control.

Proposition 3.1. *Let X be a polyhedron, κ be a block system for X satisfying the contractibility condition. Suppose $p : X \to Z$ is a control map, and assume that the diameter of the image in Z of each block in κ is at most ϵ. Then there is a $\mathbf{Z}[\mathbf{Z}_2]$-module chain map from $\mathrm{Hom}^{\kappa}_{\mathbf{Z}}(C(X)^*, C(X))$ to $\overline{\mathrm{GMor}}_{p^{-1}(2\epsilon)}(G(X)^*, G(X))$ (or $\overline{\mathrm{GMor}}_{p^{-1}(\epsilon)}(G(X)^*, G(X))$ if κ is standard) which sends $f_{\sigma,\tau}$ to the class of any path connecting the representing points of σ and τ inside a block containing both σ and τ. Here $C(X)$ denotes the ordinary $\mathbf{Z}$-module chain complex of X, and $G(X)$ denotes the geometric module chain complex of X.*

Proof: Immediate from the contractibility condition. ☐

4. Geometric symmetric/quadratic complexes.

In this section we associate to a *PL* manifold a geometric symmetric complex, and to a degree 1 normal map between *PL* manifolds a geometric quadratic complex.

Definition. Let X be a topological space equipped with a control map $p : X \to Z$. An *n-dimensional geometric* $\begin{cases} \textit{symmetric complex} & (G,\phi) \\ \textit{quadratic complex} & (G,\psi) \end{cases}$ *of radius ϵ* is an n-dimensional ϵ chain complex G on X together with a representative $\begin{cases} \phi \\ \psi \end{cases}$ of an n-cycle

$$\begin{cases} [\phi] \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \overline{\mathrm{GMor}}_\epsilon(G^*, G)). \\ [\psi] \in W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \overline{\mathrm{GMor}}_\epsilon(G^*, G). \end{cases}$$

It is ϵ *Poincaré* if

$$\begin{cases} \phi_0 & : G^{n-*} \longrightarrow G \\ (1+T)\psi_0 & : G^{n-*} \longrightarrow G \end{cases}$$

is an ϵ chain equivalence.

Let X be an oriented *PL* manifold of dimension n, and fix a *PL* triangulation of X and a metric on X. As before $C(X)$ will denote the cellular $\mathbf{Z}$-module chain complex of X, and $G(X)$ will denote the geometric module chain complex of X. Let $\phi_X : C(X) \longrightarrow W^{\%}_{\kappa}(C(X))$ be the symmetric construction on X with respect to the standard block system κ. If the diameter of each simplex of X is at most ϵ, we have a $\mathbf{Z}[\mathbf{Z}_2]$-module chain map

$$\mathrm{Hom}^{\kappa}_{\mathbf{Z}}(C(X)^*, C(X)) \longrightarrow \overline{\mathrm{GMor}}_\epsilon(G(X)^*, G(X))$$

by 3.1, and it induces a $\mathbf{Z}$-module chain map

$$\iota : W^{\%}_{\kappa}(C(X)) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \overline{\mathrm{GMor}}_\epsilon(G(X)^*, G(X))).$$

Let $[X] \in C(X)_n$ denote the n-cycle representing the fundamental class. We define the geometric module symmetric complex $\sigma^*(X)$ by

$$\sigma^*(X) = (G(X), \phi),$$

where $[\phi] = \iota\phi_X[X]$.

We shall show that $\sigma^*(X)$ is ϵ Poincaré. (If ϵ is sufficiently small, this defines the "controlled symmetric signature" $\sigma^*(X) \in L^n_c(X)$ in the "controlled symmetric L-group" of X).

To calculate ϕ_0, fix an order on the set of the vertices of X, give each simplex the orientation induced from this order, and use the Alexander-Whitney-Steenrod diagonal approximation. If $[X] = \sum n_\sigma \sigma$ ($n_\sigma = \pm 1$), then

$$(\Delta_X[X])_0 = \sum_\sigma \sum_{i+j=n} n_\sigma({}_i\sigma \otimes \sigma_j),$$

where ${}_i\sigma$ and σ_j denote the front i-face and the back j-face of the n-simplex σ, respectively. Therefore $(\phi_X[X])_0 : C(X)^{n-*} \longrightarrow C(X)$ is given by

$$\tau^* \longmapsto \sum_{{}_{(n-j)}\sigma = \tau} n_\sigma \sigma_j$$

for $\tau \in C(X)_{n-j}$. If we denote the geometric module chain complex of the dual cell complex of X by $D(X)$, then $\phi_0 : D(X) = G(X)^{n-*} \longrightarrow G(X)$ can be identified with the ϵ chain equivalence $D(X) \longrightarrow G(X)$ induced by a cellular approximation of the identity map from X (with the dual cell structure) to X (with the original triangulation) which sends the barycenter $\widehat{\tau}$ of a simplex τ of X to the largest vertex of τ (with respect to the given order). Thus $\sigma^*(X)$ is ϵ Poincaré.

Next let M, X be oriented PL manifolds of dimension n, and let $(f, b) : M \longrightarrow X$ be a degree 1 normal map. Assume that f is simplicial with respect to some PL triangulations and assume that the diameter of each simplex of X is at most ϵ.

One can form the geometric Umkehr map

$$G : \Sigma^p X_+ \longrightarrow \Sigma^p M_+$$

with p even and large. Let κ be the standard block system for X, and λ be the regular block system for M. $\Sigma^p X_+$, $\Sigma^p M_+$ are given the corresponding suspension block systems. We may assume that G is cellular and preserves the block system. Let

$$\psi_G \ : \ C(X) \longrightarrow W^\lambda_\%(C(M))$$

be the quadratic construction on G.

Next define the Umkehr chain map $f^! : G(X) \longrightarrow f_\sharp G(M)$ by:

$$f^! : G(X) \simeq_\epsilon G(X)^{n-*} \xrightarrow{f_\%^*} (f_\sharp G(M))^{n-*} \simeq_\epsilon f_\sharp G(M),$$

where $f_\% : f_\sharp G(M) \longrightarrow G(X)$ is the chain map induced by f. Let $e : f_\sharp G(M) \to \mathcal{C}(f^!)$ denote the inclusion map.

Let $[\psi] \in W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \overline{\mathrm{GMor}_{6\epsilon}}(\mathcal{C}(f^!)^*, \mathcal{C}(f^!))$ be the image of $[X] \in C(X)$ by the following composite chain map

$$C(X) \xrightarrow{\psi_G} W^\lambda_\%(C(M)) = W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \mathrm{Hom}^\lambda_{\mathbf{Z}}(C(M)^*, C(M))$$
$$\xrightarrow{3.1} W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \overline{\mathrm{GMor}}_{f^{-1}(6\epsilon)}(G(M)^*, G(M)) \longrightarrow W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \overline{\mathrm{GMor}_{6\epsilon}}(f_\sharp G(M)^*, f_\sharp G(M))$$
$$\xrightarrow{e_\%} W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \overline{\mathrm{GMor}_{6\epsilon}}(\mathcal{C}(f^!)^*, \mathcal{C}(f^!)),$$

where $e_\%$ denotes the map induced by the map $h \mapsto ehe^*$. Now define a 6ϵ Poincaré quadratic complex $\sigma_*(f,b)$ of radius 6ϵ by $(\mathcal{C}(f^!), \psi)$. This ends the construction.

APPENDIX

A1. Summary of chain complexes.

Let A be a ring. In this paper we only deal with the cases $A = \mathbf{Z}$ and $A = \mathbf{Z}[\mathbf{Z}_2]$, so assume that A is commutative.

- The *algebraic mapping cone* $\mathcal{C}(f)$ of an A-module chain map $f : C \to D$ is the A-module chain complex defined by

$$d_{\mathcal{C}(f)} = \begin{pmatrix} d_D & (-)^{r-1} f \\ 0 & d_C \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

- Given A-module chain complexes C and D, $C \otimes_A D$ and $\mathrm{Hom}_A(C, D)$ are the $\mathbf{Z}$-module chain complexes defined by

$$d_{C\otimes_A D} : (C \otimes_A D)_n = \sum_{p+q=n} C_p \otimes_A D_q \to (C \otimes_A D)_{n-1};$$
$$x \otimes y \mapsto x \otimes d_D(y) + (-)^q d_C(x) \otimes y,$$

$$d_{\mathrm{Hom}_A(C,D)} : \mathrm{Hom}_A(C, D)_n = \sum_{q-p=n} \mathrm{Hom}_A(C_p, D_q) \to \mathrm{Hom}_A(C, D)_{n-1};$$
$$f \mapsto d_D f + (-)^q f d_C \quad (: C_{q-n+1} \to C_q).$$

- C^* is the A-module chain complex defined by

$$d_{C^*} = (d_C)^* : (C^*)_r = C^{-r} \to (C^*)_{r-1} = C^{-r+1},$$

and C^{n-*} ($n \in \mathbf{Z}$) is the A-module chain complex defined by

$$d_{C^{n-*}} = (-)^r (d_C)^* : (C^{n-*})_r = C^{n-r} \to (C^{n-*})_{r-1} = C^{n-r+1}.$$

- The *suspension* SC and *desuspension* ΩC of an A-module chain complex C are the A-module chain complexes defined by

$$\begin{cases} d_{SC} = d_C : (SC)_r = C_{r-1} \longrightarrow (SC)_{r-1} = C_{r-2} \\ d_{\Omega C} = d_C : (\Omega C)_r = C_{r+1} \longrightarrow (\Omega C)_{r+1} = C_r \end{cases}$$

A2. Summary of geometric algebra.

In this section X, Y denote topological spaces, and ϵ denotes a positive number.

- Let $S : |S| \longrightarrow X$ ($|s| \mapsto [s]$) be a function. We identify a function with its graph; thus, S represents also a subset of $|S| \times X$. The free $\mathbf{Z}$-module on the graph S is called the *geometric module on X generated by S*, and is denoted $\mathbf{Z}[S]$.

- Let $\mathbf{Z}[S]$ and $\mathbf{Z}[T]$ be geometric modules on X. Consider triples (s, ρ, t) consisting of elements $s \in S$, $t \in T$ and a path $\rho : [0, \tau] \to X$ ($\tau \geq 0$) such that $\rho(0) = [s]$ and $\rho(\tau) = [t]$. Such a triple (s, ρ, t) will be called a *path from s to t*. A *geometric morphism* $f : \mathbf{Z}[S] \to \mathbf{Z}[T]$ is a formal linear combination $\sum_{\lambda \in \Lambda} m_\lambda (s_\lambda, \rho_\lambda : [0, \tau_\lambda] \to X, t_\lambda)$ of paths from generators of $\mathbf{Z}[S]$ to generators of $\mathbf{Z}[T]$, with integer coefficients. Here Λ is some index set, and the number of paths starting from each generator is required to be finite. Two geometric morphisms $f = \sum_{\lambda \in \Lambda} m_\lambda (s_\lambda, \rho_\lambda, t_\lambda)$ and $f' = \sum_{\gamma \in \Gamma} m'_\gamma (s'_\gamma, \rho'_\gamma, t'_\gamma)$ from $\mathbf{Z}[S]$ to $\mathbf{Z}[T]$ are the *same* ($f = f'$) if there exists a bijection $\varphi : \Lambda \to \Gamma$ such that

$$m'_{\varphi(\lambda)} = m_\lambda \quad \text{and} \quad (s'_{\varphi(\lambda)}, \rho'_{\varphi(\lambda)}, t'_{\varphi(\lambda)}) = (s_\lambda, \rho_\lambda, t_\lambda) \quad (\text{for all } \lambda \in \Lambda),$$

after deleting terms with zero coefficient.

- The *sum* of two geometric morphisms is defined by formally combining the two linear combinations. The *integer multiplication* of a geometric morphism is defined by termwise integer multiplication. The *difference* $f - g$ of f and g is defined by $f + (-1)g$. The *composition* gf of two consecutive geometric morphisms

$$f = \sum_{\lambda \in \Lambda} m_\lambda (s_\lambda, \rho_\lambda, t_\lambda) : \mathbf{Z}[S] \longrightarrow \mathbf{Z}[T], \quad g = \sum_{\gamma \in \Gamma} n_\gamma (t'_\gamma, \sigma_\gamma, u_\gamma) : \mathbf{Z}[T] \longrightarrow \mathbf{Z}[U]$$

is defined to be

$$\sum_{\lambda\in\Lambda,\gamma\in\Gamma,t_\lambda=t'_\gamma} n_\gamma m_\lambda(s_\lambda,\sigma_\gamma\rho_\lambda,u_\gamma),$$

where $\sigma_\gamma\rho_\lambda : [0,\tau_\lambda+\tau'_\gamma] \to X$ is the composite path

$$\sigma_\gamma\rho_\lambda(x) = \begin{cases} \rho_\lambda(x) & \text{if } 0\le x\le\tau_\lambda, \\ \sigma_\gamma(x-\tau_\lambda) & \text{if } \tau_\lambda\le x\le\tau_\lambda+\tau'_\gamma, \end{cases}$$

of two paths $\rho_\lambda : [0,\tau_\lambda] \to X$, $\sigma_\gamma : [0,\tau'_\gamma] \to X$ with $\rho_\lambda(\tau_\lambda) = \sigma_\gamma(0)$.

- A geometric morphism with no term is called the *zero geometric morphism*, and is denoted 0.

- Let $\mathbf{Z}[S]$ be a geometric module on X and define a "one-point" path $c_s : \{0\} \to X$ by $c_s(0) = [s]$, for $s \in S$. The geometric morphism

$$\sum_{s\in S} 1(s,c_s,s) : \mathbf{Z}[S] \longrightarrow \mathbf{Z}[S]$$

is called the *identity geometric morphism on* $\mathbf{Z}[S]$, and is denoted $1_{\mathbf{Z}[S]}$ or simply 1.

- A *homotopy* of a path (s,ρ,t) is a homotopy of the path ρ with both ends fixed. Here a homotopy is allowed to change continuously the interval on which the path is defined. A *homotopy* ($\sim$) of a geometric morphism is a finite sequence of the following operations:
  1. homotopies of the paths,
  2. combining two terms $m(s,\rho,t)+n(s,\rho,t)$ into $(m+n)(s,\rho,t)$, and its inverse.

- Let $\varphi : X \to Y$ be a continuous map. For a geometric module $A = \mathbf{Z}[S]$ on X, its *direct image* $\varphi_\sharp A$ is defined to be the geometric module $\mathbf{Z}[\varphi S : |S| \to X \to Y]$ on Y. If $s = (|s|,[s])$ is an element of the graph S, φs will denote the element $(|s|,\varphi[s])$ of the graph of $\varphi S : |S| \to Y$. If $f = \sum m_\lambda(s_\lambda,\rho_\lambda,t_\lambda) : A \to B$ is a geometric morphism between geometric modules A, B on X, then $\varphi_\sharp f : \varphi_\sharp A \to \varphi_\sharp B$ will denote the geometric morphism

$$\sum m_\lambda(\varphi s_\lambda, \varphi\rho_\lambda : [0,\tau_\lambda] \xrightarrow{\rho_\lambda} X \xrightarrow{\varphi} Y, \varphi t_\lambda).$$

If $f\sim g$, then $\varphi_\sharp f \sim \varphi_\sharp g$.

- A map $p : X \longrightarrow Z$ to a metric space is called a *control map*.

In the following we assume that X is equipped with a control map $p : X \longrightarrow Z$.

- A geometric morphism f *has radius* ϵ if, for each path (s, ρ, t) appearing with non-zero coefficient in f, the image of the path $p\rho$ in Z is contained in the closed ϵ neighbourhoods of $p[s]$ and $p[t]$. A homotopy of geometric morphisms of radius ϵ is an ϵ *homotopy* $(\sim_\epsilon)$ if
  1. each homotopy of a path (s, ρ, t) has image via p in the closed ϵ neighbourhoods of $p[s]$ and $p[t]$ in operation 1, and
  2. each path (s, ρ, t) in the combined terms (or split term) has image via p in the closed ϵ neighbourhoods of $p[s]$ and $p[t]$.

- A sequence of morphisms of geometric modules on X

$$\{C, d\} : \ldots \to C_{r+1} \xrightarrow{d_{r+1}} C_r \xrightarrow{d_r} C_{r-1} \to \ldots$$

  is called an ϵ *chain complex* on X if all d_r's have radius ϵ and $d_r d_{r+1} \sim_{2\epsilon} 0$.

- An ϵ *chain map* $f : C \to D$ between chain complexes on X is a collection $f = \{f_r\}$ of geometric morphisms $f_r : (C_r, p_r) \to (D_r, q_r)$ of radius ϵ such that $d_r f_r \sim_\epsilon f_{r-1} d_r$.

- An ϵ *chain homotopy* $h : f \simeq_\epsilon g$ between chain maps $f, g : C \to D$ of radius ϵ is a collection $h = \{h_r\}$ of geometric morphisms $h_r : C_r \to D_{r+1}$ of radius ϵ such that $d_{r+1} h_r + h_{r-1} d_r \sim_{2\epsilon} g_r - f_r$.

- An ϵ chain map $f : C \to D$ is an ϵ *chain equivalence* if there exists an ϵ chain map $g : D \to C$ such that $gf \simeq_\epsilon 1$ and $fg \simeq_\epsilon 1$.

- The dual $\mathbf{Z}[S]^*$ of $\mathbf{Z}[S]$ is defined to be $\mathbf{Z}[S]$ itself.

- The dual $f^* : \mathbf{Z}[T]^* \longrightarrow \mathbf{Z}[S]^*$ of a geometric morphism $f : \mathbf{Z}[S] \longrightarrow \mathbf{Z}[T]$ is obtained by reversing the paths. (f^* may not satisfy the finiteness condition when T is an infinite set.) f and f^* have the same radius.

- If X is a *CW* complex and satisfies a certain technical condition, then one can form a cellular geometric module chain complex $G(X)$ on X. (This condition is satisfied if X is a polyhedron.)

- If a cellular map $f : Y \longrightarrow X$ satisfies a certain technical condition, then one can construct a chain map $f_\% : f_\sharp G(Y) \longrightarrow G(X)$. (This condition is satisfied if f is a simplicial map between polyhedra.)

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